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Anisotropic inverse problems in two dimensions

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Abstract
Let g be a Riemannian metric on a bounded domain in two dimensions with a Lipschitz boundary. We show that one can determine the equivalent class of g and β in the $W^{1,p}$ topology, $p > 2$, from knowledge of the associated Dirichlet-to-Neumann (DN) map $\Lambda_{g,\beta}$ to the elliptic equation $\operatorname{div}_g(\beta\nabla_g u) = 0$. The DN map encodes all the voltage and current measurements at the boundary.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary and let $g = (g_{ij})$ be a Riemannian metric on Ω in the $W^{1,p}(\Omega)$ class with $p > 2$. Let $\beta \in W^{1,p}(\Omega)$ be a scalar function with a positive lower bound. Consider the following elliptic differential operator associated with the metric g:

$$L_{g,\beta}(u) = \operatorname{div}_g(\beta\nabla_g u) = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\sqrt{|g|}\beta g^{ij}\frac{\partial u}{\partial x_j}\right), \tag{1.1}$$

where (g^{ij}) is the inverse of g and $|g| = \det(g_{ij})$. Then for every $f \in W^{2-1/p,p}(\Omega)$, the boundary value problem

$$L_{g,\beta}(u) = 0, \qquad u|_{\partial\Omega} = f, \tag{1.2}$$

has a unique solution $u \in W^{2,p}(\Omega)$. The Dirichlet-to-Neumann (DN) map associated with (1.2) is defined as the map $f \to \Lambda_{g,\beta} f \in W^{1-1/p,p}(\Omega)$ where

$$\Lambda_{g,\beta} f = (\beta\nabla_g u)\rfloor \, \mathrm{d}V_g|_{\partial\Omega} = \nu \cdot \left(\sqrt{|g|}\beta\nabla_g u\right)\Big|_{\partial\Omega} = \sum_{i,j=1}^{2} \sqrt{|g|}\beta\nu_i g^{ij}\frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}, \tag{1.3}$$

with u the unique solution of (1.2) and ν the outer normal of $\partial\Omega$. See endnote 1

Physically β models the electrical conductivity of the domain Ω provided with the metric g. The DN map encodes the current and voltage measurements at the boundary.

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Clearly, both the Dirichlet boundary value problem (1.2) and the DN map (1.3) are conformally invariant. In fact, if $\tilde{g} = cg$ for a scalar function $c \in W^{1,p}(\Omega)$ with a positive lower bound, then

$$L_{\tilde{g},\beta} = c^{-1} L_{g,\beta}, \qquad \Lambda_{\tilde{g},\beta} = \Lambda_{g,\beta}. \tag{1.4}$$

In addition, the DN map $\Lambda_{g,\beta}$ has an invariance property when changing variables in Ω. Let $\Phi : \Omega \to \tilde{\Omega}$ be a $W^{2,p}$ diffeomorphism. The push forward of g under Φ is given by

$$\tilde{g} = \Phi_* g = [(\mathrm{D}\Phi)^{-1} g ((\mathrm{D}\Phi)^{-1})^{\mathrm{T}}] \circ \Phi^{-1}, \tag{1.5}$$

where A^{T} denotes the transpose of the matrix A.

Then the pull back of $\tilde{g}$, given by

$$\Phi^* \tilde{g} = ((\mathrm{D}\Phi) \tilde{g} (\mathrm{D}\Phi)^{\mathrm{T}}) \circ \Phi,$$

is identical to g. By writing the equation $L_{g,\beta} u = 0$ in the integral form

$$\int_\Omega \sum_{i,j=1}^{2} \frac{\partial \phi}{\partial x_i} \left(\sqrt{|g|} \beta g^{ij} \frac{\partial u}{\partial x_j} \right) \mathrm{d}x = 0, \qquad \forall \phi \in C_0^\infty(\Omega),$$

and making the change of variables $y = \Phi(x)$, it is easy to show that u is a solution of $L_{g,\beta} u = 0$ in Ω if and only if $\tilde{u} = u \circ \Phi^{-1}$ is a solution of $L_{\tilde{g},\tilde{\beta}} \tilde{u} = 0$ in $\tilde{\Omega}$, where

$$\tilde{\beta} = \beta \circ \Phi^{-1} = \Phi_* \beta. \tag{1.6}$$

Furthermore, the DN maps $\Lambda_{g,\beta}$ and $\Lambda_{\tilde{g},\tilde{\beta}}$ are related by the following identity:

$$\int_{\partial\Omega} \phi \Lambda_{g,\beta}(f) \, \mathrm{d}s = \int_{\partial\tilde{\Omega}} \tilde{\phi} \Lambda_{\tilde{g},\tilde{\beta}}(\tilde{f}) \, \mathrm{d}\tilde{s}, \tag{1.7}$$

where $f \in W^{2-1/p,p}(\tilde{\Omega})$, $\tilde{f} = f \circ \Phi^{-1}$ and $\tilde{\Omega} = \Omega \circ \Phi^{-1}$. Here $\mathrm{d}s$ and $\mathrm{d}\tilde{s}$ denote the measures on $\partial\Omega$, $\partial\tilde{\Omega}$ respectively.

Identity (1.7) implies that if the diffeomorphism above is the identity on $\partial\Omega$, then

$$\Lambda_{g,\beta} = \Lambda_{\tilde{g},\tilde{\beta}}.$$

This shows that the DN map $\Lambda_{g,\beta}$ is also invariant under the above transformation in g and β defined in (1.5) and (1.6). Therefore we have

$$\Lambda_{c\Phi_* g, \Phi_* \beta} = \Lambda_{g,\beta} \tag{1.8}$$

for any diffeomorphism $\Phi \in W^{2,p}(\Omega)$ with $\Phi|_{\partial\Omega} =$ identity and any scalar function $c \in W^{1,p}(\Omega)$ with a positive lower bound.

Given (g_1, β_1) and (g_2, β_2), we define $(g_1, \beta_1) \sim (g_2, \beta_2)$ if there is a diffeomorphism $\Phi \in W^{2,p}(\Omega)$ with $\Phi|_{\partial\Omega} =$ identity and a scalar function $c \in W^{1,p}(\Omega)$ with a positive lower bound such that $g_2 = c\Phi_* g_1$ and $c_2 = \Phi_* c_1$. Then from (1.8) we see that the map

$$\Lambda : [(g, \beta)] \to \Lambda_{g,\beta}$$

is well defined where $[(g, \beta)]$ stands for the equivalent class under the equivalence relation $\sim$. See endnote 2

In this paper, we prove that the map Λ is injective. In other words, we show that one can determine $[(g, \beta)]$ from knowledge of $\Lambda_{g,\beta}$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary. Let g_1 and g_2 be two Riemannian metrics in $W^{1,p}(\Omega)$ with $g_1 - g_2 \in W_0^{1,p}(\Omega)$. Let β_1 and β_2 be two scalar functions in $W^{1,p}(\Omega)$ with positive lower bounds. If*

$$\Lambda_{g_1,\beta_1} = \Lambda_{g_2,\beta_2},$$

then there exists a diffeomorphism $\Phi : \Omega \to \Omega$ *in the* $W^{2,p}$ *class with* $\Phi|_{\partial\Omega} = \text{identity}$ *such that*

$$g_2 = c\Phi_* g_1$$

for some positive function $c \in W^{1,p}(\Omega)$ *and*

$$\beta_2 = \Phi_* \beta_1.$$

Remarks.

(a) We remark that if the metric g is $C^{2,1}(\Omega)$ near the boundary and the domain Ω is $C^{1,1}(\Omega)$, it seems possible to remove the assumption that the metrics coincide to order one at the boundary. The boundary determination of the metric g and β and its derivatives would follow by using the method of singular solutions of Alessandrini [A] combined with the use of boundary normal coordinates as in [LU].

(b) The method of proof of theorem 1.1 is based on the reduction to a first-order system as in [BU] for isotropic conductivities and isothermal coordinates [Ah]. The uniqueness proof in [BU] was developed into a reconstruction method in [KT] for conductivities in $C^{1+\epsilon}$. By solving the Beltrami equation (see section 2) and using [KT], it is likely that one can also develop a reconstruction algorithm for slightly smoother β. Stability estimates were derived in [BBR] using the uniqueness proof of [BU] for $C^{1+\epsilon}$ conductivities. We also expect that stability estimates can be proven under slightly smoother assumptions on β for the case considered in this paper.

In the case where $\beta_1 = \beta_2 = 1$ on Ω and the Riemannian metric is smooth, theorem 1.1 was proven in [LU] and was extended to general connected, compact Riemannian manifolds with a boundary in [LaU]. In the case where the Riemannian metric is Euclidean, this problem is the electrical impedance tomography problem for isotropic conductivities. Uniqueness was proven in [N] for $\beta \in W^{2,p}$, $p > 1$ and extended in [BU] to conductivities in $W^{1,p}$, $p > 2$. An immediate consequence of theorem 1.1 is the extension of the [BU] result when the background metric is not Euclidean. More precisely we have

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with a Lipschitz boundary. Let* g *a Riemannian metric in* $W^{1,p}(\Omega)$. *Let* β_1 *and* β_2 *be two scalar functions in* $W^{1,p}(\Omega)$ *with positive lower bounds. If*

$$\Lambda_{g,\beta_1} = \Lambda_{g,\beta_2},$$

then

$$\beta_1 = \beta_2.$$

We now discuss an application of theorem 1.1 to anisotropic conductivities. See [U2] for a recent survey. Let $\gamma = (\gamma^{ij})$ be a positive definite symmetric matrix on $\bar{\Omega}$ in the $W^{1,p}$ class, $p > 2$. The conductivity equation is given by

$$L_\gamma(u) = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) = 0, \qquad u|_{\partial\Omega} = f, \tag{1.9}$$

and the DN map is defined as before by $f \to \Lambda_\gamma f \in W^{1-1/p,p}(\Omega)$ where

$$\Lambda_\gamma f = \sum_{i,j=1}^{2} \nu_i \gamma^{ij} \frac{\partial u}{\partial x_j}\bigg|_{\partial\Omega}, \tag{1.10}$$

with u the unique solution of (1.9) and ν the unit outer normal of $\partial\Omega$.

Let $\Phi : \Omega \to \tilde{\Omega}$ be a $W^{2,p}$ diffeomorphism. The push forward of γ under Φ is given by

$$\Phi_*\gamma = \left(\frac{[(\mathrm{D}\Phi)^{-1}g((\mathrm{D}\Phi)^{-1})^{\mathrm{T}}]}{|\det \mathrm{D}\Phi|}\right) \circ \Phi^{-1}. \tag{1.11}$$

A direct consequence of theorem 1.1 is the following.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary. Let γ_1 and γ_2 be two anisotropic conductivities in $W^{1,p}(\Omega)$ with $\gamma_1 - \gamma_2 \in W_0^{1,p}(\Omega)$. Assume*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2};$$

then there exists a diffeomorphism $\Phi : \Omega \to \Omega$ in the $W^{2,p}$ class with $\Phi|_{\partial\Omega} =$ identity such that

$$\gamma_2 = \Phi_*\gamma_1.$$

This result follows from theorem 1.1 on taking $\gamma_i = g_i^{-1}$ and $\beta_i = \sqrt{|g_i|}$, $i = 1, 2$.

This result was previously known for $C^3(\Omega)$ anisotropic conductivities. It follows by combining the result of [S], which reduces the anisotropic problem to the isotropic one by using isothermal coordinates [Ah], and the result of Nachman [N] for isotropic conductivities.

The following theorem shows that the smoothness of the diffeomorphism Φ depends only on the smoothness of the metric g.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a $C^{k,\alpha}$ boundary, where k is a positive integer and $0 < \alpha < 1$. Let g_1 and g_2 be two Riemannian metrics in $C^{k,\alpha}(\bar{\Omega})$ with $\mathrm{D}^\gamma g_1 = \mathrm{D}^\gamma g_2$ on $\partial\Omega$, $|\gamma| \leqslant k$. Let β_1 and β_2 be two scalar functions in $W^{1,p}(\Omega)$ with positive lower bounds. If*

$$\Lambda_{g_1,\beta_1} = \Lambda_{g_2,\beta_2},$$

then there exists a diffeomorphism $\Phi : \Omega \to \Omega$ in the $C^{k+1,\alpha}(\bar{\Omega})$ class with $\Phi|_{\partial\Omega} =$ identity such that

$$g_2 = c\Phi_* g_1$$

for some scalar function $c \in C^{k,\alpha}(\bar{\Omega})$ with positive lower bound and

$$\beta_2 = \Phi_*\beta_1.$$

For a description of other results in anisotropic inverse boundary problems, we refer the reader to the survey papers [U1] and [U2].

2. Lemmas

Lemma 2.1. *Let g_1 and g_2 be two Riemannian metrics in $W^{1,p}(\Omega)$ with $g_1 - g_2 \in W_0^{1,p}(\Omega)$. Let β_1 and β_2 be two positive scalar functions in $W^{1,p}(\Omega)$. If*

$$\Lambda_{q_1,\beta_1} = \Lambda_{q_2,\beta_2},$$

then on $\partial\Omega$,

$$\beta_1 = \beta_2.$$

Proof. Since $g_1 - g_2 \in W_0^{1,p}(\Omega)$, we can extend g_1 and g_2 outside Ω so that

$$\begin{aligned} &g_1(x) = g_2(x), \qquad x \in \Omega^c, \\ \text{and} \\ &g_1(x) = g_2(x) = e, \qquad \text{for } |x| \text{ large enough}, \end{aligned} \tag{2.1}$$

and the extended metrics, which we still denote by g_i, $i = 1, 2$, are in $W^{1,p}(\mathbb{R}^2)$. Let $\Phi \in W^{2,p}(\Omega)$ be a conformal diffeomorphism: $\Omega \to \tilde{\Omega}$ (see (2.4) below) so that $\Phi_* g_1$ is the Euclidean metric on $\tilde{\Omega}$. Then $\Phi_* g_2$ is also the Euclidean metric on $\partial\tilde{\Omega}$. From the proof in [A] we see that on $\partial\tilde{\Omega}$,

$$\Phi_*\beta_1 = \Phi_*\beta_2.$$

This, together with $g_1 - g_2 \in W_0^{1,p}(\Omega)$, implies the result. □

According to lemma 2.1, we can extend β_1 and β_2 outside Ω so that

$$\beta_1(x) = \beta_2(x), \qquad x \in \Omega^c,$$

and (2.2)

$$\beta_1(x) = \beta_2(x) = 1, \qquad \text{for } |x| \text{ large enough},$$

and the extended function, which we still denote by β_i, $i = 1, 2$, is in $W^{1,p}(\mathbb{R}^2)$, where e stands for the Euclidean metric. We shall assume (2.1) and (2.2) throughout the rest of the paper.

Let $g, \beta \in W^{1,p}(\mathbb{R}^2)$ with

$$g(x) = e, \qquad \beta(x) = 1, \qquad \text{for } |x| \text{ large enough.} \tag{2.3}$$

We use the notation

$$\mu_g = \frac{g_{11} - g_{22} + 2\mathrm{i}g_{12}}{g_{11} + g_{22} + 2\sqrt{|g|}} < 1$$

and consider as in [Ah] the Beltrami equation

$$\bar{\partial}\Phi_g = \mu_g \partial\Phi_g. \tag{2.4}$$

Then any diffeomorphism solution of (2.4) corresponds to an isothermal coordinate for the metric g. More precisely,

$$(\Phi_g)_* g = a_g e$$

for some $a_g > 0$, and the equation $L_{g,\beta}u = 0$ is transformed to

$$L_{(\Phi_g)_*g,(\Phi_g)_*\beta} v = \nabla \cdot (\beta \circ \Phi_g^{-1} \nabla v) = 0,$$

where $v = u \circ \Phi_g^{-1}$. In the next lemma, we construct a diffeomorphism Φ_g that behaves like $z = x_1 + \mathrm{i}x_2$ as $|z| \to \infty$ in an appropriate sense. We denote by $L_1^\infty(\mathbb{R}^2)$ the space of functions satisfying $|f(z)| \leqslant C|z|^{-1}$ for some constant C, or equivalently,

$$L_1^\infty(\mathbb{R}^2) = \{f \in L^\infty(\mathbb{R}^2) : zf(z) \in L^\infty(\mathbb{R}^2)\}.$$

Lemma 2.2. *Let $g \in W^{1,p}(\mathbb{R}^2)$ satisfying equation (2.3). Then there exists a diffeomorphism $\Phi_g \in W_{\mathrm{loc}}^{2,p}(\mathbb{R}^2)$, a diffeomorphism of $\mathbb{R}^2$, which solves (2.4) and satisfies*

$$\Phi_g - z \in L_1^\infty(\mathbb{R}^2). \tag{2.5}$$

Moreover, Φ_g^{-1}, $\mathrm{D}\Phi_g - I$ and $\mathrm{D}\Phi_g^{-1} - I$ are all bounded in the $L_1^\infty(\mathbb{R}^2)$ norm.

Proof. We use the method of isothermal coordinates [Ah] although we need the solvability of the Beltrami equation in different spaces to the ones used in [Ah]. We will use the solvability of the Beltrami equation in weighted L^p spaces as was done in [S].

Since $g \in W_{\mathrm{loc}}^{1,p}(\mathbb{R}^2) \subset L_{\mathrm{loc}}^\infty(\mathbb{R}^2)$, we can construct $\Phi_g = z + F$ with F solving the equation

$$\bar{\partial}F - \mu_g \partial F = \mu_g \tag{2.6}$$

in the weighted space $L^{\gamma}_{\delta}(\mathbb{R}^2)$ for some γ and δ, which satisfy (2.18) in [S] (with $\gamma = p$). Clearly, since μ_g is in the $W^{1,p}$ class, we have that F and thus Φ_g is in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$. From the argument following (2.6) in [S], we see that Φ_g is a diffeomorphism from $\mathbb{R}^2$ to itself. We shall show that this diffeomorphism carries the property of (2.5). We recall that if h is a function in $L^{\gamma}(\mathbb{R}^2)$, $\gamma \geqslant 1$, with compact support, then

$$\bar{\partial}^{-1}h = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}^2} \frac{h(w)}{z-w}\, \mathrm{d}w \wedge \mathrm{d}\bar{w} \in L^{\infty}_{1}(\mathbb{R}^2). \tag{2.7}$$

Since μ_g has compact support (note that $g = e$ for $|z|$ large enough) and $\mu_g(\partial F + 1) \in L^p(\mathbb{R}^2)$, it follows from (2.7) that

$$F = \bar{\partial}^{-1}(\mu_g(\partial F + 1)) \in L^{\infty}_{1}(\mathbb{R}^2),$$

which leads to (2.5). To see that Φ_g with (2.5) is unique, let $\tilde{\Phi}_g$ be another diffeomorphism satisfying (2.4) and (2.5). Then $h = \Phi_g - \tilde{\Phi}_g$ is in $L^{\infty}_{1}(\mathbb{R}^2)$ and solves $\bar{\partial}h = \mu_g \partial h$. Since h is uniformly bounded in $\mathbb{R}^2$, it follows from Liouville's theorem (see for instance [BU], section 3) that $h = 0$.

From (2.5) it is easy to see that $\Phi_g^{-1} - z \in L^{\infty}_{1}(\mathbb{R}^2)$. To show that $\mathrm{D}\Phi_g - I \in L^{\infty}_{1}(\mathbb{R}^2)$, let H be one of the derivatives of $F = \Phi_g - z$, say, $H = \partial F$, then, by differentiating (2.6), we have

$$\bar{\partial}H - \mu_g \partial H = \partial\mu_g(1 + \partial F).$$

Again, since μ_g and therefore $\partial\mu_g$ has compact support and $\mu_g\partial H + \partial\mu_g(1+\partial F) \in L^p(\mathbb{R}^2)$, it follows from (2.7) that

$$H = \bar{\partial}^{-1}(\mu_g\partial H + \partial\mu_g(\partial F + 1)) \in L^{\infty}_{1}(\mathbb{R}^2).$$

It remains to show that $\mathrm{D}\Phi_g^{-1} - I \in L^{\infty}_{1}(\mathbb{R}^2)$. We note that

$$\mathrm{D}\Phi_g^{-1} = (\mathrm{D}\Phi_g)^{-1} \circ \Phi_g^{-1}.$$

Since $\Phi_g^{-1} - z \in L^{\infty}_{1}(\mathbb{R}^2)$, Φ_g^{-1} behaves like $z + \mathrm{O}(z^{-1})$ as $|z| \to \infty$, we only have to show that See endnote 3

$$(\mathrm{D}\Phi_g)^{-1} - I \in L^{\infty}_{1}(\mathbb{R}^2). \tag{2.8}$$

For $|z|$ large enough, we have

$$(\mathrm{D}\Phi_g)^{-1} - I = (\mathrm{D}\Phi_g - I + I)^{-1} - I = (\mathrm{D}\Phi_g - I)\sum_{n=0}^{\infty}(-1)^{n+1}(\mathrm{D}\Phi_g - I)^n,$$

and

$$\left|\sum_{n=0}^{\infty}(-1)^{n+1}(\mathrm{D}\Phi_g - I)^n\right| \leqslant \sum_{n=0}^{\infty} C^n|z|^{-n} = (1 + (C/|z|)^{-n})^{-1}.$$

So

$$|(\mathrm{D}\Phi_g)^{-1} - I| \leqslant C|z|^{-1}(1 + (C/|z|)^{-n})^{-1} \leqslant 2C|z|^{-1}$$

for $|z|$ large enough. This proves (2.8). □

Lemma 2.3. *Let $g, \beta \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ satisfying (2.3). Then for each $k \in C$ there exists a pair of solutions $u_{g,\beta}(z,k)$ and $v_{g,\beta}(z,k)$ of (1.1) in $\mathbb{R}^2$ such that*

$$\begin{pmatrix} \partial u_{g,\beta} & \partial v_{g,\beta} \\ \bar{\partial} u_{g,\beta} & \bar{\partial} v_{g,\beta} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}zk} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\bar{z}k} \end{pmatrix} - I \in L^q(\mathbb{R}^2), \qquad \forall q > 2. \tag{2.9}$$

Moreover, this pair of solutions is unique modulo constants.

Proof. Let Φ_g be the diffeomorphism constructed in lemma 2.2. Under Φ_g, the equation $L_{g,\beta}u = 0$ is transformed to $\nabla \cdot (\beta \circ \Phi_g^{-1} \nabla w) = 0$. As in [BU], this equation can be reduced to the first-order elliptic system

$$\left[\begin{pmatrix} \bar\partial & 0 \\ 0 & \partial \end{pmatrix} - \begin{pmatrix} 0 & Q \\ \bar Q & 0 \end{pmatrix}\right]\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0, \tag{2.10}$$

where

$$Q = -\tfrac{1}{2}\partial \log(\beta \circ \Phi_g^{-1}), \qquad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = (\beta \circ \Phi_g^{-1})^{1/2}\begin{pmatrix} \partial w \\ \bar\partial w \end{pmatrix}. \tag{2.11}$$

For each $k \in C$, this system carries a unique matrix solution in the form

$$\Psi(z,k) = m(z,k)\begin{pmatrix} \mathrm{e}^{\mathrm{i}zk} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar z k} \end{pmatrix} \tag{2.12}$$

with

$$m(\cdot,k) - I \in L^q(\mathbb{R}^2), \qquad \forall q > 2. \tag{2.13}$$

Here, m is a matrix function of z and k [BU].

By using Φ_g^{-1}, we can transform $\Psi(z,k)$ to obtain a unique pair of solutions $u_{g,\beta}$ and $v_{g,\beta}$ (modulo constants) and, according to (2.10) and (2.11),

$$\beta^{1/2}\begin{pmatrix} \partial u_{g,\beta} & \partial v_{g,\beta} \\ \bar\partial u_{g,\beta} & \bar\partial v_{g,\beta} \end{pmatrix} = H_{\Phi_g}(m \circ \Phi_g)\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_g k} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_g k} \end{pmatrix}. \tag{2.14}$$

Here H_{Φ_g} is the gradient transformation matrix associated with the diffeomorphism Φ_g:

$$H_{\Phi_g} = \begin{pmatrix} \partial\Phi_g & \partial\bar\Phi_g \\ \bar\partial\Phi_g & \bar\partial\bar\Phi_g \end{pmatrix}. \tag{2.15}$$

From (2.14) we get

$$\begin{pmatrix} \partial u_{g,\beta} & \partial v_{g,\beta} \\ \bar\partial u_{g,\beta} & \bar\partial v_{g,\beta} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}zk} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\bar z k} \end{pmatrix} - I = \beta^{-1/2}H_{\Phi_g}(m \circ \Phi_g)\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_g k - \mathrm{i}zk} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_g k + \mathrm{i}\bar z k} \end{pmatrix} - I. \tag{2.16}$$

Since $\beta^{-1/2} = 1$ for $|z|$ large enough, it is enough to show that

$$H_{\Phi_g}(m \circ \Phi_g)\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_g k - \mathrm{i}zk} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_g k + \mathrm{i}\bar z k} \end{pmatrix} - I \in L^q(\mathbb{R}^2), \qquad \forall q > 2.$$

We rewrite this as

$$\begin{aligned} H_{\Phi_g}(m \circ \Phi_g)\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_g k - \mathrm{i}zk} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_g k + \mathrm{i}\bar z k} \end{pmatrix} - I &= H_{\Phi_g}(m \circ \Phi_g - I)\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_g k - \mathrm{i}zk} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_g k + \mathrm{i}\bar z k} \end{pmatrix} \\ &+ (H_{\Phi_g} - I)\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_g k - \mathrm{i}zk} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_g k + \mathrm{i}\bar z k} \end{pmatrix} + \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_g k - \mathrm{i}zk} - 1 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_g k + \mathrm{i}\bar z k} - 1 \end{pmatrix}. \end{aligned} \tag{2.17}$$

From (2.5) it is clear that

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_g k - \mathrm{i}zk} - 1 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_g k + \mathrm{i}\bar z k} - 1 \end{pmatrix} \in L_1^\infty(\mathbb{R}^2), \tag{2.18}$$

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_g k - \mathrm{i}zk} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_g k + \mathrm{i}\bar z k} \end{pmatrix} \in L^\infty(\mathbb{R}^2) \tag{2.19}$$

and

$$H_{\Phi_g} - I \in L_1^\infty(\mathbb{R}^2), \qquad H_{\Phi_g} \in L^\infty(\mathbb{R}^2). \tag{2.20}$$

So, the second and the third terms on the right-hand side of (2.17) are in $L_1^\infty(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$, $\forall q > 2$. From (2.13) and (2.5), it is easy to show that

$$m \circ \Phi_g - I \in L^q(\mathbb{R}^2), \qquad \forall q > 2.$$

This, together with (2.19) and (2.20), implies that the first term on the right-hand side of (2.17) is also in $L^q(\mathbb{R}^2)$, $\forall q > 2$. This proves that the solution pair constructed above satisfies the property (2.9).

To prove the uniqueness (modulo constants), let $u_{g,\beta}$ and $v_{g,\beta}$ be a pair of solutions satisfying $L_{g,\beta}w = 0$ and (2.9). Then

$$(\beta \circ \Phi_g^{-1})^{1/2} H_{\Phi_g^{-1}} \begin{pmatrix} \partial u_{g,\beta} & \partial v_{g,\beta} \\ \bar\partial u_{g,\beta} & \bar\partial v_{g,\beta} \end{pmatrix} \circ \Phi_g^{-1}$$

is a matrix solution to the system (2.10). Using the properties of $\mathrm{D}\Phi_g$ and $\mathrm{D}\Phi_g^{-1}$ and the argument above that leads to (2.9), one can show that this matrix solution takes the form (2.12) with (2.13), which is unique. This completes the proof. □

3. Proof of theorems

We extend g_1, g_2, β_1 and β_2 as we did in (2.1) and (2.2). Let Φ_{g_1} and Φ_{g_2} be the diffeomorphisms in lemma 2.2 associated with g_1 and g_2, respectively.

We shall first prove that

$$\Phi_{g_1}(z) = \Phi_{g_2}(z), \qquad z \in \Omega^c. \tag{3.1}$$

To this end we consider the solution pairs constructed in lemma 2.3: (u_{g_1}, v_{g_1}) and (u_{g_2}, v_{g_2}). From (2.14), we have

$$\beta^{1/2} \begin{pmatrix} \partial u_{g_i,\beta_i} & \partial v_{g_i,\beta_i} \\ \bar\partial u_{g_i,\beta_i} & \bar\partial v_{g_i,\beta_i} \end{pmatrix} = H_{\Phi_{g_i}}(m_i \circ \Phi_{g_i}) \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_{g_i}k} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_{g_i}k} \end{pmatrix}, \tag{3.2}$$

for $i = 1, 2$, where m_i corresponds to the matrix function m in (2.12) under the diffeomorphism Φ_{g_i}. Notice that m_i satisfies (2.13). We claim that

$$u_{g_1,\beta_1}(z) = u_{g_2,\beta_2}(z), \qquad v_{g_1,\beta_1}(z) = v_{g_2,\beta_2}(z), \qquad z \in \Omega^c. \tag{3.3}$$

To see this, we construct a new solution pair $(u^*_{g_1,\beta_1}, v^*_{g_1,\beta_1})$ as follows. For $z \in \Omega$, $u^*_{g_1,\beta_1}$ solves (1.2) with $g = g_1$, $\beta = \beta_1$ and $f = u_{g_2,\beta_2}|_{\partial\Omega}$. Similarly, $v^*_{g_1,\beta_1}$ solves (1.2) with $g = g_1$, $\beta = \beta_1$ and $f = v_{g_2,\beta_2}|_{\partial\Omega}$. For $z \in \Omega^c$, $u^*_{g_1,\beta_1} = u_{g_2,\beta_2}$ and $v^*_{g_1,\beta_1} = v_{g_2,\beta_2}$. Since $\Lambda_{g_1,\beta_1} = \Lambda_{g_2,\beta_2}$, a well known argument shows that $(u^*_{g_1,\beta_1}, v^*_{g_1,\beta_1})$ is a solution pair of (1.1) with $g = g_1$ and $\beta = \beta_1$ in $W^{1,p}(\mathbb{R}^2)$. Since

$$(u^*_{g_1,\beta_1}, v^*_{g_1,\beta_1}) = (u_{g_2,\beta_2}, v_{g_2,\beta_2}) \tag{3.4}$$

for $z \in \Omega^c$, it is clear that $(u^*_{g_1,\beta_1}, v^*_{g_1,\beta_1})$ satisfies the condition in (2.9). Thus it is the unique solution pair claimed by lemma 2.3 with $g = g_1$ and $\beta = \beta_1$. Therefore,

$$(u^*_{g_1,\beta_1}, v^*_{g_1,\beta_1}) = (u_{g_1,\beta_1}, v_{g_1,\beta_1}). \tag{3.5}$$

Combining (3.4) with (3.5) yields (3.3).

Fix $z \in \Omega^c$. From (3.2) and (3.3) we conclude that

$$H_{\Phi_{g_1}}(m_1 \circ \Phi_{g_1}) \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_{g_1}k} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_{g_1}k} \end{pmatrix} = H_{\Phi_{g_2}}(m_2 \circ \Phi_{g_2}) \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_{g_2}k} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\bar\Phi_{g_2}k} \end{pmatrix}. \tag{3.6}$$

From theorem 2.3 in [BU] we know that $m_i(z, k)$ satisfies

$$\sup_z \|m_i(z, \cdot) - I\|_{L^q(\mathbb{R}^2)} \leqslant C$$

for some constant C and $q > 2$. Then it is easy to show that there exist a sequence $\{k_n\} \subset C$ with $\lim_{n\to\infty} |k_n| = \infty$ such that

$$\lim_{n\to\infty} (m_i(z, k_n) - I) = 0, \qquad i = 1, 2. \tag{3.7}$$

Let us use the notation

$$H_{\Phi_{g_i}} = \begin{pmatrix} h_{11}^{(i)} & h_{12}^{(i)} \\ h_{21}^{(i)} & h_{22}^{(i)} \end{pmatrix}, \qquad m_i = \begin{pmatrix} m_{11}^{(i)} & m_{12}^{(i)} \\ m_{21}^{(i)} & m_{22}^{(i)} \end{pmatrix},$$

for $i = 1, 2$. Then, by setting the (1, 1) entries on either side of (3.6) equal, we get See endnote 4

$$(h_{11}^{(1)} m_{11}^{(1)} + h_{12}^{(1)} m_{21}^{(1)}) \mathrm{e}^{\mathrm{i}\Phi_{g_1} k} = (h_{11}^{(2)} m_{11}^{(2)} + h_{12}^{(2)} m_{21}^{(2)}) \mathrm{e}^{\mathrm{i}\Phi_{g_2} k}. \tag{3.8}$$

Replacing k by k_n in (3.8) and letting n be large enough, we conclude from (3.7) and the fact that $h_{11}^{(i)} \neq 0$ (if $h_{11}^{(i)} = 0$, then by (2.15), the definition of the matrix and (2.4), we would have that $h_{12}^{(i)} = 0$, implying that the matrix H is not invertible) that

$$h_{11}^{(i)} m_{11}^{(i)}(z, k_n) + h_{12}^{(i)} m_{21}^{(i)}(z, k_n) \neq 0, \qquad i = 1, 2.$$

Thus we can take the logarithm of both sides of (3.8) (restricted on $\{k_n\}$ with large n) to get

$$\begin{aligned} &\log(h_{11}^{(1)} m_{11}^{(1)}(z, k_n) + h_{12}^{(1)} m_{21}^{(1)}(z, k_n)) + \mathrm{i}\Phi_{g_1} k_n \\ &\quad = \log(h_{11}^{(2)} m_{11}^{(2)}(z, k_n) + h_{12}^{(2)} m_{21}^{(2)}(z, k_n)) + \mathrm{i}\Phi_{g_2} k_n. \end{aligned} \tag{3.9}$$

Dividing by k_n on both sides of (3.9) and then letting $n \to \infty$ yields

$$\Phi_{g_1}(z) = \Phi_{g_2}(z).$$

This gives (3.1).

Equation (3.1) implies that both Φ_{g_1,β_1} and Φ_{g_2,β_2} send Ω to the same open set Ω^*. Then by (1.7),

$$\Lambda_{(\Phi_{g_1})_* g_1, (\Phi_{g_1})_* \beta_1} = \Lambda_{(\Phi_{g_2})_* g_2, (\Phi_{g_2})_* \beta_2}$$

for the equation $\nabla \cdot (\beta \circ \Phi_{g_i}^{-1} \nabla w) = 0$ with $i = 1, 2$. Thus, by the uniqueness result in [BU],

$$\beta_1 \circ \Phi_{g_1}^{-1} = \beta_2 \circ \Phi_{g_2}^{-1}.$$

If we define $\Phi = \Phi_{g_2}^{-1} \Phi_{g_1}$, then $\Phi|_{\partial\Omega} =$ identity and

$$\beta_2 = \beta_1 \circ \Phi^{-1}.$$

But $(\Phi_{g_i})_* g_i = a_{g_i} e$ for some scalar function $a_{g_i} \in W^{1,p}(\mathbb{R}^2)$, $i = 1, 2$, with a positive lower bound, so we have

$$(a_{g_1} a_{g_2}^{-1}) g_2 = (\Phi_{g_2})_*^{-1} \circ (\Phi_{g_1})_* g_1 = \Phi_* g_1.$$

In other words,

$$g_2 = (a_{g_1}^{-1} a_{g_2}) \Phi_* g_1.$$

This completes the proof of theorem 1.1. See endnote 5

To prove theorem 1.3, we only need to show that $\Phi \in C^{k+1,\alpha}(\bar{\Omega})$. Since Φ_{g_i}, $i = 1, 2$, solves (2.4) with $\mu_{g_i} \in C^{k,\alpha}(\mathbb{R}^2)$ (we can extend the g_i smoothly in $C^{k,\alpha}$ outside Ω so that they satisfy (2.1)), we have, by elliptic regularity, that $\Phi_{g_i} \in C^{k+1,\alpha}(\mathbb{R}^2)$. Therefore $\Phi = \Phi_{g_2}^{-1} \Phi_{g_1} \in C^{k+1,\alpha}(\bar{\Omega})$.

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See endnote 6

See endnote 7

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