Contemporary Mathematics

Oscillating-decaying solutions for elliptic systems

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ABSTRACT. We present a general framework for constructing oscillating-decaying solutions for general inhomogeneous elliptic systems using semiclassical microlocal analysis. We give an application to the determination of inclusions and cavities embedded in an anisotropic elastic medium.

1. Introduction

Special solutions for elliptic scalar equations or systems have played an important role in inverse problems since the pioneering work of Calderón [**C**]. In 1987, Sylvester and Uhlmann [**SU**] introduced complex geometrical optics solutions to solve the inverse boundary value problem for the conductivity equation in dimension $n \geq 3$. Recently, Ikehata used Calderón type solutions in the inverse problem of identifying embedded objects [**I**] in an isotropic conducting medium. It should be pointed out that the complex geometrical optics solutions considered in those papers are available only for operators and system of operators whose leading part is the Laplacian and diagonal operators with Laplacian in the diagonal entries, respectively.

To consider inverse problems for more general systems we look for a substitute of the complex geometrical optics solutions. In this paper we show that the *oscillating-decaying solutions* we construct can also be useful in inverse problems. Roughly speaking, given a hyperplane, the oscillating-decaying solutions are rapidly oscillating along this plane and decaying exponentially in the direction transverse to the same plane. They are also complex geometrical optics solutions but with the imaginary part of the phase function non-negative. Previously, the oscillating-decaying solutions for general elasticity systems were constructed in a rather elementary way in [**NW I**] and [**NUW I**]. The method used in those papers is to convert the PDEs into systems of ODEs with an appropriate definition of "order". In this paper we present a more systematic way of constructing the oscillating-decaying solutions for general elliptic systems. Our method here is based on semiclassical microlocal analysis techniques. The decaying property of the oscillating-decaying

2000 *Mathematics Subject Classification.* Primary 35J45; Secondary 35R30.
Key words and phrases. Oscillating-decaying solutions, elliptic systems, inverse problems.

solutions will emerge once we invoke the stationary phase method. In the last section, we will briefly describe how these solutions can be used to detect inclusions and cavities embedded in an anisotropic, inhomogeneous elastic medium.

2. Description of the oscillating-decaying solutions

In this section we give a detailed description of the oscillating-decaying solutions for general elliptic systems. Assume that

$$C^{j\ell}_{\alpha\beta}(x) \in B^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \ \partial^\gamma f \in L^\infty(\mathbb{R}^n), \forall \gamma \in \mathbb{Z}^n_+\}, \quad \forall\ \alpha, \beta, j, \ell$$

and

$$C^{j\ell}_{\alpha\beta}(x) = C^{\ell j}_{\beta\alpha}(x), \quad \forall\ x \in \mathbb{R}^n, \quad \forall\ \alpha, \beta, j, \ell. \tag{2.1}$$

Unless otherwise indicated, here and below, α, β are from 1 to N and all Roman indices are set to vary from 1 to n. Furthermore, we assume that there exists $\delta > 0$ such that for any vectors $a = (a_1, \cdots, a_N)^\top \in \mathbb{R}^N$ and $b = (b_1, \cdots, b_n)^\top \in \mathbb{R}^n$

$$\sum_{\alpha\beta}\sum_{j\ell} C^{j\ell}_{\alpha\beta}(x) a_\alpha b_j a_\beta b_\ell \ge \delta |a|^2 |b|^2 \quad \forall\ x \in \mathbb{R}^n \quad \text{(strong ellipticity)}. \tag{2.2}$$

Denote L the $N \times N$ system

$$(Lu)_\alpha(x) = \sum_{j,\ell=1}^{n}\sum_{\beta=1}^{N} C^{j\ell}_{\alpha\beta}(x)\partial^2_{j\ell} u_\beta \quad \forall\ 1 \le \alpha \le N,$$

where $u(x) = (u_1(x), \cdots, u_N(x))^\top$ is a vector-valued function of size N. Here, for simplicity, we only consider the pure second operator L. However, our method is valid for L with lower order terms.

Before going to the main theme of the section, we want to define several notations. Assume that $\Omega \subset \mathbb{R}^n$ is an open set with smooth boundary and $\omega \in \mathbb{S}^{n-1}$ is given. Let $\eta_1, \cdots, \eta_{n-1} \in \mathbb{S}^{n-1}$ be chosen so that $\{\eta_1, \cdots, \eta_{n-1}, \omega\}$ forms an orthonormal system of $\mathbb{R}^n$. We then denote $x' = (x \cdot \eta_1, \cdots, x \cdot \eta_{n-1})$ for any $x \in \mathbb{R}^n$. Let $t \in \mathbb{R}$, $\Omega_t(\omega) = \Omega \cap \{x \cdot \omega > t\}$ and $\Sigma_t(\omega) = \{x \cdot \omega = t\} \cap \Omega \ne \emptyset$. In the next section we will construct special solutions satisfying

$$\begin{cases} Lu = 0 \quad \text{in} \quad \Omega_t(\omega) \\ u|_{x\cdot\omega=t} = e^{ix\cdot\theta/h}\chi(x')b \end{cases} \tag{2.3}$$

with $\theta \in \mathbb{S}^{n-1}$ lying in the span of $\{\eta_1, \cdots, \eta_{n-1}\}$ is fixed, $\chi(x') \in \mathbb{C}^\infty_0(\mathbb{R}^{n-1})$ with $\operatorname{supp}(\chi) \subset \Sigma_t(\omega)$ and $0 \ne b \in \mathbb{C}^N$. Here $h > 0$ is a small parameter. Since the solution u involves a parameter h, it is quite natural to construct u by semiclassical microlocal analysis techniques. Some properties of u will emerge when we apply the stationary phase method to its integral representation. It turns out the solution obtained by the semiclassical microlocal analysis does not satisfy the initial condition in (2.3) exactly, but the error is $O(h^\infty)$, which is typical in the semiclassical microlocal approach.

3. Construction of oscillating-decaying solutions

Without loss of generality, we consider the special case where $t = 0$, $\omega = e_n$ and choose $\eta_j = e_j$ for $j = 1, \cdots, n-1$, where $e_1, \cdots, e_n$ denotes the canonical basis of $\mathbb{R}^n$. In this case, we write $x' = (x_1, \cdots, x_{n-1})$ and $\theta = (\theta', 0)$. Now in the canonical Euclidean coordinates, the operator L is written as

$$L = T(x)\partial_{x_n}^2 + A(x, \partial_{x'})\partial_{x_n} + Q(x, \partial_{x'}),$$

where

$$T(x) = \big(C_{\alpha\beta}^{nn}(x); \alpha \downarrow 1, \cdots, N,\ \beta \to 1 \cdots, N\big),$$
$$A(x, \partial_{x'}) = R(x, \partial_{x'}) + R^t(x, \partial_{x'}),$$
$$R(x, \partial_{x'}) = \big(\sum_{j=1}^{n-1} C_{\alpha\beta}^{jn}(x)\partial_{x_j}; \alpha \downarrow 1, \cdots, N,\ \beta \to 1 \cdots, N\big),$$
$$Q(x, \partial_{x'}) = \big(\sum_{j,\ell=1}^{n-1} C_{\alpha\beta}^{j\ell}(x)\partial_{x_j}\partial_{x_\ell}; \alpha \downarrow 1, \cdots, N,\ \beta \to 1 \cdots, N\big).$$

To work in the semiclassical setting, we write $-L$ as

$$-L = h^{-2}\{T(x)(hD_{x_n})^2 + A(x, hD_{x'})(hD_{x_n}) + Q(x, hD_{x'})\},$$

where $D_{x'} = -i\partial_{x'}$ and $D_{x_n} = -i\partial_{x_n}$. Denote $\tilde{L}_h = -h^2 L$. The principal symbol of $\tilde{L}_h$ is given by

$$\sigma(\tilde{L}_h)(x, \xi) := M(x, \xi', \xi_n) := T(x)\xi_n^2 + A(x, \xi')\xi_n + Q(x, \xi'),$$

where (ξ', ξ_n) are variables in the semiclassical Fourier transform with $\xi' = (\xi_1, \cdots, \xi_{n-1})$. More precise, we define the semiclassical Fourier transform of u by

$$\mathcal{F}_h u(\xi) = \frac{1}{(2\pi h)^{n/2}} \int e^{-ix\cdot\xi/h} u(x)dx.$$

It follows from the symmetric properties of C that $T(x)$ and $Q(x, \xi')$ are symmetric matrices. By virtue of the strong ellipticity condition (2.2), we find that $T(x)$ and $M(x, \xi', \xi_3)$ are positive definite matrices for $x \in \mathbb{R}^n$, $(\xi', \xi_n) \in \mathbb{R}^n \setminus 0$. Therefore for any fixed (x, ξ'), the polynomial $\det(M(x, \xi', \xi_n))$ in ξ_n admits n roots $\xi_n = \lambda_j$ $(1 \le j \le n)$ with positive imaginary parts and another n roots $\xi_n = \bar{\lambda}_j$ $(1 \le j \le n)$. The following factorization of M is well-known.

Theorem 3.1. [**GLR**] *Let*

$$B_0(x, \xi') := \big(\oint_\Gamma \lambda M(x, \xi', \lambda)^{-1} d\lambda\big)\big(\oint_\Gamma M(x, \xi', \lambda)^{-1} d\lambda\big)^{-1},$$

where $\Gamma \subset \mathbb{C}_+ := \{\lambda \in \mathbb{C} :\ \mathrm{Im}\lambda > 0\}$ *is a close contour enclosing all* λ_j *for* $1 \le j \le n$. *Then we have that*

$$M(x, \xi', \xi_n) = (\xi_n - B_0^*(x, \xi'))T(x)(\xi_n - B_0(x, \xi'))$$

with

$$\mathrm{Spec}(B_0(x, \xi')) \subset \mathbb{C}_+, \tag{3.1}$$

where $\mathrm{Spec}(B_0(x, \xi'))$ *denotes the spectrum of* $B_0(x, \xi')$.

Based on Theorem 3.1, we would like to factorize $\tilde{L}$ into a product of the backward and forward "heat" operators. A similar factorization of $-L$ in the classical setting $(h = 1)$ was proved in [**LU**] for scalar operators and [**NU**] for systems. Here we will give the proof in the semiclassical setting. Before doing so, we first recall several definitions and notation used in semiclassical analysis (see for instance [**M**])
.

Definition 3.2. *A function $a = a(y, \eta; h)$ defined in $\mathbb{R}^n \times \mathbb{R}^n \times (0, h_0]$ for some $h_0 > 0$ is said to be in $S_{2n}(< \eta >^m)$ if a depends smoothly on $(y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ and satisfies*

$$\sup_{(y,\eta,h)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,h_0]} |\partial^\gamma_{y,\eta} a| \leq c_\gamma < \eta >^m \quad \textit{for some} \quad c_\gamma > 0,$$

where γ is any multi-index and $< \eta > = \sqrt{1 + |\eta|^2}$.

Given $a(y, \eta; h) \in S_{2n}(< \eta >^m)$, we define an operator $\mathrm{Op}_h(a)$ by

$$(\mathrm{Op}_h(a)u)(x) = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\cdot\eta/h} a(y, \eta; h) u(y) dy d\eta.$$

The operator $\mathrm{Op}_h(a)$ is called h-pseudodifferential operator with symbol $a(y, \eta; h)$.

Definition 3.3. *A function $a(y, \eta; h) \in S_{2n}(< \eta >^m)$ is called an h-classical symbol of degree $m \in \mathbb{R}$ if $a(y, \eta; h)$ admits an asymptotic expansion of the type*

$$a(y, \eta; h) \sim \sum_{j\geq 0} h^j a_j(y, \eta),$$

where $a_j(y, \eta) \in S_{2n}(< \eta >^m)$ are independent of h for all j. Also, we denote $a = O(h^\infty)$ in $S_{2n}(< \eta >^m)$ if all a_j's are zero.

Let a be an h-classical symbol of degree m. Then the operator $\mathrm{Op}_h(a)$ defined by

$$(\mathrm{Op}_h(a)u)(x) = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\cdot\eta/h} a(y, \eta; h) u(y) dy d\eta$$

is called a classical h-pseudodifferential operator of degree m. Similar to the classical case, the composition rule for two h-pseudodifferential operators is described as follows. Let $a(y, \eta; h)$ and $b(y, \eta; h)$ be two h-classical symbols of degree m and m'. Then $\mathrm{Op}_h(a) \circ \mathrm{Op}_h(b) = \mathrm{Op}_h(c)$, where $c(y, \eta; h)$ is an h-classical symbol of degree $m + m'$ and is written as

$$c(y, \eta; h) \sim \sum_{|\gamma|\geq 0} \frac{h^{|\gamma|}}{\gamma!} \partial^\gamma_\eta a(y, \eta; h) D^\gamma_y b(y, \eta; h) =: (a\sharp b)(y, \eta; h). \tag{3.2}$$

Notice that $c(y, \eta; h)$ is unique up to $O(h^\infty)$ in $S_{2n}(< \eta >^m)$. The formal adjoint $\mathrm{Op}^*_h(a)$ of $\mathrm{Op}_h(a)$ is also a classical h-pseudodifferential operator of degree m with full symbol given by

$$\tilde{\sigma}(\mathrm{Op}^*_h(a)) \sim \sum_{|\gamma|\geq 0} \frac{h^{|\gamma|}}{\gamma!} \partial^\gamma_\eta D^\gamma_y a^*(y, \eta; h).$$

Hereafter, we use $\tilde{\sigma}(P_h)$ to denote the full symbol of a h-pseudodifferential operator P_h. Now we can show that

Theorem 3.4. *There exist classical h-pseudodifferential operators $B_h(x, D_{x'})$ and $G_h(x, D_{x'})$ of degree 1 and 0 respectively depending smoothly on x_n $(0 \leq x_n)$ such that*

$$\tilde{L}_h = (hD_{x_n} - B^*_h(x, D_{x'}) + hG_h(x, D_{x'}))T(x)(hD_{x_n} - B_h(x, D_{x'})) + R_h(x, D_x). \tag{3.3}$$

Here the operator

$$R_h(x, D_x) = R^{(1)}_h(x, D_{x'})(hD_{x_n}) + R^{(2)}_h(x, D_{x'}),$$

where $R_h^{(1)}$ and $R_h^{(2)}$ are classical h-pseudodifferential operators depending smoothly on x_n and the full symbol of $R_h^{(k)}$

$$\tilde{\sigma}(R_h^{(k)}(x,\xi')) \sim \sum_{j\geq 0} h^j R_j^{(k)}(x,\xi')$$

with $R_j^{(k)}(x,\xi') = 0$ for $|\xi'| \geq 1/2$ for all j and $k = 1,2$.

REMARK 3.5. *In the microlocal setting, a similar factorization for L is quite well-known (see [**NU**]). In this situation, the error term (corresponding to R_h here) is a smoothing operator. However, in (3.3) the error term R_h is merely order zero in h, not $O(h^\infty)$. This is due to the regularization in ξ' variables.*

Proof. As defined in (3.3), we aim to compute

$$\begin{aligned} R_h &= \tilde{L}_h - (hD_{x_n} - B_h^*(x,D_{x'}) + hG_h(x,D_{x'}))T(x)(hD_{x_n} - B_h(x,D_{x'})) \\ &= \big\{A + TB_h - h(D_{x_n}T) + B_h^*T - hG_hT\big\}(hD_{x_n}) \\ &\quad + \big\{Q + h(D_{x_n}T)B_h + hT(D_{x_n}B_h) - B_h^*TB_h + hG_hTB_h\big\}. \end{aligned}$$

We will look for B_h and G_h such that $R = O(h^\infty)$, i.e., B_h and G_h satisfy

$$-TB_h + h(D_{x_n}T) - B_h^*T + hG_hT = A \tag{3.4}$$

and

$$-h(D_{x_n}T)B_h - hT(D_{x_n}B_h) + B_h^*TB_h - hG_hTB_h = Q \tag{3.5}$$

Multiplying (3.4) by B_h and eliminating G_h term using (3.5), we obtain that

$$h(D_{x_n}B_h) + T^{-1}AB_h + B_h^2 + T^{-1}Q = 0. \tag{3.6}$$

The formula (3.6) can be treated as a semiclassical Riccati equation for B_h. In view of the composition formula (3.2), the full symbol of (3.6) gives

$$\begin{aligned} &h\tilde{\sigma}(D_{x_n}B_h) + \textstyle\sum_{|\gamma|\geq 0} \frac{h^{|\gamma|}}{\gamma!}\partial_{\xi'}^{\gamma}\tilde{\sigma}(T^{-1}A)D_{x'}^{\gamma}\tilde{\sigma}(B_h) \\ &+ \textstyle\sum_{|\gamma|\geq 0} \frac{h^{|\gamma|}}{\gamma!}\partial_{\xi'}^{\gamma}\tilde{\sigma}(B_h)D_{x'}^{\gamma}\tilde{\sigma}(B_h) + T^{-1}\tilde{\sigma}(Q) \sim 0. \end{aligned} \tag{3.7}$$

It is clear that

$$\tilde{\sigma}(A) = A(x,\xi') \quad \text{and} \quad \tilde{\sigma}(Q) = Q(x,\xi').$$

Now we assume that B_h is a classical h-pseudodifferential operator of degree 1 with symbol

$$\tilde{\sigma}(B_h) \sim \sum_{j\geq 0} h^j B_j(x,\xi').$$

We will solve for B_j recursively by matching the formula (3.7). We first look at the $O(h^0)$ term. It is readily seen that

$$B_0^2 + T^{-1}AB_0 + T^{-1}Q = 0 \tag{3.8}$$

which is valid by Theorem 3.1. We can see that B_0 is homogeneous of degree 1 in ξ' and is not smooth at $\xi' = 0$. To avoid this singularity, we only solve (3.8) for $|\xi'| \geq 1/2$ and extend the resulting solution to be smooth in $|\xi'| < 1/2$. For simplicity, we still denote this new solution by $B_0(x,\xi')$. Clearly, $B_0(x,\xi')$ in $|\xi'| < 1/2$ can be chosen such that the spectral condition (3.1) holds. With this in mind, we only require that (3.7) be satisfied for $|\xi'| \geq 1/2$.

Therefore, grouping the $O(h^1)$ terms we get that

$$B_0B_1+B_1B_0+T^{-1}AB_1=-\sum_{j=1}^{n-1}\partial_{\xi_j}B_0D_{x_j}B_0-\sum_{j=1}^{n-1}T^{-1}\partial_{\xi_j}AD_{x_j}B_0-D_{x_n}B_0 \tag{3.9}$$

for $|\xi'|\geq 1/2$. Moreover, grouping the $O(h^r)$ terms for $r\in\mathbb{N}$ leads to

$$\begin{aligned} B_0B_r+B_rB_0+T^{-1}AB_r&=-\sum_{\substack{j+k=r\\ j,k\geq 1}}B_jB_k-\sum_{\substack{|\gamma|\geq 0\\ j+k=r-|\gamma|\\ j,k\geq 0}}\tfrac{1}{\gamma!}\partial_{\xi'}^{\gamma}B_jD_{x'}^{\gamma}B_k\\ &\quad-\textstyle\sum_{j=1}^{n-1}T^{-1}\partial_{\xi_j}AD_{x_j}B_{r-1}-D_{x_n}B_{r-1}\quad \text{for } |\xi'|\geq 1/2.\end{aligned} \tag{3.10}$$

To verify that (3.9) and (3.10) are solvable, we observe that for $|\xi'|\geq 1/2$

$$B_0B_r+B_rB_0+T^{-1}AB_r=(-T^{-1}QB_0^{-1})B_r-B_r(-B_0). \tag{3.11}$$

Using the fact that $\operatorname{Spec}(B_0)\subset\mathbb{C}_+$ and the strong ellipticity condition (2.2), we can see that

$$\operatorname{Spec}(-B_0)\subset\mathbb{C}_-\quad\text{and}\quad \operatorname{Spec}(-T^{-1}QB_0^{-1})\subset\mathbb{C}_+. \tag{3.12}$$

By virtue of (3.11) and (3.12), we conclude that (3.9) and (3.10) are uniquely solvable.

Once B_h is determined, we can compute the full symbol of G_h iteratively from (3.4) in the sense of symbol for $|\xi'|\geq 1/2$ by setting

$$\tilde{\sigma}(G_h)\sim\sum_{j\geq 0}h^jG_j(x,\xi'),$$

where $G_j(x,\xi')$ is smooth in $\mathbb{R}^n\times\mathbb{R}^{n-1}$ for all j. Precisely, G_0 and G_j $(j>0)$ are required to satisfy

$$G_0=\sum_{j=1}^{n-1}T\partial_{\xi_j}B_0D_{x_j}T^{-1}-(D_{x_n}T)T^{-1}+\sum_{j=1}^{n-1}\partial_{\xi_j}D_{x_j}B_0^*$$

and

$$G_j=\sum_{|\gamma|\leq j+1}\frac{1}{\gamma!}(T\partial_{\xi'}^{\gamma}B_{j+|\gamma|}D_{x'}^{\gamma}T^{-1}+\partial_{\xi'}^{\gamma}D_{x'}^{\gamma}B_{j+|\gamma|}^*)\quad\forall\ j\geq 1$$

for $|\xi'|\geq 1/2$.

Now with such choices of B_h and G_h, we note that the operator equations (3.4) and (3.5) are satisfied up to error terms $(R_h^{(1)}(x,D_{x'}))(hD_{x_n})$ and $R_h^{(2)}(x,D_{x'})$, respectively, where

$$\tilde{\sigma}(R_h^{(1)}(x,\xi'))\sim\sum_{j\geq 0}h^jR_j^{(1)}(x,\xi')\quad\text{and}\quad\tilde{\sigma}(R_h^{(2)}(x,\xi'))\sim\sum_{j\geq 0}h^jR_j^{(2)}(x,\xi').$$

Moreover, it is easy to see that

$$R_j^{(1)}(x,\xi')=R_j^{(2)}(x,\xi')=0\quad\text{for } |\xi'|\geq 1/2.$$

Let $R_h(x,D_{x'})=(R_h^{(1)}(x,D_{x'}))(hD_{x_n})+R_h^{(2)}(x,D_{x'})$. Then R_h satisfies the requirements in the theorem. □

REMARK 3.6. *By carefully investigating the derivations of B_h and G_h, we can see that*

$$\tilde\sigma(B_h)\sim\sum_{j\geq 0}h^jB_j(x,\xi')\quad \textit{with } B_j \textit{ homogeneous of degree } 1-j \textit{ in } \xi' \textit{ for } |\xi'|\geq 1/2$$

and

$$\tilde\sigma(G_h)\sim\sum_{j\geq 0}h^jG_j(x,\xi')\quad \textit{with } G_j \textit{ homogeneous of degree } -j \textit{ in } \xi' \textit{ for } |\xi'|\geq 1/2.$$

With the decomposition of $\tilde L_h$ in (3.3), we are now ready to construct a function u_h satisfying

$$\begin{cases} Lu_h=0 \quad \text{in} \quad \Omega_0(e_n)\\ u_h|_{x_n=0}=e^{ix'\cdot\theta'/h}\chi(x')b+O(h^\infty).\end{cases}\tag{3.13}$$

Our strategy is as follows. We will first construct the "parametrix" of the forward "heat" operator $hD_n-B_h(x,D_{x'})$, i.e. find $U_h(x,D_{x'})$ such that

(3.14)

$$\begin{cases}(hD_{x_n}-B_h(x,D_{x'}))U_h=O(h^\infty)\quad \text{in } S_{2n-2}(<\xi'>^{-\infty})\quad \text{for } (x',x_n)\in\mathbb{R}^{n-1}\times[0,d)\\ U_h|_{x_n=0}-I=O(h^\infty),\end{cases}$$

where $d>0$ is a fixed constant. The notation $S_{2n-2}(<\xi'>^{-\infty})$ represents a symbol class belonging to $S_{2n-2}(<\xi'>^{-m})$ for all $m>0$. Then a solution u_h can be derived from

$$u_h=U_h(e^{ix'\cdot\theta'/h}\chi(x')b)+r_h,$$

where r_h can be obtained and estimated by the Lax-Milgram theorem.

We will construct U_h following [**T**] (see [**T**, Chapter III]). To begin, we write

$$(U_hw)(x',x_n)=\frac{1}{(2\pi h)^{n-1}}\int e^{i(x'-y')\cdot\xi'/h}U(x',x_n,\xi';h)w(y')dy'd\xi'.$$

Arguing formally, the symbol $U(x',x_n,\xi';h)$ is required to satisfy

$$hD_{x_n}U-\tilde\sigma(B_h)\sharp U\sim 0\quad\text{for}\quad 0\leq x_n<d\tag{3.15}$$

and

$$U(x',0,\xi';h)\sim I.\tag{3.16}$$

Let K' be a compact set in $\mathbb{R}^{n-1}$ and $0<d_0<d$. Then we set $K:=K'\times[0,d_0]$. In view of the spectral property of B_0, we get that there exists a compact set $\widetilde K\subset\mathbb{C}_+$ such that

$$zI-B_0(x',x_n,\xi')\varrho^{-1}\quad\text{is invertible}$$

for all $z\in\mathbb{C}_+\setminus\widetilde K$, $(x',x_n)\in K$, and $\xi'\in\mathbb{R}^{n-1}$, where $\varrho(\xi')=<\xi'>$. Now let Γ be a closed contour in $\mathbb{C}_+$ enclosing $\widetilde K$. We look for

$$U(x',x_n,\xi';h)=\frac{1}{2\pi i}\oint_\Gamma e^{i\varrho x_nz/h}\mathcal{U}(x',x_n,\xi',z;h)\,dz.$$

Therefore, (3.15) is equivalent to

$$\frac{1}{2\pi i}\oint_\Gamma e^{i\varrho x_nz/h}\mathcal{L}\mathcal{U}(x',x_n,\xi',z;h)\,dz=0\quad\text{in the sense of full symbol,}\tag{3.17}$$

where

$$\mathcal{L}\mathcal{U}=hD_{x_n}\mathcal{U}+\varrho z\mathcal{U}-\tilde\sigma(B_h)\sharp\mathcal{U}.$$

To specify $\mathcal{U}$, we will solve the equation

$$\mathcal{L}\mathcal{U} = \varrho I, \tag{3.18}$$

which immediately implies (3.17). We look for $\mathcal{U}(x', x_n, \xi', z; h) = \sum_{j \geq 0} h^j \mathcal{U}_j(x', x_n, \xi', z)$ and $\mathcal{U}_j$ satisfy

$$\mathcal{U}_0 = (zI - B_0\varrho^{-1})^{-1},$$

and

$$\mathcal{U}_j = \mathcal{U}_0\varrho^{-1}\{\sum_{0 \leq k \leq j-1} B_{j-k}\mathcal{U}_k + \sum_{\substack{1 \leq |\gamma| \leq j \\ k+l=j-|\gamma|}} \frac{1}{\gamma!}\partial_{\xi'}^{\gamma} B_k D_{x'}^{\gamma}\mathcal{U}_l - D_{x_n}\mathcal{U}_{j-1}\} \quad \text{for } j \geq 1.$$

As in [**T**], we can see that for $(x', x_n) \in K$ and $\xi' \in \mathbb{R}^{n-1}$

$$\frac{1}{2\pi i}\oint_{\Gamma} \mathcal{U}_0 dz = \frac{1}{2\pi i}\oint_{\Gamma} (zI - B_0)^{-1} dz = I \tag{3.19}$$

and

$$\frac{1}{2\pi i}\oint_{\Gamma} \mathcal{U}_j dz = 0 \quad (j \geq 1). \tag{3.20}$$

Putting $x_n = 0$ in (3.19) and (3.20) we have (3.16).

Having constructed the parametrix U_h, we set $v_h = U_h(e^{i\bullet\cdot\theta'/h}\chi(\bullet)b)$. Then v_h satisfies

$$\widetilde{L}_h v_h = \widetilde{R}_h v_h + R_h v_h =: \mathcal{R} \tag{3.21}$$

and

$$v_h|_{x_n=0} = e^{ix'\cdot\theta'/h}\chi(x')b + O(h^{\infty}),$$

where $\widetilde{R}_h$ is a classical h-pseudodifferential operator with symbol $\widetilde{\sigma}(\widetilde{R}_h) = O(h^{\infty})$ in
$S_{2n-2}(<\xi'>^{-\infty})$. Ideally, it is desirable to solve $\widetilde{L}_h v_h = 0$ up to an $O(h^{\infty})$ error. However, at a first look the right-hand side of (3.21) is $O(h^0)$ since the operator R_h is order zero in h (see Theorem 3.4). Fortunately, combining the structure of R_h and the stationary phase method, we can show that $R_h v_h$ is in fact $O(h^{\infty})$. Indeed, if we write
(3.22)

$$\begin{aligned} R_h v_h(x', x_n) &= (R_h \circ U_h)(e^{i\bullet\cdot\theta'/h}\chi(\bullet)b) \\ &= \tfrac{1}{(2\pi h)^{n-1}}\int e^{i(x'-y')\cdot\xi'/h} V_h(y', x_n, \xi'; h) e^{iy'\cdot\theta'/h}\chi(y')b \; dy' d\xi' \\ &= \tfrac{1}{(2\pi h)^{n-1}}\int e^{i[x'\cdot\xi' + y'\cdot(\theta'-\xi')]/h} V_h(y', x_n, \xi'; h)\chi(y')b \; dy' d\xi', \end{aligned}$$

where the symbol $V_h \sim \sum_{j \geq 0} h^j V_j = \widetilde{\sigma}(R_h^{(1)})\sharp(hD_{x_n}U) + \widetilde{\sigma}(R_h^{(2)})\sharp U$. In view of the structures of $\widetilde{\sigma}(R_h^{(1)})$ and $\widetilde{\sigma}(R_h^{(2)})$, we obtain that $V_j = 0$ for $|\xi'| \geq 1/2$ for all j. Note that the phase function of the oscillatory integral in (3.22) is given by

$$\phi(x', y', \xi', \theta') = x'\cdot\xi' + y'\cdot(\theta' - \xi').$$

So the set of critical points of ϕ is

$$F := \{(y', \xi') : d_{y',\xi'}\phi = 0\} = \{(y', \xi') : y' = x', \ \xi' = \theta'\}.$$

Since $|\theta'| = 1$, it is clear that V_j vanishes in a neighborhood of F uniformly in (x', x_n) for all j. By the stationary phase method, we get that

$$R_h v_h(x', x_n) = O(h^{\infty})$$

uniformly in (x', x_n) and thus

$$\|R_h v_h\|_{L^2(K)} = O(h^\infty), \tag{3.23}$$

where $K = K' \times [0, d_0]$ for $0 < d_0 < d$ and compact set $K' \subset \mathbb{R}^{n-1}$. On the other hand, by the Calderón-Vaillancourt theorem (see [**M**, Theorem 2.8.1]), we get that

$$\|\widetilde{R}_h v_h\|_{L^2(K)} = O(h^\infty). \tag{3.24}$$

Thus, combining (3.23) and (3.24) yields

$$\|\mathcal{R}\|_{L^2(K)} = O(h^\infty). \tag{3.25}$$

Now we look for r_h satisfying

$$\begin{cases} Lr_h = h^{-2}\widetilde{L}_h v_h = h^{-2}\mathcal{R} \quad \text{in} \quad \Omega_0(e_n) \\ r_h = 0 \quad \text{on} \quad \partial\Omega_0(e_n). \end{cases} \tag{3.26}$$

Here we choose d (and d_0) large enough such that $\Omega_0(e_n) \subset K$. Using (3.25) and the Lax-Milgram theorem, we can conclude that (3.26) is uniquely solvable and r_h satisfies

$$\|r_h\|_{H^1(\Omega_0(e_3))} = O(h^\infty). \tag{3.27}$$

Now let $u_h = v_h + r_h$ then $u_h(x) = U_h(e^{i\tau\bullet\cdot\theta'}\chi(\bullet)b) + r_h$ satisfies

$$\begin{cases} Lu_h = 0 \quad \text{in} \quad \Omega_0(e_n) \\ u_h|_{x_n=0} = e^{ix'\cdot\theta'/h} + O(h^\infty). \end{cases}$$

Using the stationary phase method again, we can see that

$$\begin{aligned} u_h(x) = u_h(x', x_n) \quad &\sim U(x', x_n, \theta'; \tau^{-1})\chi(x')b + O(h^\infty) \\ &= \frac{1}{2\pi i}\oint_\Gamma e^{i\tau\varrho(\theta')x_n z}\mathcal{U}(x', x_n, \theta', z; h)\chi(x')b\, dz + O(h^\infty) \quad \text{as} \quad h \to 0. \end{aligned}$$

Since $\Gamma \subset \mathbb{C}_+$, we observe that $U(x', x_3, \theta'; \tau^{-1})\chi(x')b$ decays exponentially in $x_n > 0$. In other words, u_h decays faster than any polynomial power of h for $x_n > 0$.

4. Applications

In this section we will briefly describe how to use oscillating-decaying solutions to solve the inverse problem of identifying an inclusion or a cavity embedded in an inhomogeneous, anisotropic elastic body. For the details, we refer the reader to our recent article [**NUW I**]. The main result is that one can reconstruct the convex hull of an inclusion or a cavity by boundary measurements provided the background information is known and the Runge property is satisfied.

To be precise, let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with smooth boundary. Assume that D is an open subset of Ω with $\bar{D} \subset \Omega$. The domain D represents the inclusion or cavity inside Ω. When D is the inclusion, we suppose that the elasticity tensor $C(x)$ takes the form

$$C(x) = C_0(x) + \chi_D H(x),$$

where $C_0(x) \in B^\infty(\mathbb{R}^3)$ is an anisotropic elasticity tensor and χ_D is the characteristic function of D and $H(x) \in L^\infty(D)$ is a fourth-order tensor. Here we assume that $C_0(x)$ and $C(x)$ satisfy the symmetry property and the strong convexity condition.

Then, for any $f \in H^{1/2}(\partial\Omega)$, there exists a unique weak solution to the boundary value problem

$$\begin{cases} L_C w = \nabla \cdot (C(x)\nabla w) = 0 \quad \text{in} \quad \Omega, \\ w|_{\partial\Omega} = f. \end{cases}$$

We now define the Dirichlet-to-Neumann map $\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by the formula

$$< \Lambda_D(f), g >= \int_\Omega C\nabla w \cdot \nabla v dx,$$

where $g \in H^{1/2}(\partial\Omega)$ and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. Here we assume that both D and H are unknown. We are interested in the inverse problem of determining the inclusion D from the knowledge of $\Lambda_D(f)$ for infinitely many $f's$.

On the other hand, when D is a cavity, we have that for any $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution v to

$$\begin{cases} L_{C_0} v = \nabla \cdot (C_0(x)\nabla u) = 0 \quad \text{in} \quad \Omega \setminus \bar{D}, \\ v|_{\partial\Omega} = f, \quad (C_0\nabla v)\nu|_{\partial D} = 0. \end{cases} \tag{4.1}$$

Associated with (4.1), we can define the Dirichlet-to-Neumann map Λ_D, respectively, by

$$< \Lambda_D(f), g >= \int_{\Omega\setminus\bar{D}} C_0\nabla v \cdot \nabla w \; dx$$

where $w \in H^1(\Omega \setminus \bar{D})$ with $w|_{\partial\Omega} = g$. The inverse problem considered here is to determine D from $\Lambda_D(f)$ for infinitely many $f \in H^{1/2}(\partial\Omega)$.

In both cases, determination of an inclusion or cavity, we will make use of the Runge approximation property. We want to point out that the Runge property for general elasticity systems is still an open problem. It is known that the Runge property is an easy consequence of the *unique continuation property.* The unique continuation property for elliptic systems (including anisotropic elasticity) with some restrictions has been proved recently. We refer the reader to [**NW II**], [**NUW II**], and [**NUW III**]. So, for simplicity, we assume that the background operator L_{C_0} in Ω possesses this property. . The idea of reconstructing the convex hull of D is as follows. Let $\omega \in \mathbb{S}^2$ and t be a parameter. Denote $u_{h,t}(x)$ the oscillating-decaying solution with initial data given on $x \cdot \omega = t$. Suppose that $D \subset \{x \cdot \omega < t\}$. In view of the Runge property, we approximate $u_{h,t}(x)$ in the neighborhood of D by a sequence of $\{v_{h,t,j}(x)\}$ in the H^1 topology, where $v_{h,t,j}$ satisfies $L_{C_0} v_{h,t,j} = 0$ in Ω. Using $v_{h,t,j}$, we define

$$E(h,t) = \lim_{j\to\infty} | < (\Lambda_D - \Lambda_0) v_{h,t,j}|_{\partial\Omega}, \overline{v_{h,t,j}}|_{\partial\Omega} > |,$$

where Λ_0 is the Dirichlet-to-Neumann map for the unperturbed system, i.e.,

$$< \Lambda_0(f), g >= \int_\Omega C_0\nabla u \cdot \nabla v dx,$$

where u is the solution to

$$\begin{cases} L_{C_0} u = \nabla \cdot (C_0\nabla u) = 0 \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = f \end{cases}$$

and $g \in H^{1/2}(\partial\Omega)$ and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. From the formula of $E(h,t)$, we see that it is determined by boundary measurements $\{v_{h,t,j}|_{\partial\Omega}, \Lambda_D(v_{h,t,j}|_{\partial\Omega})\}_{j=1}^{\infty}$.

Now the main idea in determining the convex hull of the inclusion or cavity D is to look at the behaviors of $E(h,t)$ as $h \to 0$ in the cases of $\partial D \cap \{x \cdot \omega = t\} = \emptyset$ and $\partial D \cap \{x \cdot \omega = t\} \neq \emptyset$. Integrating by parts, we can derive certain integral inequalities for $E(h,t)$ containing $u_{h,t}$. Using the exponentially decaying property of $u_{h,t}$, one can see that in the former case, $E = O(h^N)$ for any $N \in \mathbb{N}$. While in the latter case, i.e., the hyperplane $\{x \cdot \omega = t\}$ touches ∂D, we get $E = O(h^d)$ for some finite d provided we impose some a priori assumptions on H and ∂D. Therefore, from the distinct behavior of $E(h,t)$, we can determine when the hyperplane $\{x \cdot \omega = t\}$ touches ∂D. In other words, we can identify the convex hull of D.

Acknowledgments

Gen Nakamura was partially supported by Grant-in-Aid for Scientific Research (B)(2)(No. 14340038) of Japan Society for Promotion of Science. Gunther Uhlmann was partly supported by NSF grant DMS-0245414. Jenn-Nan Wang was supported in part by the National Science Council of Taiwan (NSC 94-2115-M-002-003).

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