Inverse problems with partial data in a slab

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Abstract

In this paper we consider several inverse boundary value problems with partial data on an infinite slab. We prove the unique determination results of the coefficients for the Schrödinger equation and the conductivity equation when the corresponding Dirichlet and Neumann data are given either on the different boundary hyperplanes of the slab or on the same single hyperplane.

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1 Introduction

Inverse boundary value problems arise when one tries to recover internal parameters of a medium from data obtained by boundary measurements. The physical situation of many of these problems is modeled by partial differential equations, and the goal is to determine the coefficients of the underlying equations from measurements of the solutions at the boundary. Since Calderón's pioneer contribution [3], a key method in inverse boundary problems has been the construction of complex geometrical optics solutions with a large parameter which was introduced by Sylvester and Uhlmann [22]. These solutions were used in [22] to show, in three or higher dimensions, that the

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Dirichlet-to-Neumann (DN) map determines uniquely the conductivity coefficient in the conductivity equation and the potential in the Schrödinger equation. This was the first breakthrough for the non-linear problem in three or higher dimensions and led to several other developments (see, for example, [23, 24, 25]).

In recent years, inverse problems with partial data, that is when the measurements are made on part of the boundary, have received a lot of attention. Bukhgeim and Uhlmann [2] proved uniqueness for the potential in the Schrödinger equation in three or higher dimensions when the boundary measurements are given by the Dirichlet data on the whole boundary but the Neumann data only on (roughly speaking) half of the boundary. In [14] the regularity assumption in [2] on the conductivity was improved. Stability estimates in the [2] setting were proved in [6]. Kenig, Sjöstrand and Uhlmann [13] improved significantly on the [2] result by showing unique identifiability when the Dirichlet data is given on any (possible very small) open subset of the boundary and the Neumann data is given on a slightly larger part of its complement. A reconstruction method has been proposed in [18] for the latter result. Isakov [12] proved a uniqueness result in dimension three or higher when the DN map is given on an arbitrary part of the boundary assuming that the remaining part is an open subset of a plane or a sphere. In two dimensions it has been shown recently [10] that one can uniquely recover the potential and conductivity if the the DN map is measured on any subset of the boundary with Dirichlet data supported in the same set.

In this paper, we consider the inverse boundary value problem with partial data in an infinite slab in three or higher dimensions. We prove the unique determination result for the Schrödinger equation and the conductivity equation when the Dirichlet and Neumann data are given either on the two different boundary hyperplanes or on the same single hyperplane. The infinite slab is an important and interesting geometry. For example, it models important problems of wave propagation in marine acoustics. It is also a simple geometrical setting in medical imaging. Inverse boundary value problems in a slab were investigated by several authors. The inverse coefficient problems for wave guides were studies in [1, 8, 9, 4, 5] and the references cited therein under various settings. The inverse conductivity problems of identifying an embedded object were considered in [7, 20]. And the inverse problems of optical tomography in the diffusion approximation were investigated in [16, 17].

To deal with inverse boundary value problems with partial data we use not only complex geometrical optics solutions but also Carleman estimates. If the data is given on the whole boundary, Green's formula gives an identity involving the unknown parameters and these solutions. In partial data inverse problems, Green's formula gives an identity, which involves not only the unknown parameters and the corresponding solutions but also the boundary terms of the solutions. This is because one only has the knowledge of the solutions on part of the boundary and the solutions on the remaining part of the boundary are unknown. Thus one needs to show that the unknown information can be neglected. A suitable Carleman estimate is needed to control the unknown boundary terms. To fit the slab geometry and the boundary information, we carefully construct the complex geometrical optics solutions, especially the phase functions, to control the behavior of the solution when the large parameter goes to infinity.

This paper is organized as follows. In Section 2, we will state the inverse problems with partial data in an infinite slab and our main results. In Section 3, we prove the unique determination of the potential in the Schrödinger equation when the Dirichlet and Neumann data are given on the two different boundary hyperplanes. In Section 4, we prove the unique determination when the data are given on the same single hyperplane. The results for the inverse conductivity problems will be proved in Section 5. In Appendix, we will discuss the solvability of the direct problem.

2 Inverse problems in a slab

Suppose $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is an infinite slab between two parallel hyperplanes Γ_1 and Γ_2. Without loss of generality, we assume

$$\Omega = \{x = (x', x_n) \in \mathbb{R}^n : x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, 0 < x_n < L\}$$

and

$$\Gamma_1 = \{x \in \mathbb{R}^n : x_n = L > 0\},\ \Gamma_2 = \{x \in \mathbb{R}^n : x_n = 0\}$$

Consider the following Dirichlet problem:

$$(-\Delta + q(x) - k^2)u(x) = 0 \quad in\ \Omega \tag{2.1}$$

$$u(x) = f(x) \quad on\ \Gamma_1 \tag{2.2}$$

$$u(x) = 0 \quad on\ \Gamma_2 \tag{2.3}$$

where $k > 0$, the compactly supported potential $q(x) \in L^\infty(\Omega)$, and $f(x) \in H^{1/2}(\Gamma_1)$ with compact support in Γ_1. We also require u satisfies the partial radiation condition introduced by Sveshnikov [21]

$$\left(\frac{\partial}{\partial \rho} - ikm\right) u_m(x') = o(\rho^{\frac{1-n}{2}}) \tag{2.4}$$

where $u_m(x') = \dfrac{1}{L}\displaystyle\int_0^L u(x) \sin \frac{m\pi x_n}{L} dx_n$, $\rho = |x'|$, $m = 1, 2, \cdots$.

The existence of the weak solution $u \in H^1(\Omega)$ can be proved using Lax-Phillips method (see [11]) together with the following assumptions.
Assumption 1: There is only zero solution to the homogeneous equations (2.1)–(2.3) with $f = 0$ satisfying the partial radiation condition (2.4).
Assumption 2: If $n = 3$, we also require that $k \neq m\frac{\pi}{L}$, $m = 1, 2, \cdots$.
We give the proof in the Appendix.

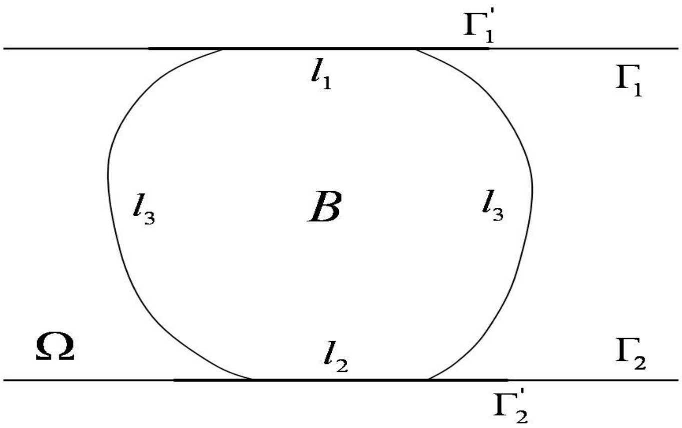
After discussing the well-posedness of the direct problem, we then define the boundary measurements. For an open set Γ_2' on Γ_2, we define the Cauchy data for $q(x)$ by

$$C^D_{q,\ \Gamma_2'} := \{(u|_{\Gamma_1}, \frac{\partial u}{\partial \nu}|_{\Gamma_2'}) : \text{ u is a solution of } (2.1)(2.2)(2.3)(2.4)\}$$

where ν is the unit outer normal vector. Similarly, for an open set Γ_1' on Γ_1, we define the Cauchy data

$$C^S_{q,\ \Gamma_1'} := \{(u|_{\Gamma_1}, \frac{\partial u}{\partial \nu}|_{\Gamma_1'}) : \text{ u is a solution of } (2.1)(2.2)(2.3)(2.4)\}$$

The superscripts D and S represent the data are given in the different hyperplanes and the same hyperplane, respectively. Both $C^D_{q,\ \Gamma_2'}$ and $C^S_{q,\ \Gamma_1'}$ contain only partial measurements on the boundary. The inverse boundary value problems consist of the recovery of $q(x)$ from the knowledge of $C^D_{q,\ \Gamma_2'}$ or $C^S_{q,\ \Gamma_1'}$.



Now we state our main results.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a slab and $q_j(x) \in L^\infty(\Omega)$, $j = 1, 2$. Suppose a compact set B contains both supports of $q_1(x)$ and $q_2(x)$. For any Γ_2' such that $B \cap \Gamma_2 \subset \Gamma_2'$, if $C^D_{q_1,\ \Gamma_2'} = C^D_{q_2,\ \Gamma_2'}$, then $q_1(x) = q_2(x)$ in Ω.*

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a slab and $q_j(x) \in L^\infty(\Omega)$, $j = 1, 2$. Suppose a compact set B contains both supports of $q_1(x)$ and $q_2(x)$. For any Γ_1' such that $B \cap \Gamma_1 \subset \Gamma_1'$, if $C^S_{q_1,\ \Gamma_1'} = C^S_{q_2,\ \Gamma_1'}$, then $q_1(x) = q_2(x)$ in Ω.*

Remark 3. *If the support of the potential $q(x)$ is strictly contained in Ω, then $B \cap \Gamma_1 = B \cap \Gamma_2 = \emptyset$. So Γ_1' and Γ_2' can be any arbitrary small open sets on Γ_1 and Γ_2, respectively, in the above theorems.*

Theorem 1 and Theorem 2 have immediate consequences in electrical impedance tomography. Instead of (2.1), one considers the conductivity equation

$$-\mathrm{div}(\gamma \nabla u) - k^2 \gamma u = 0 \quad \textit{in } \Omega \tag{2.5}$$

where $\gamma \in C^2(\bar{\Omega})$, $\gamma > 0$ in $\bar{\Omega}$, and $\gamma = 1$ outside a compact set. If $k = 0$, the Lax-Milgram theorem guarantees the existence and uniqueness of the solution to (2.5)(2.2)(2.3). If $k > 0$, we also require the partial radiation condition (2.4) and the assumptions as for the Schrödinger equations. The well-posedness of the conductivity problem is then the same as that of the Schrödinger equations. We define two sets of Cauchy data for γ:

$$C^D_{\gamma,\ \Gamma_2'} := \left\{ (u|_{\Gamma_1}, \gamma \frac{\partial u}{\partial \nu}|_{\Gamma_2'}) : \begin{array}{l} \text{u is a solution of } (2.5)(2.2)(2.3)(2.4), \text{ if } k > 0 \\ \text{u is a solution of } (2.5)(2.2)(2.3), \text{ if } k = 0 \end{array} \right\}$$

and

$$C^S_{\gamma,\ \Gamma_1'} := \left\{ (u|_{\Gamma_1}, \gamma\frac{\partial u}{\partial \nu}|_{\Gamma_1'}) : \begin{array}{l} \text{u is a solution of (2.5)(2.2)(2.3)(2.4), if } k>0 \\ \text{u is a solution of (2.5)(2.2)(2.3), if } k=0 \end{array} \right\}$$

where Γ_j' is any open set on Γ_j, $j = 1, 2$, and ν is the unit outer normal vector. We have the following results for electrical impedance tomography problems.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a slab. Suppose $\gamma_j \in C^2(\bar{\Omega})$, $\gamma_j > 0$ in $\bar{\Omega}$, and $\gamma_j = 1$ outside a compact set. Denote B the compact set containing both supports of $\gamma_j - 1$, $j = 1, 2$. For any Γ_2' such that $B \cap \Gamma_2 \subset \Gamma_2'$, if $C^D_{\gamma_1,\ \Gamma_2'} = C^D_{\gamma_2,\ \Gamma_2'}$ and*

$$\gamma_1 = \gamma_2 \quad on\ \partial\Omega, \qquad \frac{\partial \gamma_1}{\partial \nu} = \frac{\partial \gamma_2}{\partial \nu} \quad on\ \Gamma_2', \tag{2.6}$$

then $\gamma_1(x) = \gamma_2(x)$ in Ω.

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a slab. Suppose $\gamma_j \in C^2(\bar{\Omega})$, $\gamma_j > 0$ in $\bar{\Omega}$, and $\gamma_j = 1$ outside a compact set. Denote B the compact set containing both supports of $\gamma_j - 1$, $j = 1, 2$. For any Γ_1' such that $B \cap \Gamma_1 \subset \Gamma_1'$, if $C^S_{\gamma_1,\ \Gamma_1'} = C^S_{\gamma_2,\ \Gamma_1'}$, then $\gamma_1(x) = \gamma_2(x)$ in Ω.*

Remark 6. *We do not need any further restriction about the conductivity on the boundary in Theorem 5. Kohn and Vogelius [15] showed that a C^1 conductivity together with its first derivatives on Γ_1' can be uniquely obtained from the Cauchy data $C^S_{\gamma,\ \Gamma_1'}$.*

3 Determination of the potential from $C^D_{q,\ \Gamma_2'}$

In this section, we shall prove Theorem 1. Let $q_1(x)$ and $q_2(x)$ be two potentials with the same Cauchy data $C^D_{q_1,\ \Gamma_2'} = C^D_{q_2,\ \Gamma_2'}$. We will derive an key inequality involving these two potentials and the corresponding solutions. Let $u_1(x) \in H^1(\Omega)$ be a solution of

$$(-\Delta + q_1(x) - k^2)u_1(x) = 0 \quad in\ \Omega \tag{3.1}$$

$$u_1(x) = 0 \quad on\ \Gamma_2 \tag{3.2}$$

satisfying the partial radiation condition (2.4) together with $u_1|_{\Gamma_1}$ having compact support in Γ_1. Let $v(x) \in H^1(\Omega)$ be a solution of

$$(-\Delta + q_2(x) - k^2)v(x) = 0 \quad in\ \Omega \tag{3.3}$$

$$v(x) = u_1(x) \quad on\ \Gamma_1 \cup \Gamma_2 \tag{3.4}$$

satisfying the partial radiation condition (2.4).
Then we define $w(x) = v(x) - u_1(x)$. Obviously, $w(x)$ satisfies the equation

$$(-\Delta + q_2(x) - k^2)w(x) = (q_1(x) - q_2(x))u_1(x) \quad in\ \Omega. \tag{3.5}$$

Since $v(x)$ and $u_1(x)$ have the same boundary value on Γ_1 and both vanish on Γ_2, we conclude from $C^D_{q_1,\ \Gamma'_2} = C^D_{q_2,\ \Gamma'_2}$ that

$$\frac{\partial v}{\partial \nu} = \frac{\partial u_1}{\partial \nu} \quad on\ \Gamma'_2.$$

Therefore $w(x)$ satisfies the boundary conditions

$$w(x) = 0 \quad on\ \Gamma_1 \cup \Gamma_2, \qquad \frac{\partial w(x)}{\partial \nu} = 0 \quad on\ \Gamma'_2.$$

To get more information about $w(x)$, we consider the region $\Omega \setminus B$. Denote $l_1 = B \cap \Gamma_1$, $l_2 = B \cap \Gamma_2$ and $l_3 = \partial B \cap \Omega$. Since supp $q_2(x) \subset B$, we know $w(x)$ is a solution of

$$(-\Delta - k^2)w(x) = 0 \quad in\ \Omega \setminus B, \quad w(x) = \frac{\partial w(x)}{\partial \nu} = 0 \quad on\ \Gamma'_2 \setminus l_2.$$

By unique continuation, $w(x) = 0$ in $\Omega \setminus B$. Particularly, $w(x) = \frac{\partial w(x)}{\partial \nu} = 0$ on l_3.

Let $u_2(x) \in H^1(\Omega \cap B)$ be a solution of

$$(-\Delta + q_2(x) - k^2)u_2(x) = 0 \quad in\ \Omega \cap B \tag{3.6}$$

Note that $u_2(x)$ does not need to satisfy the equation on the whole domain Ω and we do not impose any boundary condition for u_2. We shall take this advantage later. In view of (3.5)(3.6) and supp $(q_1(x) - q_2(x)) \subset B$ we get

$$\begin{aligned}
&\int_\Omega (q_1 - q_2)u_1 u_2 dx = \int_{\Omega\cap B} (q_1 - q_2)u_1 u_2 dx \\
= &\int_{\Omega\cap B} (-\Delta + q_2(x) - k^2)w u_2 dx \\
= &\int_{\Omega\cap B} w(-\Delta + q_2(x) - k^2)u_2 dx - \int_{\partial(\Omega\cap B)} \frac{\partial w}{\partial \nu} u_2 ds + \int_{\partial(\Omega\cap B)} w \frac{\partial u_2}{\partial \nu} ds
\end{aligned}$$

where we use Green's formula in the last step. We know $\partial(\Omega\cap B) = l_1\cup l_2\cup l_3$. But we already proved that

$$w(x) = 0 \quad on\ l_1\cup l_2\cup l_3, \qquad \frac{\partial w(x)}{\partial\nu} = 0 \quad on\ l_2\cup l_3.$$

Together with (3.6), we obtain the following lemma.

Lemma 7. *Under the above notations, we have the identity*

$$\int_\Omega (q_1 - q_2)u_1u_2dx = -\int_{l_1} \frac{\partial w}{\partial\nu}u_2ds. \tag{3.7}$$

In order to control the boundary term, we need a Carleman estimate. We recall the Carleman estimate derived in [2] (Corollary 2.3) for a bounded domain Q.

Lemma 8. *For $q \in L^\infty(Q)$, there exists $\tau_0 > 0$ and $C > 0$ such that for all $u \in C^2(\bar{Q})$ with $u = 0$ on ∂Q, and $\tau \geq \tau_0$ we have the estimate*

$$C\tau^2\int_Q |e^{-\tau x\cdot\eta}u|^2dx + \tau\int_{\partial Q}(\eta\cdot\nu)\ |e^{-\tau x\cdot\eta}\frac{\partial u}{\partial\nu}|^2ds \leq \int_Q |e^{-\tau x\cdot\eta}(-\Delta + q - k^2)u|^2dx$$

The above inequality holds obviously for all $H^2(Q)$ functions. We apply this estimate to our case with $Q = \Omega\cap B$. In view of (3.5) and standard elliptic theory, $w \in H^2(\Omega\cap B)$. Since $\frac{\partial w}{\partial\nu} = 0$ on $l_2\cup l_3$, we get

$$\tau\int_{l_1}(\eta\cdot\nu)\ |e^{-\tau x\cdot\eta}\frac{\partial w}{\partial\nu}|^2ds \leq \int_{\Omega\cap B} |e^{-\tau x\cdot\eta}(-\Delta + q_2 - k^2)w|^2dx \tag{3.8}$$

In order to make use of this estimate, we must choose η such that $\eta\cdot\nu > 0$ on l_1, otherwise, the left hand side will be a non-positive number. We also note that the unit outer normal vector ν is invariant on l_1 since Γ_1 is a hyperplane. Thus $\eta\cdot\nu$ can be moved out the integration, and using (3.5) we obtain

$$\int_{l_1} |e^{-\tau x\cdot\eta}\frac{\partial w}{\partial\nu}|^2ds \leq \frac{1}{\tau(\eta\cdot\nu)}\int_{\Omega\cap B} |e^{-\tau x\cdot\eta}(q_1 - q_2)u_1|^2dx \tag{3.9}$$

Then, from (3.7)(3.9), we get

$$
\begin{aligned}
&\left|\int_\Omega (q_1-q_2)u_1u_2dx\right| = \left|\int_{l_1}\frac{\partial w}{\partial \nu}u_2ds\right| \\
= \ &\left|\int_{l_1} e^{-\tau x\cdot\eta}\frac{\partial w}{\partial \nu}e^{\tau x\cdot\eta}u_2ds\right| \leq \int_{l_1}\left|e^{-\tau x\cdot\eta}\frac{\partial w}{\partial \nu}e^{\tau x\cdot\eta}u_2\right|ds \\
\leq \ &\left(\int_{l_1}\left|e^{-\tau x\cdot\eta}\frac{\partial w}{\partial \nu}\right|^2 ds\right)^{\frac{1}{2}} \left(\int_{l_1}\left|e^{\tau x\cdot\eta}u_2\right|^2 ds\right)^{\frac{1}{2}} \\
\leq \ &\left(\frac{1}{\tau(\eta\cdot\nu)}\right)^{\frac{1}{2}}\left(\int_{\Omega\cap B}|e^{-\tau x\cdot\eta}(q_1-q_2)u_1|^2dx\right)^{\frac{1}{2}} \left(\int_{l_1}\left|e^{\tau x\cdot\eta}u_2\right|^2 ds\right)^{\frac{1}{2}}
\end{aligned} \tag{3.10}
$$

We shall get the relationship between q_1 and q_2 from (3.10) by choosing suitable η and constructing complex geometrical optics solutions to (3.1)(3.2) and (3.6).

For $j = 1, 2$, we define

$$\mathcal{V}_j(\Omega\cap B) := \{u \in H^1(\Omega\cap B) : (-\Delta + q_j(x) - k^2)u(x) = 0 \ in \ \Omega\cap B\},$$

$$\mathcal{W}_j(\Omega) := \{u \in H^1(\Omega) : (-\Delta + q_j(x) - k^2)u(x) = 0 \ in \ \Omega, \ u = 0 \ on \ \Gamma_2\},$$

$$\mathcal{W}_j(\Omega\cap B) := \{u \in H^1(\Omega\cap B) : (-\Delta + q_j(x) - k^2)u(x) = 0 \ in \ \Omega\cap B, \ u = 0 \ on \ \Gamma_2\}.$$

So (3.10) holds for any $u_1 \in \mathcal{W}_1(\Omega)$ and any $u_2 \in \mathcal{V}_2(\Omega\cap B)$. The solution we will construct later for u_1 grows exponentially at infinity. We need a Runge type approximation.

Lemma 9. *$\mathcal{W}_j(\Omega)$ is dense in $\mathcal{W}_j(\Omega\cap B)$ with respect to $L^2(\Omega\cap B)$ norm, for $j = 1, 2$.*

Proof: We only need to show that $\mathcal{W}_1(\Omega)$ is dense in $\mathcal{W}_1(\Omega\cap B)$ with respect to $L^2(\Omega\cap B)$ norm. If not, then by Hahn-Banach theorem there is g in $L^2(\Omega)$, $g = 0$ in $\Omega\setminus B$, such that

$$\int_\Omega gudx = 0 \text{ for any } u \in \mathcal{W}_1(\Omega),$$

but

$$\int_\Omega gu_0dx = 0 \text{ for some } u_0 \in \mathcal{W}_1(\Omega\cap B). \tag{3.11}$$

Let U be the solution to the problem

$$(-\Delta + q_1(x) - k^2)U(x) = g \quad in\ \Omega,$$
$$U(x) = 0 \quad on\ \partial\Omega = \Gamma_1 \cup \Gamma_2.$$

Using the Green's formula and boundary conditions, we get

$$\begin{aligned} 0 &= \int_\Omega gudx = \int_\Omega [(-\Delta + q_1(x) - k^2)U]udx \\ &= \int_\Omega U(-\Delta + q_1(x) - k^2)udx - \int_{\partial\Omega} \frac{\partial U}{\partial \nu} uds + \int_{\partial\Omega} U\frac{\partial u}{\partial \nu} ds \\ &= -\int_{\partial\Omega} \frac{\partial U}{\partial \nu} uds = -\int_{\Gamma_1} \frac{\partial U}{\partial \nu} uds, \quad \text{for any } u \in \mathcal{W}_1(\Omega). \end{aligned}$$

Then we know $\frac{\partial U}{\partial \nu} = 0$ on Γ_1 since u can be arbitrary smooth function on Γ_1. Hence U is a solution to

$$(-\Delta - k^2)U(x) = 0 \quad in\ \Omega \setminus B, \quad U(x) = \frac{\partial U}{\partial \nu} = 0 \quad on\ \Gamma_1.$$

By the unique continuation, we conclude that $U = 0$ in $\Omega \setminus B$. Then for any $u_0 \in \mathcal{W}_1(\Omega \cap B)$, using the Green's formula again, we have

$$\begin{aligned} &\int_\Omega gu_0dx = \int_{\Omega\cap B} gu_0dx = \int_{\Omega\cap B} [(-\Delta + q_1(x) - k^2)U]u_0dx \\ = &\int_{\Omega\cap B} U(-\Delta + q_1(x) - k^2)u_0dx - \int_{\partial(\Omega\cap B)} \frac{\partial U}{\partial \nu} u_0ds + \int_{\partial(\Omega\cap B)} U\frac{\partial u_0}{\partial \nu} ds \\ = &\ 0. \end{aligned}$$

This contradicts to (3.11), and the proof is complete.

Summary up, we obtain the following important inequality.

Lemma 10. *For any $\eta \in \mathbb{R}^n$ such that $\eta \cdot \nu > 0$ where ν is the unit outer normal of Γ_1, and for any $\tau >> 0$, if $C^D_{q_1,\ \Gamma_2'} = C^D_{q_2,\ \Gamma_2'}$, then*

$$\begin{aligned} &\left|\int_\Omega (q_1 - q_2)u_1u_2dx\right| \\ \le &\left(\frac{1}{\tau(\eta\cdot\nu)}\right)^{\frac{1}{2}} \left(\int_{\Omega\cap B} |e^{-\tau x\cdot\eta}(q_1 - q_2)u_1|^2dx\right)^{\frac{1}{2}} \left(\int_{l_1} \left|e^{\tau x\cdot\eta}u_2\right|^2 ds\right)^{\frac{1}{2}} \end{aligned} \tag{3.12}$$

for all $u_1 \in \mathcal{W}_1(\Omega \cap B)$ and all $u_2 \in \mathcal{V}_2(\Omega \cap B)$.

Next we construct complex geometrical optics solutions. We only study the case $n = 3$. The proof in the case $n > 3$ is similar.

Denote $x^* = (x_1, x_2, -x_3)$ for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, and $\Omega^* = \{x^* : x \in \Omega\}$. For any function f, denote $f^*(x) = f(x^*) = f(x_1, x_2, -x_3)$.

For any $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ with $\xi_{1e} = \sqrt{\xi_1^2 + \xi_2^2} > 0$, we introduce

$$e(1) = \frac{1}{\xi_{1e}}(\xi_1, \xi_2, 0), \quad e(3) = (0, 0, 1)$$

and $e(2)$ such that $e(1)$, $e(2)$ and $e(3)$ form a orthogonal normal basis in $\mathbb{R}^3$. We also denote the coordinate of $x \in \mathbb{R}^3$ in this basis by $(x_{1e}, x_{2e}, x_{3e})_e$. A similar choice was done in [12]. We have,

$$\xi = (\xi_{1e}, 0, \xi_3)_e$$

For $\tau >> 0$, we choose

$$\rho_1 = (\frac{i}{2}\xi_{1e} - \tau\xi_3, i|\xi|\sqrt{\tau^2 - \frac{1}{4}}, \frac{i}{2}\xi_3 + \tau\xi_{1e})_e$$

$$\rho_2 = (\frac{i}{2}\xi_{1e} + \tau\xi_3, -i|\xi|\sqrt{\tau^2 - \frac{1}{4}}, \frac{i}{2}\xi_3 - \tau\xi_{1e})_e$$

A direct computation gives that

$$\rho_1 \cdot \rho_1 = \rho_2 \cdot \rho_2 = 0, \quad \rho_1 + \rho_2 = (i\xi_{1e}, 0, i\xi_3)_e = i\xi. \tag{3.13}$$

We look for u_2 in the form

$$u_2(x) = e^{x\cdot\rho_2}(1 + \psi_2(x, \rho_2))$$

By [22],

$$\|\psi_2(x, \rho_2)\|_{H^s(\Omega\cap B)} \leq \frac{C}{\tau^{1-s}}, \quad 0 \leq s \leq 2. \tag{3.14}$$

For $u_1(x)$, one needs to consider the boundary restriction $u_1(x) = 0$ on Γ_2. We do an even extension about x_3 for $q_1(x)$. By [22] there is a complex geometrical optics solution

$$e^{x\cdot\rho_1}(1 + \psi_1(x, \rho_1))$$

satisfying the estimate

$$\|\psi_1(x, \rho_1)\|_{H^s((\Omega\cap B)\cup(\Omega^*\cap B^*))} \leq \frac{C}{\tau^{1-s}}, \quad 0 \leq s \leq 2.$$

We look for $u_1(x)$ in the form

$$u_1(x) = e^{x\cdot\rho_1}(1+\psi_1(x,\rho_1)) - e^{x^*\cdot\rho_1}(1+\psi_1^*(x,\rho_1))$$

Then we have that $u_1(x)=0$ on Γ_2 satisfying the estimates

$$\|\psi_1(x,\rho_1)\|_{H^s(\Omega\cap B)} \le \frac{C}{\tau^{1-s}}, \quad 0\le s\le 2. \tag{3.15}$$

$$\|\psi_1^*(x,\rho_1)\|_{H^s(\Omega\cap B)} \le \frac{C}{\tau^{1-s}}, \quad 0\le s\le 2. \tag{3.16}$$

It is obvious that $u_1 \in \mathcal{W}_1(\Omega\cap B)$ and $u_2 \in \mathcal{V}_2(\Omega\cap B)$.

The choice of the phases ρ_1 and ρ_2 are quite different from those in [12]. Here we need a large negative parameter in the third component of the real part of $x\cdot\rho_2$ in order to use the inequality (3.12) in Lemma 10 (essentially, to use the Carleman estimate (3.8)). It plays an important role that u_2 does not need to vanish on Γ_2. Thus we do not need the reflection of ρ_2 as a phase function in the construction of the special solutions u_2, which in turn to guarantee the property we need. On the other hand, we also need to study the product of the phases corresponding to u_1 and u_2. By our construction of these phases, it is easy to show that

$$\mathrm{Re}\,(x^*\cdot\rho_1 + x\cdot\rho_2) = -2\tau x_3\xi_{1e}.$$

We know $x_3>0$ for any $x\in\Omega$. Using that $\xi_{1e}>0$, as $\tau\to+\infty$, we get $|e^{x^*\cdot\rho_1+x\cdot\rho_2}| = e^{-2\tau x_3\xi_{1e}} \to 0$ which is essential in our proof.

Next we will apply the special solutions we constructed above to the inequality (3.12) in Lemma 10. Denote $\xi^\perp = (-\xi_3, 0, \xi_{1e})_e$. Note that the third component of $\xi^\perp$ is positive. Then $\eta=\xi^\perp$ satisfies the condition in Lemma 10. We first show the right hand side goes to 0 as τ goes to infinity. For computational convenience we separate the real part and imaginary part of ρ_1 and ρ_2,

$$\rho_1 = i\big(\frac{1}{2}\xi + |\xi|\sqrt{\tau^2-\frac{1}{4}}e(2)\big) + \tau\xi^\perp, \quad \rho_2 = i\big(\frac{1}{2}\xi - |\xi|\sqrt{\tau^2-\frac{1}{4}}e(2)\big) - \tau\xi^\perp.$$

We compute

$$\left(\int_{l_1} \left|e^{\tau x\cdot\eta} u_2\right|^2 ds\right)^{\frac{1}{2}} = \left(\int_{l_1} \left|e^{\tau x\cdot\xi^\perp} e^{x\cdot\rho_2}(1+\psi_2)\right|^2 ds\right)^{\frac{1}{2}}$$

$$= \left(\int_{l_1} \left|(1+\psi_2)\right|^2 ds\right)^{\frac{1}{2}} \le C\left([area(l_1)]^{\frac{1}{2}} + \|\psi_2\|_{L^2(l_1)}\right)$$

$$\le C\left([area(l_1)]^{\frac{1}{2}} + \|\psi_2\|_{H^1(\Omega\cap B)}\right) \le C$$

and

$$\left(\int_{\Omega\cap B} \left|e^{-\tau x\cdot\eta}(q_1-q_2)u_1\right|^2 dx\right)^{\frac{1}{2}}$$

$$= \left(\int_{\Omega\cap B} \left|e^{-\tau x\cdot\xi^\perp}(q_1-q_2)[e^{x\cdot\rho_1}(1+\psi_1(x,\rho_1)) - e^{x^*\cdot\rho_1}(1+\psi_1^*(x,\rho_1))]\right|^2 dx\right)^{\frac{1}{2}}$$

$$\le \left(\int_{\Omega\cap B} \left|q_1-q_2\right|^2 \left[(1+|\psi_1(x,\rho_1)|) + e^{-2\tau x_3\xi_{1e}}(1+|\psi_1^*(x,\rho_1)|)\right]^2 dx\right)^{\frac{1}{2}}$$

$$\le C(\|q_1\|_{L^\infty} + \|q_1\|_{L^\infty})\cdot$$

$$\left[\left(\int_{\Omega\cap B}(1+|\psi_1(x,\rho_1)|^2)dx\right)^{\frac{1}{2}} + \left(\int_{\Omega\cap B} e^{-4\tau x_3\xi_{1e}}(1+|\psi_1^*(x,\rho_1)|^2)dx\right)^{\frac{1}{2}}\right]$$

Since $x_3 > 0$ in Ω and $\xi_{1e} > 0$, as $\tau \to +\infty$, we know $e^{-4\tau x_3\xi_{1e}} \to 0$. Then by the Lebesgue's dominated convergence theorem we obtain, as $\tau \to +\infty$,

$$\int_{\Omega\cap B} e^{-4\tau x_3\xi_{1e}}(1+|\psi_1^*(x,\rho_1)|^2)dx \to 0.$$

Together with (3.15), we get

$$\left(\int_{\Omega\cap B} \left|e^{-\tau x\cdot\eta}(q_1-q_2)u_1\right|^2 dx\right)^{\frac{1}{2}} \le C$$

We also know $\eta\cdot\nu = \xi_{1e} > 0$, therefore as $\tau \to +\infty$

$$\left(\frac{1}{\tau(\eta\cdot\nu)}\right)^{\frac{1}{2}} \left(\int_{\Omega\cap B} |e^{-\tau x\cdot\eta}(q_1-q_2)u_1|^2 dx\right)^{\frac{1}{2}} \left(\int_{l_1} \left|e^{\tau x\cdot\eta}u_2\right|^2 ds\right)^{\frac{1}{2}} \to 0.$$

We then compute the left hand side

$$
\begin{aligned}
& \left|\int_\Omega (q_1 - q_2) u_1 u_2 dx\right| \\
= & \left|\int_\Omega (q_1 - q_2) e^{ix\cdot\xi}(1+\psi_1)(1+\psi_2)dx - \right. \\
& \left. \int_\Omega (q_1 - q_2) e^{ix_{1e}\cdot\xi_{1e}} e^{-2\tau x_3 \xi_{1e}}(1+\psi_1^*)(1+\psi_2)dx\right|
\end{aligned}
$$

Since $x_3 > 0$ in Ω and $\xi_{1e} > 0$, as before, the second term goes to 0 as $\tau \to +\infty$. In view of (3.14)(3.15), we know

$$
\left|\int_\Omega (q_1 - q_2) u_1 u_2 dx\right| \to \left|\int_\Omega e^{ix\cdot\xi}(q_1 - q_2)dx\right|, \quad as\ \tau \to +\infty.
$$

We eventually obtain

$$
\left|\int_\Omega e^{ix\cdot\xi}(q_1 - q_2)dx\right| = 0
$$

or equivalently,

$$
\int_\Omega e^{ix\cdot\xi}(q_1 - q_2)dx = 0
$$

for all ξ with $\xi_{1e} > 0$. Since ξ_{1e} is always non-negative, then using continuity, we have

$$
\int_\Omega e^{ix\cdot\xi}(q_1 - q_2)dx = 0
$$

for any ξ. Therefore

$$
q_1(x) - q_2(x) = 0 \quad in\ \Omega.
$$

4 Determination of the potential from $C^S_{q,\ \Gamma_1'}$

In this section, we shall prove Theorem 2. We only prove the case $n = 3$ since the proof in the case $n > 3$ is similar. Let $q_1(x)$ and $q_2(x)$ be two potentials with the same Cauchy data $C^S_{q_1,\ \Gamma_1'} = C^S_{q_2,\ \Gamma_1'}$. Similarly as the proof in last section, we can obtain the identity

$$
\int_\Omega (q_1 - q_2) u_1 u_2 dx = -\int_{l_2} \frac{\partial w}{\partial \nu} u_2 ds,
$$

where $u_1(x)$ is a solution of (3.1)(3.2), $u_2(x)$ is a solution of (3.6), and $w(x) = v(x) - u_1(x)$ with $v(x)$ being a solution of (3.3)(3.4). If we also require that $u_2(x) = 0$ on Γ_2, then we have the orthogonality relation

$$\int_\Omega (q_1 - q_2)u_1u_2 dx = 0. \tag{4.1}$$

In this case, we do not need to use Carleman estimates, but we have to construct solutions of u_1 and u_2 both vanishing on Γ_2. From Lemma 9, (4.1) holds for any $u_1 \in \mathcal{W}_1(\Omega \cap B)$ and any $u_2 \in \mathcal{W}_2(\Omega \cap B)$. We use the same argument as in [12]. For any $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ with $\xi_{1e} = \sqrt{\xi_1^2 + \xi_2^2} > 0$, define

$$\rho_1 = (\frac{i}{2}\xi_{1e} - i\tau\xi_3, -|\xi|\sqrt{\tau^2 + \frac{1}{4}}, \frac{i}{2}\xi_3 + i\tau\xi_{1e})_e$$

$$\rho_2 = (\frac{i}{2}\xi_{1e} + i\tau\xi_3, |\xi|\sqrt{\tau^2 + \frac{1}{4}}, \frac{i}{2}\xi_3 - i\tau\xi_{1e})_e$$

We do an even extension about x_3 for $q_1(x)$ and $q_2(x)$. By [22] there are complex geometrical optics solutions

$$e^{x\cdot\rho_j}(1 + \psi_j(x, \rho_j))$$

to the equation

$$(-\Delta + q_j(x) - k^2)u(x) = 0$$

satisfying the estimate

$$\|\psi_j(x, \rho_j)\|_{H^s((\Omega\cap B)\cup(\Omega^*\cap B^*))} \le \frac{C}{\tau^{1-s}}, \quad 0 \le s \le 2.$$

for $j = 1, 2$. We look for $u_j(x)$ in the form

$$u_j(x) = e^{x\cdot\rho_j}(1 + \psi_j(x, \rho_j)) - e^{x^*\cdot\rho_j}(1 + \psi_j^*(x, \rho_j))$$

Then $u_j(x) = 0$ $(j = 1, 2)$ on Γ_2 is automatically satisfied and we have the estimates

$$\|\psi_j(x, \rho_j)\|_{H^s(\Omega\cap B)} \le \frac{C}{\tau^{1-s}}, \quad \|\psi_j^*(x, \rho_j)\|_{H^s(\Omega\cap B)} \le \frac{C}{\tau^{1-s}}, \quad 0 \le s \le 2. \tag{4.2}$$

We get

$$\begin{aligned}
0 &= \int_\Omega (q_1 - q_2) u_1 u_2 dx \\
&= \int_\Omega (q_1 - q_2) e^{ix\cdot\xi}(1+\psi_1)(1+\psi_2) dx - \\
&\quad \int_\Omega (q_1 - q_2) e^{ix_{1e}\cdot\xi_{1e}} e^{-2i\tau x_3 \xi_{1e}}(1+\psi_1^*)(1+\psi_2) dx - \\
&\quad \int_\Omega (q_1 - q_2) e^{ix_{1e}\cdot\xi_{1e}} e^{2i\tau x_3 \xi_{1e}}(1+\psi_1)(1+\psi_2^*) dx + \\
&\quad \int_\Omega (q_1 - q_2) e^{ix^*\cdot\xi}(1+\psi_1)(1+\psi_2) dx
\end{aligned}$$

Since $x_3 > 0$ in Ω and $\xi_{1e} > 0$, by the Riemann-Lebesgue lemma, the middle two terms converge to zero as $\tau \to +\infty$. Using the even extension of q_1 and q_2, we get

$$\int_{\Omega\cup\Omega^*} (q_1 - q_2) e^{ix\cdot\xi} dx = 0.$$

for all ξ with $\xi_{1e} > 0$. Since ξ_{1e} is always non-negative, using continuity, we obtain

$$\int_{\Omega\cup\Omega^*} e^{ix\cdot\xi}(q_1 - q_2) dx = 0$$

for any ξ. Therefore

$$q_1(x) - q_2(x) = 0 \quad in\ \Omega.$$

5 Determination of the conductivity

In this section, we prove Theorem 4 and Theorem 5. We will transform the conductivity equation to the Schrödinger equation by the well known transformation

$$\omega = \gamma^{1/2} u. \tag{5.1}$$

We consider the case $k > 0$ first. If u solves the conductivity equation (2.5), then ω solves the Schrödinger equation (2.1) with

$$q(x) = \gamma^{-1/2} \Delta \gamma^{1/2}. \tag{5.2}$$

We prove Theorem 4 for $k > 0$. From (2.6)(5.1), we know that if $C^D_{\gamma_1,\ \Gamma'_2} = C^D_{\gamma_2,\ \Gamma'_2}$ then $C^D_{q_1,\ \Gamma'_2} = C^D_{q_2,\ \Gamma'_2}$. So $q_1(x) = q_2(x)$ in Ω from Theorem 1. We rewrite (5.2) as

$$-\Delta\gamma^{1/2} + q(x)\gamma^{1/2} = 0. \tag{5.3}$$

The condition (2.6) implies that $\gamma_1^{1/2} = \gamma_2^{1/2}$ on Γ'_2 and $\frac{\partial\gamma_1^{1/2}}{\partial\nu} = \frac{\partial\gamma_2^{1/2}}{\partial\nu}$ on Γ'_2, then we conclude $\gamma_1 = \gamma_2$ from the unique continuation for (5.3).

The proof of Theorem 5 for $k > 0$ is similar. By [15], if $C^S_{\gamma_1,\ \Gamma'_1} = C^S_{\gamma_2,\ \Gamma'_1}$ then $\gamma_1 = \gamma_2$ on Γ'_1 and $\frac{\partial\gamma_1}{\partial\nu} = \frac{\partial\gamma_2}{\partial\nu}$ on Γ'_1. Since $\gamma_1 = \gamma_2 = 1$ on $\Gamma_1 \setminus l_1$, we know $\gamma_1 = \gamma_2$ on Γ_1. Together with (5.1), we have $C^S_{q_1,\ \Gamma'_1} = C^S_{q_2,\ \Gamma'_1}$. The rest of the proof is the same as the proof of Theorem 4.

Now we consider the case $k = 0$. The difference of the proof between $k = 0$ and $k > 0$ is only the solvability of the direct problem. When $k = 0$, if u solves (2.5), then ω, defined by (5.1), solves (2.1) with q as in (5.2) and $k = 0$. The solvability of (2.1) for such q and $k = 0$ is based on the solvability of (2.5) for $k = 0$, which is guaranteed by the Lax-Milgram theorem. Once we have the solvability for the Schrödinger equation (2.1) with $k = 0$, all the arguments in the proofs of Theorem 1 (Section 3) and Theorem 2 (Section 4) hold for $k = 0$. Then the rest of the proofs of Theorem 4 and Theorem 5 for $k = 0$ is the same as that for $k > 0$ by using the results for the corresponding Schrödinger equation (the discussion at the beginning of this section). The proofs are complete.

A The solvability of the Schrödinger equation in an infinite slab

In this appendix we will use the Lax-Phillips method (see [11]) to prove the existence of the $H^1(\Omega)$ solution to the Dirichlet problem (2.1)(2.2)(2.3) satisfying the partial radiation condition (2.4), Assumption 1, and Assumption 2 (if $n = 3$). By the trace formula, it is enough to study the equation

$$(-\Delta + q(x) - k^2)v(x) = F \quad in\ \Omega,$$

$$v(x) = 0 \quad on\ \partial\Omega.$$

where $F \in H^{-1}(\Omega)$ has compact support in $\mathbb{R}^n$.

Let the compact set B contains the support of $q(x)$. We choose an open set B_0 containing B such that k^2 is not a Dirichlet eigenvalue for both $-\Delta$

and $-\Delta + q(x)$ in $B_0 \cap \Omega$. We also choose an open set B_1 such that $B \subset B_1 \subset \overline{B_1} \subset B_0$. Let φ be a cutoff $\mathcal{C}^\infty$ function that is 1 in B and 0 outside B_1. We look for a solution of the form

$$v = \omega - \varphi(w - V)$$

where w is a solution of

$$(-\Delta - k^2)w(x) = F_1 \quad in\ \Omega, \qquad w = 0 \quad on\ \partial\Omega \tag{A.1}$$

satisfying the partial radiation condition (2.4), and V is a solution of

$$(-\Delta + q(x) - k^2)V(x) = F_1 \quad in\ B_0 \cap \Omega, \qquad V = w \quad on\ \partial(B_0 \cap \Omega). \tag{A.2}$$

The function $F_1 \in H^{-1}(\Omega)$ with compact support in $\mathbb{R}^n$ will be determined later.

We first discuss the solvability of $w \in H^1(\Omega)$ to (A.1). The uniqueness was proved in [21, 19]. The Assumption 2 is needed if $n = 3$. The Green's function for $-\Delta - k^2$ in the slab Ω with vanishing condition on the boundary is

$$G(x, y) = \sum_{m=1}^{\infty} \frac{-i}{2L} \left(\frac{k_m}{2\pi|x' - y'|} \right)^{\frac{n-3}{2}} \sin(\frac{m\pi x_n}{L}) \sin(\frac{m\pi y_n}{L}) H^1_{\frac{n-3}{2}}(k_m|x' - y'|)$$

where $k_m = k(1 - \frac{m^2\pi^2}{k^2L^2})^{\frac{1}{2}}$ and $H^1_{\frac{n-3}{2}}(\cdot)$ is the Hankel function of first kind. We then have

$$w(x) = \int_\Omega G(x, y)F_1(y)dy.$$

The existence of $V \in H^1(B_0 \cap \Omega)$ is from the uniqueness of the solution to (A.2) by Fredholm alternative. The uniqueness is based on our choice of B_0 such that k^2 is not a Dirichlet eigenvalue for $-\Delta + q(x)$ in $B_0 \cap \Omega$.

Obviously, $v = 0$ on $\partial\Omega$. Next we study F_1. We have $v = V$ in $B \cap \Omega$, so $(-\Delta + q(x) - k^2)V(x) = f$ there. In $B^c \cap \Omega$, we have $q(x) = 0$, and

$$\begin{aligned}
& (-\Delta + q(x) - k^2)v(x) = (-\Delta - k^2)v(x) \\
= \ & (-\Delta - k^2)w + (\Delta\varphi)(w - V) + 2\nabla\varphi \cdot \nabla(w - V) \\
& +\varphi[(\Delta + k^2)w - (\Delta + k^2)V] \\
= \ & F_1 + KF_1 + \varphi(F_1 - F_1) = F_1 + KF_1
\end{aligned}$$

where we define $KF_1 = (\Delta\varphi)(w-V) + 2\nabla\varphi\cdot\nabla(w-V)$. We also define KF_1 as zero in $B\cap\Omega$. We conclude that v solves the original equation if and only of F_1 solves the following equation

$$F = F_1 + KF_1 \tag{A.3}$$

We claim that the operator K is compact on $H^{-1}(B_0\cap\Omega)$. The elliptic theory shows that $(-\Delta + q(x) - k^2)^{-1}$ is a continuous operator from $H^{-1}(B_0\cap\Omega)$ to $H^1(B_0\cap\Omega)$. Since K involves only the first-order derivative, K is then a continuous operator from $H^{-1}(B_0\cap\Omega)$ to $L^2(B_0\cap\Omega)$, and therefore compact on $H^{-1}(B_0\cap\Omega)$ by the compact embedding theory. So equation (A.3) is Fredholm and its solvability follows from the uniqueness of its solution.

We show the uniqueness. Let $F = 0$. Then v is a solution to the homogeneous equation. From Assumption 1, we know that $v = 0$ in Ω. In $B\cap\Omega$, we have $F_1 = F - KF_1 = F = 0$ and $V = v = 0$. In $B^c\cap\Omega$, we have $q(x) = 0$. Thus from the equations for ω and V, we know that

$$(-\Delta - k^2)(w-V) = 0 \quad \textit{in } B_0\cap\Omega, \qquad w - V = 0 \quad \textit{on } \partial(B_0\cap\Omega)$$

Since we choose B_0 such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in $B_0\cap\Omega$, we get $w = V$ in $B_0\cap\Omega$. From the definition of KF_1, we know that $KF_1 = 0$ in Ω, and hence $F_1 = F - KF_1 = 0 - 0 = 0$ in Ω. This proves the uniqueness of the solution. And then from the Fredholm alternative, we know the existence of the solution.

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