Anisotropic inverse problems in two dimensions

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Abstract

Let \(g\) be a Riemannian metric on a bounded domain in two dimensions with a Lipschitz boundary. We show that one can determine the equivalent class of \(g\) and \(\beta\) in the \(W^{1,p}\) topology, \(p > 2\), from knowledge of the associated Dirichlet-to-Neumann (DN) map \(\Lambda_{g,\beta}\) to the elliptic equation \(\text{div}_g(\beta \nabla gu) = 0\). The DN map encodes all the voltage and current measurements at the boundary.

1. Introduction

Let \(\Omega \subset \mathbb{R}^2\) be a bounded domain with a Lipschitz boundary and let \(g = (g_{ij})\) be a Riemannian metric on \(\Omega\) in the \(W^{1,p}(\Omega)\) class with \(p > 2\). Let \(\beta \in W^{1,p}(\Omega)\) be a scalar function with a positive lower bound. Consider the following elliptic differential operator associated with the metric \(g\):

\[
L_{g,\beta}(u) = \text{div}_g(\beta \nabla gu) = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( \sqrt{|g|} \beta g_{ij} \frac{\partial u}{\partial x_j} \right),
\]

(1.1)

where \((g^{ij})\) is the inverse of \(g\) and \(|g| = \det(g_{ij})\). Then for every \(f \in W^{2-1/p, p}(\Omega)\), the boundary value problem

\[
L_{g,\beta}(u) = 0, \quad u|_{\partial \Omega} = f,
\]

(1.2)

has a unique solution \(u \in W^{2,p}(\Omega)\). The Dirichlet-to-Neumann (DN) map associated with (1.2) is defined as the map \(f \rightarrow \Lambda_{g,\beta} f \in W^{1-1/p, p}(\Omega)\) where

\[
\Lambda_{g,\beta} f = (\beta \nabla gu)|_{\partial \Omega} = \nu \cdot \left( \sqrt{|g|} \beta \nabla gu \right)|_{\partial \Omega} = \sum_{i,j=1}^2 \sqrt{|g|} \beta v_i g^{ij} \frac{\partial u}{\partial x_j} \bigg|_{\partial \Omega},
\]

(1.3)

with \(u\) the unique solution of (1.2) and \(\nu\) the outer normal of \(\partial \Omega\).

Physically \(\beta\) models the electrical conductivity of the domain \(\Omega\) provided with the metric \(g\). The DN map encodes the current and voltage measurements at the boundary.
Clearly, both the Dirichlet boundary value problem (1.2) and the DN map (1.3) are conformally invariant. In fact, if \( \bar{g} = cg \) for a scalar function \( c \in W^{1,p}(\Omega) \) with a positive lower bound, then

\[
L_{\tilde{g}, \beta} = c^{-1}L_{g, \beta}, \quad \Lambda_{\tilde{g}, \beta} = \Lambda_{g, \beta}.
\] (4.1)

In addition, the DN map \( \Lambda_{g, \beta} \) has an invariance property when changing variables in \( \Omega \). Let \( \Phi : \Omega \to \tilde{\Omega} \) be a \( W^{2,p} \) diffeomorphism. The push forward of \( g \) under \( \Phi \) is given by

\[
\bar{g} = \Phi_* g = [(D\Phi)^{-1}g((D\Phi)^{-1})^T] \circ \Phi^{-1},
\] (4.5)

where \( A^T \) denotes the transpose of the matrix \( A \).

Then the pull back of \( \bar{g} \), given by

\[
\Phi^* \bar{g} = ((D\Phi) \bar{g} (D\Phi)^T) \circ \Phi,
\]

is identical to \( g \). By writing the equation \( L_{g, \beta}u = 0 \) in the integral form

\[
\int_{\Omega} \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( \sqrt{|g|} \beta g^{ij} \frac{\partial u}{\partial x_j} \right) \, dx = 0, \quad \forall \phi \in C_0^\infty(\Omega),
\]

and making the change of variables \( y = \Phi(x) \), it is easy to show that \( u \) is a solution of \( L_{\tilde{g}, \beta} \bar{u} = 0 \) in \( \tilde{\Omega} \) if and only if \( \bar{u} = u \circ \Phi^{-1} \) is a solution of \( L_{g, \beta} u = 0 \) in \( \Omega \), where

\[
\tilde{\beta} = \beta \circ \Phi^{-1} = \Phi_* \beta.
\] (4.6)

Furthermore, the DN maps \( \Lambda_{g, \beta} \) and \( \Lambda_{\tilde{g}, \tilde{\beta}} \) are related by the following identity:

\[
\int_{\partial\Omega} \phi \Lambda_{g, \beta}(f) \, ds = \int_{\partial\tilde{\Omega}} \tilde{\phi} \Lambda_{\tilde{g}, \tilde{\beta}}(\tilde{f}) \, d\tilde{s},
\] (4.7)

where \( f \in W^{2-1/p, p}(\tilde{\Omega}) \), \( \tilde{f} = f \circ \Phi^{-1} \) and \( \tilde{\Omega} = \Omega \circ \Phi^{-1} \). Here \( ds \) and \( d\tilde{s} \) denote the measures on \( \partial \Omega \), \( \partial \tilde{\Omega} \) respectively.

Identity (4.7) implies that if the diffeomorphism above is the identity on \( \partial \Omega \), then

\[
\Lambda_{g, \beta} = \Lambda_{\tilde{g}, \tilde{\beta}}.
\]

This shows that the DN map \( \Lambda_{g, \beta} \) is also invariant under the above transformation in \( g \) and \( \beta \) defined in (1.5) and (1.6). Therefore we have

\[
\Lambda_{\Phi_* g, \Phi_* \beta} = \Lambda_{g, \beta}
\] (4.8)

for any diffeomorphism \( \Phi \in W^{2,p}(\Omega) \) with \( \Phi|_{\partial \Omega} = \text{identity} \) and any scalar function \( c \in W^{1,p}(\Omega) \) with a positive lower bound.

Given \( (g_1, \beta_1) \) and \( (g_2, \beta_2) \), we define \( (g_1, \beta_1) \sim (g_2, \beta_2) \) if there is a diffeomorphism \( \Phi \in W^{2,p}(\Omega) \) with \( \Phi|_{\partial \Omega} = \text{identity} \) and a scalar function \( c \in W^{1,p}(\Omega) \) with a positive lower bound such that \( g_2 = c \Phi_* g_1 \) and \( c_2 = \Phi_* c_1 \). Then from (4.8) we see that the map \( \Lambda : [(g, \beta)] \to \Lambda_{g, \beta} \)

is well defined where \( [(g, \beta)] \) stands for the equivalent class under the equivalence relation \( \sim \). See endnote 2.

In this paper, we prove that the map \( \Lambda \) is injective. In other words, we show that one can determine \( [(g, \beta)] \) from knowledge of \( \Lambda_{g, \beta} \).

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a Lipschitz boundary. Let \( g_1 \) and \( g_2 \) be two Riemannian metrics in \( W^{1,p}(\Omega) \) with \( g_1 = g_2 \in W^{1,p}_0(\Omega) \). Let \( \beta_1 \) and \( \beta_2 \) be two scalar functions in \( W^{1,p}(\Omega) \) with positive lower bounds. If

\[
\Lambda_{g_1, \beta_1} = \Lambda_{g_2, \beta_2},
\]

then \( g_1 = g_2 \) and \( \beta_1 = \beta_2 \).
then there exists a diffeomorphism $\Phi : \Omega \to \Omega$ in the $W^{2,p}$ class with $\Phi|_{\partial\Omega} =$ identity such that

$$g_2 = c \Phi_* g_1$$

for some positive function $c \in W^{1,p}(\Omega)$ and

$$\beta_2 = \Phi_* \beta_1.$$

Remarks.

(a) We remark that if the metric $g$ is $C^{2,1}(\Omega)$ near the boundary and the domain $\Omega$ is $C^{1,1}(\Omega)$, it seems possible to remove the assumption that the metrics coincide to order one at the boundary. The boundary determination of the metric $g$ and $\beta$ and its derivatives would follow by using the method of singular solutions of Alessandri [A] combined with the use of boundary normal coordinates as in [LU].

(b) The method of proof of theorem 1.1 is based on the reduction to a first-order system as in [BU] for isotropic conductivities and isothermal coordinates [Ah]. The uniqueness proof in [BU] was developed into a reconstruction method in [KT] for conductivities in $C^{1+\epsilon}$. By solving the Beltrami equation (see section 2) and using [KT], it is likely that one can also develop a reconstruction algorithm for slightly smoother $\beta$. Stability estimates were derived in [BBR] using the uniqueness proof of [BU] for $C^{1+\epsilon}$ conductivities. We also expect that stability estimates can be proven under slightly smoother assumptions on $\beta$ for the case considered in this paper.

In the case where $\beta_1 = \beta_2 = 1$ on $\Omega$ and the Riemannian metric is smooth, theorem 1.1 was proven in [LU] and was extended to general connected, compact Riemannian manifolds with a boundary in [LaU]. In the case where the Riemannian metric is Euclidean, this problem is the electrical impedance tomography problem for isotropic conductivities. Uniqueness was proven in [N] for $\beta \in W^{2,p}$, $p > 1$ and extended in [BU] to conductivities in $W^{1,p}$, $p > 2$. An immediate consequence of theorem 1.1 is the extension of the [BU] result when the background metric is not Euclidean. More precisely we have

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary. Let $g$ a Riemannian metric in $W^{1,p}(\Omega)$. Let $\beta_1$ and $\beta_2$ be two scalar functions in $W^{1,p}(\Omega)$ with positive lower bounds. If

$$\Lambda_{g,\beta_1} = \Lambda_{g,\beta_2},$$

then

$$\beta_1 = \beta_2.$$

We now discuss an application of theorem 1.1 to anisotropic conductivities. See [U2] for a recent survey. Let $\gamma = (\gamma^{ij})$ be a positive definite symmetric matrix on $\bar{\Omega}$ in the $W^{1,p}$ class, $p > 2$. The conductivity equation is given by

$$L_{\gamma}(u) = \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( \gamma^{ij} \frac{\partial u}{\partial x_j} \right) = 0, \quad u|_{\partial\Omega} = f,$$

and the DN map is defined as before by $f \to \Lambda_{\gamma} f \in W^{1-1/p,p}(\Omega)$ where

$$\Lambda_{\gamma} f = \sum_{i,j=1}^2 v_i \gamma^{ij} \frac{\partial u}{\partial x_j} \bigg|_{\partial\Omega},$$

(1.10)
with \( u \) the unique solution of (1.9) and \( v \) the unit outer normal of \( \partial \Omega \).

Let \( \Phi : \Omega \rightarrow \tilde{\Omega} \) be a \( W^{2,p} \) diffeomorphism. The push forward of \( \gamma \) under \( \Phi \) is given by
\[
\Phi_* \gamma = \left( \frac{[(D\Phi)^{-1} g((D\Phi)^{-1})^T]}{|\det D\Phi|} \right) \circ \Phi^{-1}.
\] (1.11)

A direct consequence of theorem 1.1 is the following.

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a Lipschitz boundary. Let \( \gamma_1 \) and \( \gamma_2 \) be two anisotropic conductivities in \( W^{1,p}(\Omega) \) with \( \gamma_1 - \gamma_2 \in W^{1,p}_0(\Omega) \). Assume
\[
\Lambda_{\gamma_1} = \Lambda_{\gamma_2};
\]
then there exists a diffeomorphism \( \Phi : \Omega \rightarrow \Omega \) in the \( W^{2,p} \) class with \( \Phi|_{\partial\Omega} = \text{identity} \) such that
\[
\gamma_2 = \Phi_* \gamma_1.
\]

This result follows from theorem 1.1 on taking \( \gamma_i = g_i^{-1} \) and \( \beta_i = \sqrt{|g_i|} \), \( i = 1, 2 \).

This result was previously known for \( C^3(\Omega) \) anisotropic conductivities. It follows by combining the result of [S], which reduces the anisotropic problem to the isotropic one by using isothermal coordinates [Ah], and the result of Nachman [N] for isotropic conductivities.

The following theorem shows that the smoothness of the diffeomorphism \( \Phi \) depends only on the smoothness of the metric \( g \).

**Theorem 1.4.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a \( C^{k,\alpha} \) boundary, where \( k \) is a positive integer and \( 0 < \alpha < 1 \). Let \( g_1 \) and \( g_2 \) be two Riemannian metrics in \( C^{k,\alpha}(\tilde{\Omega}) \) with \( D^\gamma g_1 = D^\gamma g_2 \) on \( \partial\Omega \), \( |\gamma| \leq k \). Let \( \beta_1 \) and \( \beta_2 \) be two scalar functions in \( W^{1,p}(\Omega) \) with positive lower bounds. If
\[
\Lambda_{g_1,\beta_1} = \Lambda_{g_2,\beta_2},
\]
then there exists a diffeomorphism \( \Phi : \Omega \rightarrow \Omega \) in the \( C^{k+1,\alpha}(\tilde{\Omega}) \) class with \( \Phi|_{\partial\Omega} = \text{identity} \) such that
\[
g_2 = c \Phi_* g_1
\]
for some scalar function \( c \in C^{k,\alpha}(\tilde{\Omega}) \) with positive lower bound and
\[
\beta_2 = \Phi_* \beta_1.
\]

For a description of other results in anisotropic inverse boundary problems, we refer the reader to the survey papers [U1] and [U2].

2. Lemmas

**Lemma 2.1.** Let \( g_1 \) and \( g_2 \) be two Riemannian metrics in \( W^{1,p}(\Omega) \) with \( g_1 - g_2 \in W^{1,p}_0(\Omega) \). Let \( \beta_1 \) and \( \beta_2 \) be two positive scalar functions in \( W^{1,p}(\Omega) \). If
\[
\Lambda_{g_1,\beta_1} = \Lambda_{g_2,\beta_2},
\]
then on \( \partial\Omega \),
\[
\beta_1 = \beta_2.
\]

**Proof.** Since \( g_1 - g_2 \in W^{1,p}_0(\Omega) \), we can extend \( g_1 \) and \( g_2 \) outside \( \Omega \) so that
\[
g_1(x) = g_2(x), \quad x \in \Omega^2,
\]
and
\[
g_1(x) = g_2(x) = e, \quad \text{for } |x| \text{ large enough},
\] (2.1)
and the extended function, which we still denote by $\beta/\Phi_1$ of the Beltrami equation in weighted
the Beltrami equation in different spaces to the ones used in [Ah]. We will use the solvability
in [A] we see that on $\partial\Omega$,
$$\Phi_*\beta_1 = \Phi_*\beta_2.$$  

This, together with $g_1 - g_2 \in W^{1,p}_0(\Omega)$, implies the result. □

According to lemma 2.1, we can extend $\beta_1$ and $\beta_2$ outside $\Omega$ so that
$$\beta_1(x) = \beta_2(x), \quad x \in \Omega^c,$$
and
$$\beta_1(x) = \beta_2(x) = 1, \quad \text{for } |x| \text{ large enough,}$$
and the extended function, which we still denote by $\beta_i$, $i = 1, 2$, is in $W^{1,p}(\mathbb{R}^2)$, where $e$ stands for the Euclidean metric. We shall assume (2.1) and (2.2) throughout the rest of the paper.

Let $g, \beta \in W^{1,p}(\mathbb{R}^2)$ with
$$g(x) = e, \quad \beta(x) = 1, \quad \text{for } |x| \text{ large enough.}$$

We use the notation
$$\mu_g = \frac{g_{11} - g_{22} + 2ig_{12}}{g_{11} + g_{22} + 2\sqrt{|g|}} < 1$$
and consider as in [Ah] the Beltrami equation
$$\partial\Phi_g = \mu_g \partial\Phi_g, \quad (2.4)$$

Then any diffeomorphism solution of (2.4) corresponds to an isothermal coordinate for the metric $g$. More precisely,
$$(\Phi_g)_*g = a_ge$$
for some $a_g > 0$, and the equation $L_{g,\beta}u = 0$ is transformed to
$$L_{(\Phi_g)_*g,(\Phi_g)_*\beta}v = \nabla \cdot (\beta \circ \Phi_g^{-1}\nabla v) = 0,$$
where $v = u \circ \Phi_g^{-1}$. In the next lemma, we construct a diffeomorphism $\Phi_g$ that behaves like $z = x_1 + ix_2$ as $|z| \to \infty$ in an appropriate sense. We denote by $L_1^\infty(\mathbb{R}^2)$ the space of functions satisfying $|f(z)| \leq C|z|^{-1}$ for some constant $C$, or equivalently,
$$L_1^\infty(\mathbb{R}^2) = \{f \in L^\infty(\mathbb{R}^2) : zf(z) \in L^\infty(\mathbb{R}^2)\}.$$

**Lemma 2.2.** Let $g \in W^{1,p}(\mathbb{R}^2)$ satisfying equation (2.3). Then there exists a diffeomorphism $\Phi_g \in W^{1,p}_{loc}(\mathbb{R}^2)$, a diffeomorphism of $\mathbb{R}^2$, which solves (2.4) and satisfies
$$\Phi_g - z \in L_1^\infty(\mathbb{R}^2).$$  

Moreover, $\Phi_g^{-1}, \text{D}\Phi_g - I$ and $\text{D}\Phi_g^{-1} - I$ are all bounded in the $L_1^\infty(\mathbb{R}^2)$ norm.

**Proof.** We use the method of isothermal coordinates [Ah] although we need the solvability of the Beltrami equation in different spaces to the ones used in [Ah]. We will use the solvability of the Beltrami equation in weighted $L^p$ spaces as was done in [S].

Since $g \in W^{1,p}_{loc}(\mathbb{R}^2) \subset L_1^\infty(\mathbb{R}^2)$, we can construct $\Phi_g = z + F$ with $F$ solving the equation
$$\partial F - \mu_g \partial F = \mu_g$$
(2.6)
Lemma 2.3. Let \( g, \beta \in W^{1,p}_{loc}(\mathbb{R}^2) \) satisfying (2.3). Then for each \( k \in C \) there exists a pair of solutions \( u_{g,\beta}(z,k) \) and \( v_{g,\beta}(z,k) \) of (1.1) in \( \mathbb{R}^2 \) such that

\[
\begin{pmatrix}
\partial u_{g,\beta} \\
\partial v_{g,\beta}
\end{pmatrix}
\begin{pmatrix}
\partial u_{g,\beta} \\
\partial v_{g,\beta}
\end{pmatrix}
\begin{pmatrix}
e^{-i\beta k} & 0 \\ 0 & e^{i\beta k}
\end{pmatrix}
- I \in L^q(\mathbb{R}^2),
\forall q > 2.
\]

Moreover, this pair of solutions is unique modulo constants.
Proof. Let $\Phi_g$ be the diffeomorphism constructed in lemma 2.2. Under $\Phi_g$, the equation $L_{\beta,\mu} w = 0$ is transformed to $\nabla \cdot (\beta \circ \Phi_g^{-1} \nabla w) = 0$. As in [BU], this equation can be reduced to the first-order elliptic system

\[
\begin{bmatrix}
\tilde{\beta} & 0 \\
0 & \tilde{\beta}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = 0,
\]

(2.10)

where

\[
Q = -\frac{1}{2} \beta \log (\beta \circ \Phi_g^{-1}),
\]

(2.11)

For each $k \in C$, this system carries a unique matrix solution in the form

\[
\Psi(z, k) = m(z, k) \begin{pmatrix} e^{ik} & 0 \\
0 & e^{-ik}
\end{pmatrix}
\]

(2.12)

with

\[
m(z, k) = I \in L^q(\mathbb{R}^2), \quad \forall q > 2.
\]

(2.13)

Here, $m$ is a matrix function of $z$ and $k$ [BU].

By using $\Phi^{-1}_g$, we can transform $\Psi(z, k)$ to obtain a unique pair of solutions $u_{\beta,\beta}$ and $v_{\beta,\beta}$ (modulo constants) and, according to (2.10) and (2.11),

\[
\beta^{1/2} \begin{pmatrix} \frac{\partial u_{\beta,\beta}}{\partial u_{\beta,\beta}} & \frac{\partial u_{\beta,\beta}}{\partial v_{\beta,\beta}} \\
\frac{\partial v_{\beta,\beta}}{\partial u_{\beta,\beta}} & \frac{\partial v_{\beta,\beta}}{\partial v_{\beta,\beta}}
\end{pmatrix} = H_{\beta,\beta} (m \circ \Phi_g) \begin{pmatrix} e^{ik} & 0 \\
0 & e^{-ik}
\end{pmatrix}.
\]

(2.14)

Here $H_{\beta,\beta}$ is the gradient transformation matrix associated with the diffeomorphism $\Phi_g$:

\[
H_{\beta,\beta} = \begin{pmatrix} \frac{\partial \Phi_g}{\partial \Phi_g} & \frac{\partial \Phi_g}{\partial \Phi_g} \\
\frac{\partial \Phi_g}{\partial \Phi_g} & \frac{\partial \Phi_g}{\partial \Phi_g}
\end{pmatrix}.
\]

(2.15)

From (2.14) we get

\[
\begin{pmatrix} \frac{\partial u_{\beta,\beta}}{\partial u_{\beta,\beta}} & \frac{\partial u_{\beta,\beta}}{\partial v_{\beta,\beta}} \\
\frac{\partial v_{\beta,\beta}}{\partial u_{\beta,\beta}} & \frac{\partial v_{\beta,\beta}}{\partial v_{\beta,\beta}}
\end{pmatrix} \begin{pmatrix} e^{ik} & 0 \\
0 & e^{-ik}
\end{pmatrix} = I = \beta^{-1/2} H_{\beta,\beta} (m \circ \Phi_g) \begin{pmatrix} e^{ik} & 0 \\
0 & e^{-ik}
\end{pmatrix}.
\]

(2.16)

Since $\beta^{-1/2} = 1$ for $|z|$ large enough, it is enough to show that

\[
H_{\beta,\beta} (m \circ \Phi_g) \begin{pmatrix} e^{ik} & 0 \\
0 & e^{-ik}
\end{pmatrix} - I \in L^q(\mathbb{R}^2), \quad \forall q > 2.
\]

(2.17)

We rewrite this as

\[
H_{\beta,\beta} (m \circ \Phi_g) \begin{pmatrix} e^{ik} & 0 \\
0 & e^{-ik}
\end{pmatrix} - I = H_{\beta,\beta} (m \circ \Phi_g - I) \begin{pmatrix} e^{ik} & 0 \\
0 & e^{-ik}
\end{pmatrix}.
\]

(2.18)

and

\[
H_{\beta,\beta} - I \in L_1^\infty(\mathbb{R}^2), \quad H_{\beta,\beta} \in L_1^\infty(\mathbb{R}^2).
\]

(2.19)

From (2.5) it is clear that

\[
\begin{pmatrix} e^{ik} & 0 \\
0 & e^{-ik}
\end{pmatrix} \in L_1^\infty(\mathbb{R}^2),
\]

(2.20)
So, the second and the third terms on the right-hand side of (2.17) are in \( L^\infty(\mathbb{R}^2) \subset L^q(\mathbb{R}^2) \), \( \forall q > 2 \). From (2.13) and (2.5), it is easy to show that 
\[ \Phi_g \circ I \in L^q(\mathbb{R}^2), \quad \forall q > 2. \]
This, together with (2.19) and (2.20), implies that the first term on the right-hand side of (2.17) is also in \( L^q(\mathbb{R}^2) \), \( \forall q > 2 \). This proves that the solution pair constructed above satisfies the property (2.9).

To prove the uniqueness (modulo constants), let \( u_{g,\beta} \) and \( v_{g,\beta} \) be a pair of solutions satisfying \( L_{g,\beta} w = 0 \) and (2.9). Then
\[
(\beta \circ \Phi_g^{-1})^{1/2} H_{\Phi_g^{-1}} \left( \begin{array}{c} \frac{\partial u_{g,\beta}}{\partial u_{g,\beta}} \\ \frac{\partial v_{g,\beta}}{\partial v_{g,\beta}} \end{array} \right) \circ \Phi_g^{-1}
\]
is a matrix solution to the system (2.10). Using the properties of \( D\Phi_g \) and \( D\Phi_g^{-1} \) and the argument above that leads to (2.9), one can show that this matrix solution takes the form (2.12) with (2.13), which is unique. This completes the proof.

\[ \square \]

3. Proof of theorems

We extend \( g_1, g_2, \beta_1 \) and \( \beta_2 \) as we did in (2.1) and (2.2). Let \( \Phi_{g_1} \) and \( \Phi_{g_2} \) be the diffeomorphisms in lemma 2.2 associated with \( g_1 \) and \( g_2 \), respectively.

We shall first prove that
\[
\Phi_{g_i}(z) = \Phi_{g_i}(z), \quad z \in \Omega^c.
\] (3.1)
To this end we consider the solution pairs constructed in lemma 2.3: \((u_{g_1}, v_{g_1})\) and \((u_{g_2}, v_{g_2})\). From (2.14), we have
\[
\beta^{1/2} \left( \begin{array}{c} \frac{\partial u_{g,\beta}}{\partial u_{g,\beta}} \\ \frac{\partial v_{g,\beta}}{\partial v_{g,\beta}} \end{array} \right) = H_{\Phi_{g_i}}(m_i \circ \Phi_{g_i}) \left( \begin{array}{c} e^{i\phi_{p_i}k} \\ 0 \end{array} \right),
\] (3.2)
for \( i = 1, 2 \), where \( m_i \) corresponds to the matrix function \( m \) in (2.12) under the diffeomorphism \( \Phi_{g_i} \). Notice that \( m_i \) satisfies (2.13). We claim that
\[
u_{g_1,\beta_1}(z) = u_{g_1,\beta_1}(z), \quad v_{g_1,\beta_1}(z) = v_{g_1,\beta_1}(z), \quad z \in \Omega^c.
\] (3.3)
To see this, we construct a new solution pair \((u^*_{g_1,\beta_1}, v^*_{g_1,\beta_1})\) as follows. For \( z \in \Omega \), \( u^*_{g_1,\beta_1} \) solves (1.2) with \( g = g_1 \), \( \beta = \beta_1 \) and \( f = u_{g_2,\beta_2}|_{\Omega} \). Similarly, \( v^*_{g_1,\beta_1} \) solves (1.2) with \( g = g_1 \), \( \beta = \beta_1 \) and \( f = v_{g_2,\beta_2}|_{\Omega} \). For \( z \in \Omega^c \), \( u^*_{g_1,\beta_1} = u_{g_2,\beta_2} \) and \( v^*_{g_1,\beta_1} = v_{g_2,\beta_2} \). Since \( \Lambda_{g_1,\beta_1} = \Lambda_{g_2,\beta_2} \), a well known argument shows that \((u^*_{g_1,\beta_1}, v^*_{g_1,\beta_1})\) is a solution pair of (1.1) with \( g = g_1 \) and \( \beta = \beta_1 \) in \( W^{1,p}(\mathbb{R}^2) \). Since
\[
(u^*_{g_1,\beta_1}, v^*_{g_1,\beta_1}) = (u_{g_2,\beta_2}, v_{g_2,\beta_2}),
\] (3.4)
for \( z \in \Omega^c \), it is clear that \((u^*_{g_1,\beta_1}, v^*_{g_1,\beta_1})\) satisfies the condition in (2.9). Thus it is the unique solution pair claimed by lemma 2.3 with \( g = g_1 \) and \( \beta = \beta_1 \). Therefore,
\[
(u^*_{g_1,\beta_1}, v^*_{g_1,\beta_1}) = (u_{g_1,\beta_1}, v_{g_1,\beta_1}).
\] (3.5)
Combining (3.4) with (3.5) yields (3.3).

Fix \( z \in \Omega^c \). From (3.2) and (3.3) we conclude that
\[
H_{\Phi_{g_1}}(m_1 \circ \Phi_{g_1}) \left( \begin{array}{c} e^{i\phi_{p_1}k} \\ 0 \end{array} \right) = H_{\Phi_{g_2}}(m_2 \circ \Phi_{g_2}) \left( \begin{array}{c} e^{i\phi_{p_2}k} \\ 0 \end{array} \right).
\] (3.6)
From theorem 2.3 in [BU] we know that \( m_1(z, k) \) satisfies
\[
\sup_{z} \|m_1(z, \cdot) - I\|_{L^q(\mathbb{R}^2)} \leq C
\]
for some constant $C$ and $q > 2$. Then it is easy to show that there exist a sequence $\{k_n\} \subset C$ with $\lim_{n \to \infty} |k_n| = \infty$ such that
\[
\lim_{n \to \infty} (m_i(z, k_n) - I) = 0, \quad i = 1, 2.
\] (3.7)

Let us use the notation
\[
H_{\Phi_{1n}} = \begin{pmatrix}
  h_{11}^{(i)} & h_{12}^{(i)} \\
h_{21}^{(i)} & h_{22}^{(i)}
\end{pmatrix}, \quad m_i = \begin{pmatrix}
m_{11}^{(i)} & m_{12}^{(i)} \\
m_{21}^{(i)} & m_{22}^{(i)}
\end{pmatrix},
\]
for $i = 1, 2$. Then, by setting the $(1, 1)$ entries on either side of (3.6) equal, we get
\[
(h_{11}^{(i)} m_{11}^{(i)} + h_{12}^{(i)} m_{21}^{(i)}) e^{i\Phi_{1n} k} = (h_{11}^{(i)} m_{11}^{(i)} + h_{12}^{(i)} m_{21}^{(i)}) e^{i\Phi_{2n} k}.
\] (3.8)

Replacing $k$ by $k_n$ in (3.8) and letting $n$ be large enough, we conclude from (3.7) and the fact that $h_{11}^{(i)} \neq 0$ (if $h_{11}^{(i)} = 0$, then by (2.15), the definition of the matrix and (2.4), we would have that $h_{12}^{(i)} = 0$, implying that the matrix $H$ is not invertible) that
\[
h_{11}^{(i)} m_{11}^{(i)}(z, k_n) + h_{12}^{(i)} m_{21}^{(i)}(z, k_n) \neq 0, \quad i = 1, 2.
\]

Thus we can take the logarithm of both sides of (3.8) (restricted on $[k_n]$ with large $n$) to get
\[
\log (h_{11}^{(i)} m_{11}^{(i)}(z, k_n) + h_{12}^{(i)} m_{21}^{(i)}(z, k_n)) + i\Phi_{1n} k_n
\]
\[
= \log (h_{11}^{(i)} m_{11}^{(i)}(z, k_n) + h_{12}^{(i)} m_{21}^{(i)}(z, k_n)) + i\Phi_{2n} k_n.
\] (3.9)

Dividing by $k_n$ on both sides of (3.9) and then letting $n \to \infty$ yields
\[
\Phi_{1n}(z) = \Phi_{2n}(z).
\]

This gives (3.1).

Equation (3.1) implies that both $\Phi_{1i}, \beta_1$, and $\Phi_{2i}, \beta_2$, send $\Omega$ to the same open set $\Omega^*$. Then by (1.7),
\[
\Lambda(\Phi_{1i}, \epsilon_{\Omega^*}, \beta_1) = \Lambda(\Phi_{2i}, \epsilon_{\Omega^*}, \beta_2)
\]
for the equation $\nabla \cdot (\beta \Phi_{i}^{-1} \nabla w) = 0$ with $i = 1, 2$. Thus, by the uniqueness result in [BU],
\[
\beta_1 \circ \Phi_{1i}^{-1} = \beta_2 \circ \Phi_{2i}^{-1}.
\]

If we define $\Phi = \Phi_{1i}^{-1} \Phi_{1i}$, then $\Phi|_{\partial \Omega} = \text{identity}$ and
\[
\beta_2 = \beta_1 \circ \Phi^{-1}.
\]

But $(\Phi_{gi})_{g_1} = a_{gi} e$ for some scalar function $a_{gi} \in W^{1,p}(\mathbb{R}^2)$, $i = 1, 2$, with a positive lower bound, so we have
\[
(a_{gi}, a_{gi}^{-1})g_2 = (\Phi_{gi})_{g_1}^{-1} \circ (\Phi_{gi})_{g_1} g_1 = \Phi_{gi} g_1.
\]

In other words,
\[
g_2 = (a_{gi})^{-1} a_{gi} \Phi_{gi} g_1.
\]

This completes the proof of theorem 1.1.

To prove theorem 1.3, we only need to show that $\Phi \in C^{k+1,a}(\bar{\Omega})$. Since $\Phi_{gi}, i = 1, 2$, solves (2.4) with $\mu_{gi} \in C^{k,a}(\mathbb{R}^2)$ (we can extend the $g_i$ smoothly in $C^{k,a}$ outside $\Omega$ so that they satisfy (2.1)), we have, by elliptic regularity, that $\Phi_{gi} \in C^{k+1,a}(\mathbb{R}^2)$. Therefore $\Phi = \Phi_{gi}^{-1} \Phi_{gi} \in C^{k+1,a}(\bar{\Omega})$.

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References


See endnote 6

See endnote 7
Queries for IOP paper 164073

Journal: IP
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Page 1
Query 1:
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