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Unique continuation property for elliptic systems and crack determination in anisotropic elasticity

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Abstract. We give a reconstruction procedure to determine cracks embedded in an inhomogeneous anisotropic elastic body by making traction and displacement measurements at the boundary. The procedure depends on the unique continuation property for the system of anisotropic elasticity. We describe some recent results in this direction.

1. Introduction

The aim of this paper is to describe a recent result on the *unique continuation property* (UCP) for second order strongly elliptic systems and its applications to the inverse problem of determining cracks in an elastic material by making traction and displacement measurements at an open subset of the boundary. This information is encoded in the so-called local Neumann-to-Dirichlet map. We use the probe method to determine cracks. This method depends on the Runge approximation property with constraints (see section 2.3) which is a consequence of the UCP.

The UCP is of interest in its own right. The UCP is not valid in general for scalar equations and there are many difficulties in extending the results from scalar equations to systems. We consider in section 2.2 the UCP for second order elliptic systems of two variables with Lipschitz coefficients improving the result in [**NW2**] (see Theorem 2.3).

In section 3 we consider application of the Runge property with constraints to the problem of identifying cracks embedded in an inhomogeneous, anisotropic elastic media. We sketch the proof of the result of the authors [**NUW1**] that proposes a reconstruction procedure to determine cracks by making traction and displacement measurements at the boundary.

2. Unique continuation property

2.1. Preliminaries. Here we would like to describe some known results on the UCP for second order scalar elliptic operators or some special elliptic systems and the difficulties in treating general systems. Let Ω be an open connected set in

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$\mathbb{R}^n$ with $n \geq 2$. We denote by $u(x) = (u_1(x), \cdots, u_N(x))^\top$ a vector-valued function of size $N \geq 1$. Assume that $u(x) \in C^2(\Omega)$ satisfies the differential inequality

$$|Lu(x)| \leq c \sum_{|\nu|\leq 1} |D^\nu u(x)| \quad \forall\, x \in \Omega, \quad \text{for some } c > 0, \tag{2.1}$$

where L denotes the $N \times N$ system

$$(Lu)_\alpha(x) = \sum_{j,\ell=1}^{n} \sum_{\beta=1}^{N} C_{\alpha\beta}^{j\ell}(x)\partial_j\partial_\ell u_\beta \quad \forall\, 1 \leq \alpha \leq N.$$

Throughout the paper we assume that

$$C_{\alpha\beta}^{j\ell}(x) = C_{\beta\alpha}^{\ell j}(x) \quad \forall\, x \in \Omega, \quad \forall\, \alpha, \beta, j, \ell. \tag{2.2}$$

Unless otherwise indicated, here and below, α, β are from 1 to N and all Roman indices are set to be from 1 to n. To insure that (2.1) is a strongly elliptic system, we assume that there exists $\delta > 0$ such that for any vectors $a = (a_1, \cdots, a_N)^\top \in \mathbb{R}^N$ and $b = (b_1, \cdots, b_n)^\top \in \mathbb{R}^n$

$$\sum_{\alpha\beta} \sum_{j\ell} C_{\alpha\beta}^{j\ell}(x) a_\alpha b_j a_\beta b_\ell \geq \delta |a|^2 |b|^2 \quad \forall\, x \in \Omega \qquad \text{(strong ellipticity)}. \tag{2.3}$$

In this paper we are concerned with the UCP of (2.1), i.e., if $u(x)$ is any solution of (2.1) and vanishes in a non-void open subset of Ω, then $u(x)$ vanishes identically in Ω. Another similar property for (2.1) is the *strong unique continuation property* (SUCP) stating that if $u(x)$ vanishes on infinity order at one point $x_0 \in \Omega$, i.e.

$$\int_{|x-x_0|<R} |u|^2 dx = O(R^s) \quad \text{for all } s \text{ as } R \to 0,$$

then $u \equiv 0$ in Ω.

When $N = 1$, (2.1) is a scalar second order elliptic equation. The investigation of UCP or SUCP for scalar second elliptic equations has a long history and there is a vast literature in this field. We will not try to give a full account here of all the progress on this question. See the recent article [**KT**] and the references given there for developments since Carleman's pioneer work. The UCP for differential equations is closely related to the uniqueness of the Cauchy problem. In fact, if the equation is elliptic, they are equivalent. When all the coefficients are analytic and the Cauchy problem is non-characteristic, the uniqueness for the Cauchy problem is guaranteed by Holmgren's theorem, regardless of what type of equation we are considering. Hörmander's uniqueness theorem is another classical result. He assumes that the coefficients of the differential equation can be C^∞ or even finitely smooth [**Ho1**], [**Ho2**] but the initial hypersurface for the Cauchy problem is required to be strongly pseudoconvex. Verifying the strong pseudoconvexity condition for systems is by no means an easy task. It is even difficult for higher order differential operators. For a comprehensive review of early developments of the uniqueness in the Cauchy problem, we refer readers to [**Z**]. Recently, there were several attempts to fill the gap between Holmgren's theorem and Hörmander's theorem, see for example [**Tat1**], [**Tat2**] and [**RZ**].

Another general and powerful method of proving the uniqueness of the Cauchy problem was pioneered by Calderón as an application of singular integral operator theory [**Ca**]. His result and method has been generalized by several authors. We refer to [**Z**] and references therein for more details. To get a better perspective on the problem we describe now Calderón's uniqueness theorem. Let $p(x,D) = \sum_{|\gamma|\le m} a_\gamma(x)D^\gamma$ being an m-th order differential operator with principal symbol $p_m(x,\xi) = \sum_{|\gamma|=m} a_\gamma(x)\xi^\gamma$. Assume that the leading coefficients are C^∞ and those of lower order terms are essentially bounded. Let the hypersurface S be non-characteristic for $p(x,D)$ at $x_0 \in \Omega$, i.e., $p_m(x_0,\mathcal{N}) \ne 0$ where $\mathcal{N}$ is the normal to S at x_0. The main hypotheses of Calderón's type theorem is the following assumptions on the characteristic roots:

(i): there exist m functions $\{\lambda_\ell(x,\xi)\}_{\ell=1}^m$ which are C^∞ in $(x,\xi) \in V \times (\mathbb{R}^n \setminus 0)$ with $\xi \nparallel \mathcal{N}$ such that for every $\xi \nparallel \mathcal{N}$, $p_m(x,\xi+\tau\mathcal{N})$ is written as

$$p_m(x,\xi+\tau\mathcal{N}) = p_m(x,\mathcal{N})\prod_{\ell=1}^m(\tau-\lambda_\ell(x,\xi))$$

in $V_0 \times (\mathbb{R}^n \setminus 0)$, where V_0 is a neighborhood of x_0;

(ii): the real roots are simple and the multiplicity of the complex roots is not more than two.

To insure that the assumptions in Calderón type theorem are essential for uniqueness of the Cauchy problem, we mention some interesting counterexamples. If one violates condition (i), Plis [**Pl**] constructed an example of the form

$$p(x,D)u = [(\partial_t^2+\partial_x^2+\partial_y^2)^2 + t(\partial_x^2+\partial_y^2)^2 - \frac{1}{2}\partial_x^4]u = fu_x + gu$$

with real-valued C^∞ functions f and g such that the uniqueness of the Cauchy problem near $t = 0$ fails. In this example, the characteristic roots have unbounded partial derivatives near the initial surface $t = 0$. A counterexample when condition (ii) is violated was constructed by Cohen [**Co**] of the form

$$p(x,D) = \Delta^3 + \sum_{|\gamma|\le 5} a_\gamma(x)D_x^\gamma$$

with continuous a_γ's for which the uniqueness property also fails. In this example, the multiplicity of the complex root is three. On the other hand, even though there are some extensions of Calderón's result to the case of higher multiplicity of the characteristic roots, the multiplicity is required to be constant and some restrictions on the lower orders are needed in those generalizations.

We now turn to the system (2.1). The characteristic polynomial is given by

$$\det(L(x,\xi)) = \det(\sum_{j,\ell=1}^n C_{\alpha\beta}^{j\ell}(x)\xi_j\xi_\ell),$$

which is homogeneous of degree $2N$. Since (2.1) is elliptic, the characteristic roots appear in complex conjugate pairs. The multiplicity of the characteristic root is at most N and in general is variable. For instance, for the transversely isotropic system (even if the elastic tensor is constant) in three dimensions, the multiplicity of one of the characteristic roots is three in some directions [**Tan**]. Therefore one cannot apply a Calderón type uniqueness theorem directly to (2.1). It should be noted that Calderón's multiplicity condition is satisfied for (2.1) when $N = 2$ but the characteristic roots are required to possess some smoothness in order for the uniqueness to hold.

On the positive side, a very important example of a strongly elliptic system satisfying the conditions (2.2)-(2.3) is anisotropic elasticity which arises in the mechanics of materials and geophysics. In this case $n = N$ and $C^{j\ell}_{\alpha\beta}$ is the elastic tensor. When the medium is isotropic, the UCP has been established in [**AITY**], [**DR**], [**W1**], and [**W2**]. Moreover, the SUCP for the isotropic system was proved in [**AM**]. Unlike the isotropic case, very little is known in the anisotropic case. In this direction, the UCP for an elasticity system with residual stress was shown in [**NW1**]. It is known that this system is no longer isotropic due to the existence of residual stress. The SUCP for the same residual stress system was established in [**Li**]. For the general anisotropic system, under some additional assumptions, similar to (2.17), the UCP was recently proved in two dimensions, under some additional conditions, in [**NW2**] where the elastic tensor is allowed to be Lipschitz. Besides the elasticity system, there are also some results for particular systems, for instance see [**J**] for the Dirac system and [**FL**], [**R**] for the Stokes system.

2.2. UCP for elliptic systems of two variables with Lipschitz coefficients. In this section we describe a UCP result for elliptic systems of two variables with Lipschitz coefficients under some assumptions. This is a generalization of [**NW2**].

For the problem here, we take $n = 2$ and we put (2.1) in a slightly different form

$$|Lu| \le c|u| \quad \text{in} \quad \Omega \tag{2.4}$$

with

$$Lu = \Lambda_{11}\partial_1^2 u + \Lambda_{12}\partial_1\partial_2 u + \Lambda_{22}\partial_2^2 u + Qu, \tag{2.5}$$

where

$$\Lambda_{11} = (C^{11}_{\alpha\beta}),\ \Lambda_{22} = (C^{22}_{\alpha\beta}),\ \Lambda_{12} = \Phi + {}^t\Phi \text{ with } \Phi = (C^{21}_{\alpha\beta})$$

and

$$(Qu)_\alpha = \sum_{\beta,\ell} B^\ell_{\alpha\beta}(x)\partial_\ell u_\beta.$$

We assume that the coefficients $C^{j\ell}_{\alpha\beta}(x)$ are *Lipschitz* and $B^\ell_{\alpha\beta}(x)$ are essentially bounded. Since the UCP is a local property, it is enough to assume the regularity hypotheses locally near each point $x \in \Omega$.

2.2.1. *First order elliptic system.* As in [**NW2**], a key step in proving the UCP for (2.4) is to transform (2.4) into a first order elliptic system. To do this, we introduce an invertible Lipschitz $N \times N$ matrix $E(x)$ and set

$$W = (w_1, w_2)^\top = (u, \partial_1 u + E\partial_2 u)^\top.$$

Thus, we have that

$$\begin{cases} \partial_1 w_1 = \partial_1 u = (\partial_1 u + E\partial_2 u) - E\partial_2 u = -E\partial_2 w_1 + w_2, \\ \partial_1 w_2 = \partial_1^2 u + E\partial_1\partial_2 u + \partial_1 E\partial_2 w_1. \end{cases} \tag{2.6}$$

From (2.5) we get

$$\partial_1^2 u = -\Lambda_{11}^{-1}(\Lambda_{12}\partial_1\partial_2 u + \Lambda_{22}\partial_2^2 u + Qu) + \Lambda_{11}^{-1}Lu. \tag{2.7}$$

It follows from (2.3) that Λ_{11} and Λ_{22} are positive-definite, hence, invertible. Using the definition of w_2, we immediately see that

$$\partial_2 w_2 = \partial_1\partial_2 u + E\partial_2^2 u + \partial_2 E\partial_2 w_1,$$

which implies

$$\partial_1\partial_2 u = -E\partial_2^2 u - \partial_2 E\partial_2 w_1 + \partial_2 w_2. \tag{2.8}$$

Plugging (2.8) into (2.7) yields

$$\partial_1^2 u = -\Lambda_{11}^{-1}\Lambda_{12}(-E\partial_2^2 u - \partial_2 E\partial_2 w_1 + \partial_2 w_2) - \Lambda_{11}^{-1}\Lambda_{22}\partial_2^2 u - \Lambda_{11}^{-1}Qu + \Lambda_{11}^{-1}Lu. \tag{2.9}$$

Now substituting (2.8) and (2.9) into the second equation of (2.6) we have that

$$\begin{aligned}\partial_1 w_2 &= \{-\Lambda_{11}^{-1}\Lambda_{12}(-E\partial_2^2 u - \partial_2 E\partial_2 w_1 + \partial_2 w_2) - \Lambda_{11}^{-1}\Lambda_{22}\partial_2^2 u - \Lambda_{11}^{-1}Qu + \Lambda_{11}^{-1}Lu)\}\\ &\quad + E(-E\partial_2^2 u - \partial_2 E\partial_2 w_1 + \partial_2 w_2) + \partial_1 E\partial_2 w_1\\ &= -(E^2 - \Lambda_{11}^{-1}\Lambda_{12}E + \Lambda_{11}^{-1}\Lambda_{22})\partial_2^2 u + (\Lambda_{11}^{-1}\Lambda_{12}\partial_2 E - E\partial_2 E + \partial_1 E)\partial_2 w_1\\ &\quad + (E - \Lambda_{11}^{-1}\Lambda_{12})\partial_2 w_2 - \Lambda_{11}^{-1}Qu + \Lambda_{11}^{-1}Lu.\end{aligned} \tag{2.10}$$

By the relations $\partial_1 u = \partial_1 w_1$ and $\partial_2 u = -E^{-1}\partial_1 w_1 + E^{-1}w_2$, we can see that Q is a first order linear operator containing only x_1 derivative acting on w_1. More precisely, we have that

$$\begin{aligned}Qu &= \partial_1\Lambda_{11}\partial_1 u + \partial_1{}^t\Phi\partial_2 u + \partial_2\Phi\partial_1 u + \partial_2\Lambda_{22}\partial_2 u\\ &= (\partial_1\Lambda_{11} + \partial_2\Phi)\partial_1 w_1 + (\partial_2\Lambda_{22} + \partial_1{}^t\Phi)(-E^{-1}\partial_1 w_1 + E^{-1}w_2)\\ &= (\partial_1\Lambda_{11} + \partial_2\Phi - E^{-1}\partial_2\Lambda_{22} - E^{-1}\partial_1{}^t\Phi)\partial_1 w_1 + (\partial_2\Lambda_{22} + \partial_1{}^t\Phi)E^{-1}w_2.\end{aligned} \tag{2.11}$$

Replacing Qu in (2.10) by (2.11) and choosing E such that

$$E^2 - \Lambda_{11}^{-1}\Lambda_{12}E + \Lambda_{11}^{-1}\Lambda_{22} = 0$$

or equivalently

$$\Lambda_{11}(-E)^2 + \Lambda_{12}(-E) + \Lambda_{22} = 0, \tag{2.12}$$

we conclude from (2.6) that

$$\begin{cases}\partial_1 w_1 + E\partial_2 w_1 - w_2 = 0\\ \Lambda_{11}^{-1}(\partial_1\Lambda_{11} + \partial_2\Phi - E^{-1}\partial_2\Lambda_{22} - E^{-1}\partial_1{}^t\Phi)\partial_1 w_1 + \partial_1 w_2\\ -(\Lambda_{11}^{-1}\Lambda_{12}\partial_2 E - E\partial_2 E + \partial_1 E)\partial_2 w_1 + (\Lambda_{11}^{-1}\Lambda_{12} - E)\partial_2 w_2\\ +\Lambda_{11}^{-1}(\partial_2\Lambda_{22} + \partial_1{}^t\Phi)E^{-1}w_2 = \Lambda_{11}^{-1}Lu.\end{cases}$$

equivalently, in the matrix notation, we have that

$$A\partial_1 W + B\partial_2 W = FW + (0, \Lambda_{11}^{-1}Lu)^\top, \tag{2.13}$$

where

$$\begin{aligned}A &= \begin{pmatrix} I & 0\\ \Lambda_{11}^{-1}(\partial_1\Lambda_{11} + \partial_2\Phi - E^{-1}\partial_2\Lambda_{22} - E^{-1}\partial_1{}^t\Phi) & I\end{pmatrix},\\ B &= \begin{pmatrix} E & 0\\ -(\Lambda_{11}^{-1}\Lambda_{12}\partial_2 E - E\partial_2 E + \partial_1 E) & \Lambda_{11}^{-1}\Lambda_{12} - E\end{pmatrix},\end{aligned}$$

and

$$F = \begin{pmatrix} 0 & I\\ 0 & -\Lambda_{11}^{-1}(\partial_2\Lambda_{22} + \partial_1{}^t\Phi)E^{-1}\end{pmatrix}.$$

The matrix A is obviously invertible. Then it immediately follows from (2.13) that W satisfies the following first elliptic system

$$\partial_1 W + M_1\partial_2 W = M_0 W + A^{-1}(0, \Lambda_{11}^{-1}Lu)^\top \tag{2.14}$$

with

$$M_1 = A^{-1}B = \begin{pmatrix} E & 0\\ M_{21} & \Lambda_{11}^{-1}\Lambda_{12} - E\end{pmatrix} \quad \text{and} \quad M_0 = A^{-1}F,$$

where

$$M_{21}=-\Lambda_{11}^{-1}(\partial_1\Lambda_{11}+\partial_2\Phi-E^{-1}\partial_2\Lambda_{22}-E^{-1}\partial_1{}^t\Phi)E-(\Lambda_{11}^{-1}\Lambda_{12}\partial_2E-E\partial_2E+\partial_1E).$$

It should be noted that $M_{12}(x)$ belongs to $L^\infty(\Omega)$ and so does M_0. Combining (2.4) and (2.14) yields

$$|\partial_1 W+M_1\partial_2 W|\le c|W|\quad\text{in}\quad\Omega \tag{2.15}$$

for some $c>0$.

In deriving (2.15), we rely on finding a Lipschitz invertible matrix E satisfying (2.12). The identity (2.12) is related to the matrix polynomial

$$P(\lambda)=\Lambda_{11}\lambda^2+\Lambda_{12}\lambda+\Lambda_{22}.$$

Using (2.3), we have that $P(\lambda)$ is positive-definite for all $\lambda\in\mathbb{R}$. We denote $\mathrm{Spec}(P)$ the spectrum of P, i.e., the roots of $\det(P(\lambda))=0$. It is readily seen that $\mathrm{Spec}(P)$ is closed under complex conjugation and all elements of $\mathrm{Spec}(P)$ have non-vanishing imaginary parts. We recall that a matrix E satisfies (2.12) if and only if $\lambda-(-E)$ is a *right spectral divisor* of $P(\lambda)$ (see [**GLR**]). In fact, the matrix polynomial $P(\lambda)$ can be written as

$$P(\lambda)=(\lambda-X^*)\Lambda_{11}(\lambda-X)$$

with

$$X=(\oint_{\gamma_+}\zeta P(\zeta)^{-1}d\zeta)(\oint_{\gamma_+}P(\zeta)^{-1}d\zeta)^{-1},$$

where $\gamma_+\subset\mathbb{C}_+$ is a closed contour enclosing all spectra of P in $\mathbb{C}_+$ and $\mathrm{Spec}(X)\subset\mathbb{C}_+$. Hence, we simply choose $E=-X$. With such choice of E, we now take a look at $\Lambda_{11}^{-1}\Lambda_{12}-E$. By straightforward computations, we have that

$$\begin{aligned}\Lambda_{11}^{-1}P(\lambda)&=\lambda^2+\Lambda_{11}^{-1}\Lambda_{12}\lambda+\Lambda_{11}^{-1}\Lambda_{22}\\&=\lambda^2+\Lambda_{11}^{-1}\Lambda_{12}\lambda+\Lambda_{11}^{-1}\Lambda_{12}E-E^2\\&=(\lambda+\Lambda_{11}^{-1}\Lambda_{12}-E)(\lambda+E).\end{aligned}\tag{2.16}$$

It follows from (2.16) that

$$\mathrm{Spec}(\Lambda_{11}^{-1}\Lambda_{12}-E)=\{-\lambda:\lambda\in\mathrm{Spec}(P)\text{ with }\lambda\in\mathbb{C}_-\}.$$

Thus we obtain that

$$\overline{\mathrm{Spec}(E)}=\mathrm{Spec}(\Lambda_{11}^{-1}\Lambda_{12}-E)$$

and

$$\mathrm{Spec}(P)=\mathrm{Spec}(-E)\cup\mathrm{Spec}(E-\Lambda_{11}^{-1}\Lambda_{12}).$$

For the purpose of proving the UCP for (2.4), we make the following technical assumption:

(2.17)

$\exists$ *Lipschitz invertible matrices* Y_{11},Y_{22}, *and Lipschitz scalar functions* $\lambda_1,\cdots,\lambda_N$ *such that* $Y_{11}EY_{11}^{-1}=\mathrm{diag}(\lambda_1,\cdots,\lambda_N),\ Y_{22}(\Lambda_{11}^{-1}\Lambda_{12}-E)Y_{22}^{-1}=\mathrm{diag}(\overline{\lambda_1},\cdots,\overline{\lambda_N})$.

For the UCP, it is enough to assume that (2.17) holds locally in Ω. The assumption clearly holds if $\mathrm{Spec}(P)$ consists of distinct elements. It is also satisfied in elasticity when the medium is orthotropic with generic conditions (see [**NW2**]). With the assumption (2.17), we now set $V=\mathrm{diag}(Y_{11},Y_{22})W$ and deduce from (2.15) that

$$|\partial_1V+\widetilde{M}\partial_2V|\le c|V|,\tag{2.18}$$

where

$$\widetilde{M} = \begin{pmatrix} Y_{11}EY_{11}^{-1} & 0 \\ M_{21} & Y_{22}(\Lambda_{11}^{-1}\Lambda_{12} - E)Y_{22}^{-1} \end{pmatrix}. \tag{2.19}$$

It should be remarked that M_{21} can not be removed by a similarity transform with a Lipschitz matrix since M_{21} is only essentially bounded. More precisely, we consider a matrix of the form

$$\widetilde{Y} = \begin{pmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{pmatrix}.$$

The inverse of $\widetilde{Y}$ is given by

$$\widetilde{Y}^{-1} = \begin{pmatrix} Y_{11}^{-1} & 0 \\ -Y_{22}^{-1}Y_{21}Y_{11}^{-1} & Y_{22}^{-1} \end{pmatrix}.$$

Thus, we can compute

$$\widetilde{Y}M_1\widetilde{Y}^{-1} = \begin{pmatrix} Y_{11}EY_{11}^{-1} & 0 \\ \widetilde{M}_{21} & Y_{22}(\Lambda_{11}^{-1}\Lambda_{12} - E)Y_{22}^{-1} \end{pmatrix},$$

where

$$\begin{aligned} \widetilde{M}_{21} &= (Y_{21}E + Y_{22}M_{21})Y_{11}^{-1} - Y_{22}(\Lambda_{11}^{-1}\Lambda_{12} - E)Y_{22}^{-1}Y_{21}Y_{11}^{-1} \\ &= [Y_{21}E - Y_{22}(\Lambda_{11}^{-1}\Lambda_{12} - E)Y_{22}^{-1}Y_{21}]Y_{11}^{-1} + Y_{22}M_{21}Y_{11}^{-1} \end{aligned}$$

So if we want $\widetilde{M}_{21} = 0$ then we have

$$Y_{21}E - Y_{22}(\Lambda_{11}^{-1}\Lambda_{12} - E)Y_{22}^{-1}Y_{21} = -Y_{22}M_{21}. \tag{2.20}$$

Since $\mathrm{Spec}(E) \cap \mathrm{Spec}(Y_{22}(\Lambda_{11}^{-1}\Lambda_{12} - E)Y_{22}^{-1}) = \varnothing$, Y_{21} is uniquely determined by (2.20). However, it can not be Lipschitz since $M_{21} \in L^\infty$.

To prove the UCP for (2.4), it suffices to prove that for (2.18). For the system like (2.18), the UCP has been shown in [**HP**] when the matrix $\widetilde{M}$ is normal. Unfortunately, for the $\widetilde{M}$ given by (2.19), this property is not satisfied. Similarly to the arguments in [**W1**], we will derive two Carleman estimates to deal with M_{21} in $\widetilde{M}$.

2.2.2. *Carleman estimates and the proof of UCP.* Let $L_\lambda v = \partial_1 v + \lambda(x)\partial_2 v$ be a first order scalar elliptic operator, where $\lambda(x) = \alpha(x) + i\beta(x)$ is a Lipschitz function and $\beta(x)$ never vanishes in $\mathbb{R}^2$. Given $0 < r_0 < 1$ and define $U = \{v \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}) : \mathrm{supp}(v) \subset B_{r_0}\}$, where B_{r_0} is the disk centered at the origin with radius r_0. The following two Carleman estimates for L_λ were derived in [**NW2**].

LEMMA 2.1. *There exist a constant $c > 0$ and a sufficiently large number $s_0 > 0$ such that for all $v \in U$ and $s \geq s_0$ we have that*

$$s^2 \int r^{-\sigma-s-2}\phi_s^2|v|^2 dx \leq c \int r^{-\sigma}\phi_s^2|L_\lambda v|^2 dx, \tag{2.21}$$

where $r = |x|$, $\phi_s = \exp(r^{-s})$ and $\sigma = \sigma_s = \sigma_0 + \sigma_1 s$ with $\sigma_0, \sigma_1 \in \mathbb{R}$.

LEMMA 2.2. *There exist a constant $c > 0$ and a sufficiently large number $s_0 > 0$ such that for all $v \in U$ and $s \geq s_0$ we have the following estimate*

$$\int r^{-\sigma}\phi_s^2|\partial_2 v|^2 dx \leq c \int r^{-\sigma}\phi_s^2(|L_\lambda v|^2 + s^2 r^{-2s-2}|v|^2)dx. \tag{2.22}$$

Here we will prove the UCP for (2.18) under a weaker assumption. Namely, we would like to show that if a solution V of (2.18) that vanishes at one point in Ω of any exponential order must vanish in Ω provided that the assumption (2.17) holds near every point in Ω. It is clear that the UCP follows from this version of the continuation property. We say that V vanishes at x_0 of any exponential order if

$$\lim_{|x-x_0|\to 0}\exp(|x-x_0|^{-s})V(x)=0\quad\forall\, s>0.$$

THEOREM 2.3. *Assume that the coefficients $C^{j\ell}_{\alpha\beta}$ are Lipschitz and (2.17) is satisfied near every point of Ω. Let V vanish at some $x_0\in\Omega$ of any exponential order, then $V\equiv 0$ in Ω.*

Proof. Without loss of generality, we assume $x_0=0$. Assume that (2.17) holds in $\omega_0\subset\Omega$, where ω_0 is a small neighborhood of 0. Define $r_0=\min\{1/2,\mathrm{dist}(0,\partial\omega_0)\}$. Now let $\chi\in C_0^\infty(\mathbb{R}^2)$ be a cut-off function satisfying $0\le\chi\le 1$, $\chi|_{B_{r_0/2}}=1$ and $\mathrm{supp}\,(\chi)\subset B_{r_0}$. Define $\widetilde V=\chi V=(\widetilde V_1,\widetilde V_2)^\top$, where $\widetilde V_1=(\chi v_1,\cdots,\chi v_N)$ and $\widetilde V_2=(\chi v_{N+1},\cdots,\chi v_{2N})$. Then from (2.18), we get that

$$|\partial_1\widetilde V+\widetilde M\partial_2\widetilde V|\le c|\widetilde V|+|G| \tag{2.23}$$

where $G=(G_1,G_2)^\top\in L^2(\Omega)$ is supported in $B_{r_0}\setminus B_{r_0/2}$. Denote

$$L_1\widetilde V_1=\mathrm{diag}(L_{\lambda_1},\cdots,L_{\lambda_N})\widetilde V_1$$

and

$$L_2\widetilde V_2=\mathrm{diag}(L_{\lambda_{N+1}},\cdots,L_{\lambda_{2N}})\widetilde V_2.$$

We set

$$\begin{aligned}H&=\gamma\textstyle\int r^{-s}\phi_s^2|L_1\widetilde V_1|^2dx+\int\phi_s^2|L_2\widetilde V_2|^2dx,\\ K&=\gamma\textstyle\int r^{-s}\phi_s^2|\widetilde V_1|^2dx+\int\phi_s^2|\widetilde V_2|^2dx,\\ J&=\gamma\textstyle\int r^{-s}\phi_s^2|G_1|^2dx+\int\phi_s^2|G_2|^2dx,\end{aligned}$$

where $\gamma>0$ will be chosen later. It follows from (2.23) that

$$H\le c(K+J+\int\phi_s^2|\partial_2\widetilde V_1|^2dx). \tag{2.24}$$

By standard approximation arguments, we can see that $\widetilde V_1$ and $\widetilde V_2$ satisfy the estimates (2.21) and (2.22). Now replacing the last term of (2.24) by the estimate (2.22) for L_1 with $\sigma=0$, we have that

$$H\le c(K+J+\int\phi_s^2|L_1\widetilde V_1|^2dx+s^2\int r^{-2s-2}\phi_s^2|\widetilde V_1|^2dx). \tag{2.25}$$

Substituting the estimate (2.21) for L_1 with $\sigma=s$ into the last term of (2.25) leads to

$$H\le c(K+J+\int r^{-s}\phi_s^2|L_1\widetilde V_1|^2dx). \tag{2.26}$$

In deriving (2.26), we also use the obvious inequality $r^{-s}>1$ for any $0<r<1$ and $s>0$. Taking $\gamma\gg 1$, we can absorb the last term of (2.26) and get

$$H\le c(K+J) \tag{2.27}$$

From now on we fix the parameter γ.

Next using $\sigma = s$ in (2.21) for L_1 and $\sigma = 0$ in (2.21) for L_2, we find that

$$\begin{aligned} & s^2 \int r^{-2s-2}\phi_s^2 |\widetilde{V}_1|^2 dx + s^2 \int r^{-s-2}\phi_s^2 |\widetilde{V}_2|^2 dx \\ & \le c\big(\int r^{-s}\phi_s^2 |L_1 \widetilde{V}_1|^2 dx + \int \phi_s^2 |L_2 \widetilde{V}_2|^2 dx \big) \\ & \le cH \le c(K + J). \end{aligned} \tag{2.28}$$

In view of the inequalities $1 < r^{-s} < r^{-s-2} < r^{-2s-2}$ for $0 < r < 1$, by taking $s \gg 1$, we get from (2.28) that

$$\begin{aligned} & s^2 \int r^{-s-2}\phi_s^2 |\widetilde{V}|^2 dx \\ & \le s^2 \int r^{-2s-2}\phi_s^2 |\widetilde{V}_1|^2 dx + s^2 \int r^{-s-2}\phi_s^2 |\widetilde{V}_2|^2 dx \\ & \le c\big(\int r^{-s}\phi_s^2 |\widetilde{V}_1|^2 dx + \int \phi_s^2 |\widetilde{V}_2|^2 dx \big) \\ & \quad + c\big(\int r^{-s}\phi_s^2 |G_1|^2 dx + \int \phi_s^2 |G_2|^2 dx \big) \\ & \le c \int r^{-s-2}\phi_s^2 |\widetilde{V}|^2 dx + c \int r^{-s-2}\phi_s^2 |G|^2 dx \\ & \le \tilde{c} \int r^{-s-2}\phi_s^2 |G|^2 dx, \end{aligned} \tag{2.29}$$

which leads to

$$s^2 \int_{B_{r_0/2}} r^{-s-2}\phi_s^2 |\widetilde{V}|^2 dx \le c \int_{B_{r_0} \setminus B_{r_0/2}} r^{-s-2}\phi_s^2 |G|^2 dx. \tag{2.30}$$

Here the first term before the last inequality of (2.29) has been absorbed into the left hand side of (2.30). Since $r^{-s-2}\phi_s^2$ is a strictly decreasing function, (2.30) implies that

$$s^2 \int_{B_{r_0/2}} \widetilde{V} dx \le c \int_{B_{r_0} \setminus B_{r_0/2}} |G|^2 dx < \infty$$

and thus $\widetilde{V} = V = 0$ on $B_{r_0/2}$ if we choose s sufficiently large. Now standard arguments imply that the set $\{x \in \Omega : V = 0\}$ is open and closed in Ω. Since Ω is connected, we must have $V \equiv 0$ in Ω. □

2.3. Runge approximation property. As an application of the UCP established previously, we state a version of Runge's approximation property with constraints on Dirichlet boundary values. We define the elliptic operator $\mathcal{L}$ by

$$(\mathcal{L}u)_\alpha(x) = \sum_{j,\ell=1}^{2} \sum_{\beta=1}^{N} \partial_j (C_{\alpha\beta}^{j\ell}(x) \partial_\ell u_\beta) \quad \forall\ 1 \le \alpha \le N,$$

where $C_{\alpha\beta}^{j,\ell}(x)$ are Lipschitz and satisfy (2.2), (2.3) such that (2.17) holds. Let U be a bounded open subset in $\mathbb{R}^2$ with C^1 boundary ∂U. Denote Γ_0 a non-empty open subset of ∂U.

THEOREM 2.4. (Runge approximation property with constraint) *Let O an open subset of U such that $\overline{O} \subset U$ and $U \setminus \overline{O}$ is connected. Assume that $-\omega$ is not a Dirichlet eigenvalue of $\mathcal{L}$ in U. Let $u \in H^1$ satisfy*

$$(\mathcal{L} + \omega)u = 0$$

in an open neighborhood of $\overline{O}$. Then for any $\epsilon > 0$ there exists $v \in H^1(U)$ satisfying

$$(\mathcal{L} + \omega)v = 0 \quad \text{in} \quad U, \quad \operatorname{supp}(v|_{\partial U}) \subset \Gamma_0$$

such that

$$\|v - u\|_{H^1(\overline{O})} < \epsilon.$$

Using the UCP for the Schrödinger operator, a similar theorem was proved for this operator in [**Ik2**]. The same proof works for the elliptic operator $\mathcal{L}$ provided we have the UCP. We omit the proof of Theorem 2.4 here. If $\Gamma_0 = \partial U$, Theorem 2.4 is the classical Runge approximation property [**La**]. The first application of the Runge approximation property to inverse problems was introduced by Kohn and Vogelius [**KV**].

3. Identification of cracks in anisotropic elasticity

3.1. Description of the problem. We review a recent result by the authors on the determination of cracks by boundary measurements in an anisotropic elastic medium [**NUW1**]. We assume in this section that the Runge approximation property with constraints for the stationary elasticity system in $\mathbb{R}^n$ with $n = 2, 3$ is valid, i.e. considering $\omega = 0$ in Theorem 2.4. Of course, we do not need this assumption for the two-dimensional elasticity system with (2.17).

Let $\mathcal{B}$ be an anisotropic elastic body and the reference configuration of $\mathcal{B}$ be Ω, a bounded connected domain in $\mathbb{R}^n$, $n = 2, 3$, with C^1 boundary Γ. Denote $C(x) = (C^{j\ell}_{\alpha\beta}(x)) \in C^1(\overline{\Omega})$ the elastic tensor. With the notation used before, we set $N = n(= 2, 3)$. Assume that the elastic tensor C satisfies the full symmetry properties

$$C^{j\ell}_{\alpha\beta} = C^{\alpha\ell}_{j\beta} = C^{\ell j}_{\beta\alpha} \quad \forall\ \alpha, \beta, j, \ell \tag{3.1}$$

and the strong convexity condition, i.e. there exists a constant $\delta > 0$ such that

$$C(x)E \cdot E \geq \delta |E|^2 \tag{3.2}$$

for any symmetric matrix E and all $x \in \Omega$. Here we have used the conventions

$$(CG)_{\alpha j} = \sum_{\beta\ell} C^{j\ell}_{\alpha\beta} g_{\beta\ell} \quad \text{and} \quad G \cdot H = \sum_{\alpha j} g_{\alpha j} h_{\alpha j},$$

where $G = (g_{\alpha j})$ and $H = (h_{\alpha j})$ are real matrices. It is clear that the strong convexity condition (3.2) implies the strong ellipticity condition (2.3). Let $u = (u_1, \cdots, u_n)^\top$ be the displacement vector, then the equation of equilibrium, when there are no exterior forces acting on the domain, is given by

$$\mathcal{L}_C u = \nabla \cdot \sigma_C(u) = 0 \quad \text{in} \quad \Omega, \tag{3.3}$$

where $(\nabla \cdot G)_\alpha = \sum_j \partial_j g_{\alpha j}$ for any matrix function $G = (g_{\alpha j})$ and $\sigma_C(u) = C\nabla u$. It is evident that $\sigma_C(u) = C\varepsilon(u)$ with $\varepsilon(u) = \mathrm{Sym}\nabla u = \frac{1}{2}({}^t\nabla u + \nabla u)$ if C satisfies (3.1).

Throughout the paper we use the following notations and assumptions. Let X be an open submanifold of a manifold Y. If F is a space of distributions in Y, we set

$$\bar{F}(X) := \{f|_X : f \in F\}, \quad \dot{F}(\bar{X}) := \{f \in F : \mathrm{supp}\,(f) \subset \bar{X}\},$$

where $f|_X$ is the restriction of f to X. These notations will be used for some Sobolev spaces defined in X and when X, Y and the boundary ∂X have sufficient regularities. Assume that $S \subset \Omega$ is a C^2 closed Jordan curve ($n = 2$) or closed connected surface ($n = 3$) and $\Sigma \subset S$ is an open curve or surface. When $n = 3$ we suppose that the boundary $\partial\Sigma$ of Σ is C^2. Here Σ will be considered as a crack. We can have several number of cracks. For this case our theory also works without any essential change. Let Ω_- be the open subset of Ω with boundary S and $\Omega_+ := \Omega \setminus \bar{\Omega}_-$. The trace operator to Γ is denoted by γ and those from $\Omega_\pm$ to

S is denoted by $\gamma_\pm$. The directions of the unit normal ν to Γ and S are directed into $\mathbb{R}^n \setminus \bar{\Omega}$ and Ω_+, respectively.

In our problem, we take $X = \Sigma$, $k \in \mathbb{R}$ (with $|k| \leq 1$) and define the Sobolev spaces $\bar{H}^k(\Sigma)$ and $\dot{H}^k(\bar{\Sigma})$, which are subspaces of $H^k(S)$. Also, we denote $\bar{H}^k(\Sigma)^*$ the dual space of $\bar{H}^k(\Sigma)$. For $1/2 < s$, we define $H^s(\Omega \setminus \Sigma)$ by

$$H^s(\Omega\setminus\Sigma) := \{u \in \mathfrak{D}'(\Omega) : u_\pm := u|_{\Omega_\pm} \in \bar{H}^s(\Omega_\pm); [u] := \gamma_+ u_+ - \gamma_- u_- = 0 \text{ on } S\setminus\bar{\Sigma}\}$$

with the norm $\|u\|_{H^s(\Omega\setminus\Sigma)} := \|u_+\|_{\bar{H}^s(\Omega_+)} + \|u_-\|_{\bar{H}^s(\Omega_-)}$.

To study different types of boundary measurements, we divide Γ into two parts:

$$\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N, \quad \Gamma_D \cap \Gamma_N = \varnothing$$

where $\Gamma_D, \Gamma_N \subset \Gamma$ are open subsets with C^1 boundaries $\partial\Gamma_D$, $\partial\Gamma_N$. Either Γ_D and Γ_N will be taken as the place where we perform the boundary measurements.

From the practical point of view, we consider a mixed-type boundary value problem. Assume $\Gamma_D \neq \varnothing$ and $\Gamma_N \neq \varnothing$. For any given $f \in \bar{H}^{\frac{1}{2}}(\Gamma_D)$ and $g \in \bar{H}^{-\frac{1}{2}}(\Gamma_N)$, find a solution $u \in H^1(\Omega \setminus \Sigma)$ to

$$\begin{cases} \mathcal{L}_C u = 0 \quad \text{in} \quad \Omega \setminus \bar{\Sigma} \\ \sigma_C(u)\nu = 0 \quad \text{on} \quad \Sigma \\ u = f \quad \text{on} \quad \Gamma_D, \quad \sigma_C(u)\nu = g \quad \text{on} \quad \Gamma_N. \end{cases} \tag{3.4}$$

It is not hard to show that the boundary value problem (3.4) is well-posed [**NUW1**]. We define the local Neumann-to-Dirichlet map $\Lambda_\Sigma : \bar{H}^{-\frac{1}{2}}(\Gamma_N) \to \bar{H}^{\frac{1}{2}}(\Gamma_N)$ by

$$\Lambda_\Sigma g = u|_{\Gamma_N},$$

where $u \in H^1(\Omega \setminus \Sigma)$ is the solution to (3.4) with $g \in \bar{H}^{-\frac{1}{2}}(\Gamma_N)$. Now the inverse problem is to reconstruct Σ by Λ_Σ. We will present a reconstruction algorithm along the lines of the probe method [**Ik1**]. This method has similarities with the point source method [**Po**].

In the paper we only consider a single crack. Nevertheless, the same method works without any change for multiple cracks. A similar problem is considered in [**IN**] of determination of crack embedded in an inhomogeneous isotropic conductive medium. In addition to the recent paper [**IN**], there are several related results on crack determination in different contexts. We mention Bryan and Vogelius [**BV**], Kress [**Kr**], Ben Abda et al [**ABB**], [**BBJ**], [**BB**], Brühl et al [**BHP**]. Ben Abda et al assumed the non-vanishing of the stress intensity factor for a surface breaking crack in a two dimensional medium and the non-vanishing of the displacement gap across a two dimensional crack in a plane and used the reciprocity gap principle to reconstruct the crack. Kang et al [**KLN**] analyzed a condition for the Cauchy data which guarantees the non-vanishing of the stress intensity factor and extracted more information about the crack. Brühl et al used Kirsch's linear sampling method [**KR**]. Cakoni and Colton also used the linear sampling method for determining cracks [**CC**]. Others' results reduce the problems to some optimization problems and use a Newton type algorithm to solve these.

3.2. Outline of the reconstruction method. To begin, let $r := \{r(t) \in \bar{\Omega}; 0 \leq t \leq 1\}$ be a non-selfintersecting continuous curve joining $r(0), r(1) \in \Gamma$ with $r(t) \in \Omega$ for $0 < t < 1$. This curve r is called a needle. Define

$$T(r, \Sigma) := \sup\{t : 0 < t < 1, r(s) \notin \bar{\Sigma} \text{ for } 0 < s < t\}.$$

Physically, $T(r,\Sigma)$ can be interpreted as the first hitting time of the needle r to Σ. It is clear that if $T(r,\Sigma)=1$ then the needle r does not touch the crack Σ. For any given needle r, we would like to find a characterization of $T(r,\Sigma)$. To do this, as indicated in the introduction, we define the indicator function $I(t,r)$ by

$$I(t,r) := \lim_{j\to\infty} < g_j, \overline{(\Lambda_\Sigma - \Lambda_\varnothing)g_j} >, \tag{3.5}$$

where here $<\cdot,\cdot>$ is the pairing between $\bar{H}^{-\frac{1}{2}}(\Gamma_N)$ and $\dot{H}^{\frac{1}{2}}(\bar{\Gamma}_N)$. The Neumann data g_j requires further explanations. Let $v', v_j'' \in H^1(\Omega)$ $(j\in\mathbb{N})$ be defined as follows. v' is the solution to

$$\begin{cases} \mathcal{L}_C v' = 0 \quad \text{in} \quad \Omega \\ v' = f \quad \text{on} \quad \Gamma_D,\ \sigma_C(v')\nu = 0 \quad \text{on} \quad \Gamma_N \end{cases} \tag{3.6}$$

and v_j'' satisfy

$$\begin{cases} \mathcal{L}_C v_j'' = 0 \quad \text{in} \quad \Omega \\ \text{supp}\,(v_j'') \subset \Gamma_0 \\ v_j'' \to G(\cdot, r(t))\ (j\to\infty) \quad \text{in} \quad H^1_{loc}(\Omega\setminus r_t), \end{cases} \tag{3.7}$$

where Γ_0 is a fixed open subset of Γ_N and

$$r_t := \{r(s) : 0 < s \le t\}.$$

Here the distribution $G(\cdot, x^0)$ in $x^0\in\Omega$ satisfies

$$\mathcal{L}_C G(\cdot, x^0) + \delta(x-x^0)b = 0$$

and

$$(G(\cdot,x^0) - E(\cdot,x^0)b)_{x^0\in\Omega} \quad \text{is bounded in} \quad H^1(\Omega), \tag{3.8}$$

where $0\neq b\in\mathbb{C}$ and the distribution $E(x,x^0)$ in $x^0\in\mathbb{R}^n$ satisfies

$$\mathcal{L}_{C(x^0)} E(x,x^0) + \delta(x-x^0)I_n = 0. \tag{3.9}$$

Note that $C(x^0)$ is a homogeneous elastic tensor with $C(x)=C(x^0)$ for all $x\in\Omega$ and I_n the identity matrix. The existence of $G(x,x^0)$ can be easily verified, while the existence of v_j'' is guaranteed by the Runge approximation property with constraints (see Theorem 2.4). Now we define

$$v_j = v' + v_j''$$

and

$$g_j = \sigma_C(v_j)\nu|_{\Gamma_N}.$$

To relate the indicator function $I(t,r)$ to $T(r,\Sigma)$, we define the quantity

$$t(r,\Sigma) := \sup\{0<t<1 : \sup_{0<s<t} |I(s,r)| < \infty\}.$$

Our aim now is to show that

$$t(r,\Sigma) = T(r,\Sigma) \quad \text{if} \quad r\cap\Sigma\neq\varnothing, \tag{3.10}$$

namely, the indicator function $I(t,r)$ will become unbounded once the tip of the needle touches the crack. Proving (3.10) requires some delicate analysis. It is the main technical part of the proof.

First of all, we would like to rewrite the indicator function $I(t,r)$ which involves the so-called *reflected solution* defined as follows. Let $u_j \in H^1(\Omega\backslash\Sigma)$ be the solution of

$$\begin{cases} \mathcal{L}_C u_j = 0 \quad \text{in} \quad \Omega \setminus \bar{\Sigma} \\ \sigma_C(u_j)\nu = 0 \quad \text{on} \quad \Sigma \\ u_j = f \quad \text{on} \quad \Gamma_D, \ \sigma_C(u_j)\nu = g_j \quad \text{on} \quad \Gamma_N \end{cases}$$

and $w_j = u_j - v_j \in H^1(\Omega \setminus \Sigma)$, then we can show that

LEMMA 3.1. *(reflected solution) If $r_t \cap \bar{\Sigma} = \varnothing$, then $w_j \to w'$ in $H^1(\Omega \setminus \Sigma)$ and $w' \in H^1(\Omega \setminus \Sigma)$ satisfies*

$$\begin{cases} \mathcal{L}_C w' = 0 \quad in \quad \Omega \setminus \bar{\Sigma} \\ \sigma_C(w')\nu = -\sigma_C(v' + G(\cdot, r(t)))\nu \quad on \quad \Sigma \\ w' = 0 \quad on \quad \Gamma_D, \ \sigma_C(w')\nu = 0 \quad on \quad \Gamma_N. \end{cases} \tag{3.11}$$

With the reflected solution w, we can give another form of the indicator $I(t,r)$.

LEMMA 3.2. *Assume $r_t \cap \bar{\Sigma} = \varnothing$. Then we have*

$$I(t,r) = \int_{\Omega\setminus\Sigma} \sigma_C(w') \cdot \varepsilon(\overline{w'})dx + \int_{\Gamma_D} f\sigma_C(\overline{w'})\nu ds. \tag{3.12}$$

See [**NUW1**] for the proofs of Lemma 3.1 and 3.2. By virtue of Lemma 3.2, we can show a distinct feature of $I(t,r)$ when $r \cap \Sigma \neq \varnothing$.

THEOREM 3.3. *If $r(T(r,\Sigma)) \in \Sigma$, then $|I(t,r)| \to \infty$ as $t \uparrow T(r,\Sigma)$.*

By Theorem 3.3, we can prove (3.10) in the same way as [**INT**]. The proof of Theorem 3.3 relies on the analysis of behavior of the reflected solution $w' = w'(x, r(t))$. We will sketch its proof in the next section. The reconstruction algorithm of our method is summarized as follows.

Reconstruction Algorithm.

Step 1. Given a needle $r = \{r(t) : 0 \leq t \leq 1\}$ we consider the domain $\Omega \setminus r_t$.

Step 2. Solve (3.6) for v' and find a sequence of functions v_j'' satisfying (3.7).

Step 3. Compute $g_j = \sigma_C(v' + v_j'')\nu|_{\Gamma_N}$ and evaluate the indicator function $I(t,r) := \lim_j < g_j, \overline{(\Lambda_\Sigma - \Lambda_\varnothing)g_j} >$.

Step 4. Increase t and search for t where $|I(r,t)|$ becomes very large. Denote this t by $t_a(r,\Sigma)$.

Step 5. Choose many needles r and repeat all previous steps. Draw a surface Σ_a which is close enough to the points $t_a(r,\Sigma)$ for these r. Σ_a is gives an approximation of Σ.

3.3. Proof of Theorem 3.3. We now sketch the proof of Theorem 3.3. The main step in proving Theorem 3.3 is to analyze the behavior of the reflected solution $w'(x, r(t))$ near $r(t)$ which is sufficiently close to the crack. It turns out the behavior of $|I(t,r)|$ as $t \uparrow T(r,\Sigma)$ is determined by the local property of $w'(x,r(t))$ near $r(t)$. So the underlying idea is to find a replacement of w' which "captures" the singularity of $w'(x,r(t))$ and solves the homogeneous system. To proceed with the analysis, let $x^0 = r(t) \in \Omega \setminus \bar{\Sigma}$ and $a = x(T(r,\Sigma))$. Assume that x^0 is sufficiently close to a, denoted by $x^0 \sim a$. Let $y = (y_1, \cdots, y_n) = (y_1(x, x^0), \cdots, y_n(x, x^0))$ be the boundary normal coordinates near a such that

$$y(a) = 0, \quad \frac{\partial y(x, x^0)}{\partial x}|_{x=x^0} = I_n, \quad \text{and} \quad \Omega_- = \{y_1 < 0\} \quad \text{near} \quad a.$$

Let $J(x) = \frac{\partial y(x,x^0)}{\partial x} = (q_{ij}(x))$ and $x = x(y(x, x^0))$. Denote

$$\tilde{C}^{\tilde{j}\tilde{\ell}}_{\alpha\beta} = |J(x)|^{-1} C^{j\ell}_{\alpha\beta} q_{\tilde{j}j} q_{\ell\tilde{\ell}}, \quad \tilde{C} = (\tilde{C}^{\tilde{j}\tilde{\ell}}_{\alpha\beta}) \quad \text{for } 1 \le \alpha, \tilde{j}, \beta, \tilde{\ell} \le n,$$

and

$$\tilde{u}(y) = u(x(y, x^0)), \quad y^0 = y(x^0, x^0) \quad \text{with} \quad y^0_1 > 0.$$

Then we can see that

(i): $\tilde{C}(y) \in C^1$ near $y = 0$;
(ii): $|J|^{-1}\mathcal{L}_C u = \mathcal{L}_{\tilde{C}}\tilde{u}$ near $y = 0$;
(iii): $\nu|_\Sigma = e_1 = (1, 0, \cdots, 0)$ near $y = 0$;
(iv): $\delta(x(y, x^0) - x^0) = \delta(y - y^0)$.

In view of this choice of the coordinates $\{y_1, \cdots, y_n\}$, we have that

$$\tilde{C}^{\tilde{j}\tilde{\ell}}_{\alpha\beta}(y^0) = C^{\tilde{j}\tilde{\ell}}_{\alpha\beta}(x^0) \quad \text{and} \quad \tilde{C}^{\tilde{j}\tilde{\ell}}_{\alpha\beta}(y) = \tilde{C}^{\tilde{\ell}\tilde{j}}_{\beta\alpha}(y) \quad \forall\, \alpha, \beta, \tilde{j}, \tilde{\ell}. \tag{3.13}$$

In what follows, to simplify the notation, the homogeneous tensor $\tilde{C}(y^0)$ is denoted by $\tilde{C}$. Also, we denote $y' = (y_2, \cdots, y_n)$ and $y^{0\prime} = (y^0_2, \cdots, y^0_n)$. Assume that $E_\pm(y, y^0) \in C^\infty(\{\pm(y_1 - y^0_1) \ge 0\}, \mathfrak{D}'(\mathbb{R}^{n-1}_{y'}))$ satisfies

$$\begin{cases} \mathcal{L}_{\tilde{C}} E_\pm = 0 \quad \text{in} \quad \pm (y_1 - y^0_1) > 0 \\ E_+|_{y_1 = y^0_1 + 0} - E_-|_{y_1 = y^0_1 - 0} = 0 \\ \sigma_{\tilde{C}}(E_+)e_1|_{y_1 = y^0_1 + 0} - \sigma_{\tilde{C}}(E_-)e_1|_{y_1 = y^0_1 - 0} = -\delta(y' - y^{0\prime})I_n. \end{cases} \tag{3.14}$$

Let E be defined by

$$E(y, y^0) = \begin{cases} E_+(y, y^0) \quad \text{in} \quad y_1 - y^0_1 \ge 0 \\ E_-(y, y^0) \quad \text{in} \quad y_1 - y^0_1 \le 0, \end{cases}$$

then we can see that

$$\mathcal{L}_{\tilde{C}} E + \delta(y - y^0)I_n = 0 \quad \text{in} \quad \mathbb{R}^n.$$

In fact, $E_\pm$ of (3.14) can be expressed as oscillatory integrals. To do this, we first introduce several notations:

$$Q(\xi') = (\textstyle\sum_{j,\ell=2}^n \tilde{C}^{j\ell}_{\alpha\beta}\xi_j\xi_\ell;\ {}^{\alpha\downarrow 1,\cdots,n}_{\beta\to 1,\cdots,n}), \quad \xi' = (\xi_2, \cdots, \xi_n),$$
$$R(\xi') = (\textstyle\sum_{j=2}^n \tilde{C}^{j1}_{\alpha\beta}\xi_j;\ {}^{\alpha\downarrow 1,\cdots,n}_{\beta\to 1,\cdots,n}), \quad A(\xi') = R(\xi') + R^\top(\xi'),$$
$$T = (\tilde{C}^{11}_{\alpha\beta};\ {}^{\alpha\downarrow 1,\cdots,n}_{\beta\to 1,\cdots,n}).$$

It is clear that

$$\mathcal{L}_{\tilde{C}} = T\partial_1^2 + A(\partial')\partial_1 + Q(\partial').$$

Now let $M(\xi) := T\xi_1^2 + A(\xi')\xi_1 + Q(\xi')$, i.e. the symbol of $\mathcal{L}_{\tilde{C}}$. As before (see [**GLR**]), M admits the factorization

$$M(\xi) = (\xi_1 - B_\pm(\xi')^*)T(\xi_1 - B_\pm(\xi')),$$

where

$$B_\pm(\xi') = \Big(\oint_{\gamma_\pm} \zeta M(\zeta, \xi')^{-1} d\zeta\Big)\Big(\oint_{\gamma_\pm} M(\zeta, \xi')^{-1} d\zeta\Big)^{-1}$$

$\gamma_+ \subset \mathbb{C}_+$ (or $\gamma_- \subset \mathbb{C}_-$) is a C^1 Jordan closed curves enclosing all roots of $\det M = 0$ in ξ_1 with positive (or negative) imaginary parts. As defined in [**BL**],

$$Z_\pm(\xi') := \mp i(TB_\pm(\xi') + R^\top(\xi'))$$

is called the surface impedance tensor for the half space $\mathbb{R}^n_\pm$. Also, it is known that $Z_\pm(\xi')$ are positive Hermitian matrices. Thus, $\mathrm{Re}Z_+$ is invertible. We can show that $E_\pm(y, y^0)$ are written as

(3.15)

$$E_\pm(y,y^0) = \mathrm{Os} - \int e^{i(y'-y^{0\prime})\cdot\xi'} e^{i(y_1-y_1^0)B_\pm(\xi')} (\frac{1}{2})(\mathrm{Re}Z_+)^{-1}\, đ\xi' \quad (\pm(y_1-y_1^0)>0),$$

where $đ\xi' = (2\pi)^{-(n-1)}d\xi'$ (see [**NUW1**, Lemma 4.3]).

For a constant vector $b \in \mathbb{R}^n \setminus \{0\}$, we define $\widetilde{w}^0_\pm$ satisfying

$$\begin{cases} \mathcal{L}_{\tilde{C}}\widetilde{w}^0_\pm = 0 \quad \text{in} \quad \mathbb{R}^n_\pm \\ \sigma_{\tilde{C}}(\widetilde{w}^0_\pm)e_1 = -\sigma_{\tilde{C}}(E_-(y,y^0)b)e_1 \quad \text{on} \quad \{y_1 = 0\}. \end{cases}$$

Likewise, $\widetilde{w}^0_\pm$ can be written as oscillatory integrals

(3.16)

$$\widetilde{w}^0_\pm(y,y^0) = \pm\mathrm{Os}-\int e^{i(y'-y^{0\prime})\cdot\eta'} e^{iy_1B_\pm(\eta')} e^{-iy_1^0B_-(\eta')}(\frac{1}{2})Z_\pm(\eta')^{-1}Z_-(\eta')(\mathrm{Re}Z_+(\eta'))^{-1}b\,đ\eta'.$$

After tedious computations, we can show that proving $|I(t,r)| \to \infty$ as $x^0 \to a$ is equivalent to proving that

$$\int_{\mathbb{R}^n_\pm} \chi(x(y,x^0))|\sigma_{\tilde{C}}(\widetilde{w}^0_\pm)|^2 dy \to \infty \quad \text{as} \quad y_1^0 \to 0, \tag{3.17}$$

where $0 \le \chi \in C_0^\infty(\Omega)$ satisfies $\chi = 1$ in a sufficiently small neighborhood of a and be supported near a such that $\mathrm{supp}\,(\chi) \cap S \subset \Sigma$. In view of (3.16), we can prove (3.17) with the help of the Plancherel formula. To get (3.17), it is enough to prove

Lemma 3.4.

$$\int_{\mathbb{R}^n_\pm \cap V_\delta} |\sigma_{\tilde{C}}(\widetilde{w}^0_\pm)|^2 dy \to \infty \quad \textit{as} \quad y_1^0 \to 0,$$

where $V_\delta := \{y \in \mathbb{R}^n : |y_1| \le \delta\}$ *such that* $\mathrm{supp}\,(\chi(x(y,x^0)))\cap V_\delta \subsetneqq \mathrm{supp}\,(\chi(x(y,x^0)))$.

Proof. Clearly, $\int_{\mathbb{R}^n_\pm\cap V_\delta} |\sigma_{\tilde{C}}(\widetilde{w}^0_\pm)|^2 dy \ge \int_{\mathbb{R}^n_\pm\cap V_\delta} |\sigma_{\tilde{C}}(\widetilde{w}^0_\pm)e_1|^2 dy$. We note that

$$\begin{aligned} \sigma_{\tilde{C}}(\widetilde{w}^0_\pm)e_1 &= \mathrm{Os} - \textstyle\int e^{i(y'-y^{0\prime})\cdot\eta'} e^{iy_1B_\pm(\eta')} e^{-iy_1^0B_-(\eta')}(\pm)(\frac{1}{2})(\mp)Z_\pm(\eta')Z_\pm(\eta')^{-1} \\ &\qquad \times Z_-(\eta')(\mathrm{Re}Z_+(\eta'))^{-1}b\,đ\eta' \\ &= \mathrm{Os} - \textstyle\int e^{i(y'-y^{0\prime})\cdot\eta'} e^{iy_1B_\pm(\eta')} e^{-iy_1^0B_-(\eta')}(-\frac{1}{2})Z_-(\eta')(\mathrm{Re}Z_+(\eta'))^{-1}b\,đ\eta'. \end{aligned}$$

Since $Z_-(\eta')$ and $\mathrm{Re}Z_+(\eta')$ are homogeneous of degree one in η' and nonsingular for all $\eta' \ne 0$, we obtain that

$$|Z_-(\eta')(\mathrm{Re}Z_+(\eta'))^{-1}b| \ge c|b| \quad \text{with} \quad c > 0, \quad \forall\, \eta' \ne 0.$$

On the other hand, let $\beta := \sup\{\mathrm{Im}\zeta; \zeta \in \mathrm{Spec}(B_+(\eta')|\eta'|^{-1})\}$ $(= -\inf\{\mathrm{Im}\zeta; \zeta \in \mathrm{Spec}(B_-(\eta')|\eta'|^{-1})\})$ then we can get that

$$\int_{\mathbb{R}^n_\pm\cap V_\delta} |\sigma_{\tilde{C}}(\widetilde{w}^0_\pm)e_1|^2 dy \ge \pm c\int_{\mathbb{R}^{n-1}}\int_0^{\pm\delta} e^{\mp 2\beta y_1|\eta'|} e^{-2\beta y_1^0|\eta'|} dy_1 d\eta'.$$

It is enough to compute

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}}\int_0^\delta e^{-2\beta y_1|\eta'|}e^{-2\beta y_1^0|\eta'|}dy_1d\eta'\\&=\int_{\mathbb{R}^{n-1}}(1-e^{-2\beta\delta|\eta'|})(2\beta|\eta'|)^{-1}e^{-2\beta y_1^0|\eta'|}d\eta'\\&\geq c\int_0^\infty(1-e^{-2\beta\delta\rho})e^{-2\beta y_1^0\rho}\rho^{n-3}d\rho.\end{aligned}$$

Thus, when $n=3$, we have

$$\int_0^\infty(1-e^{-2\beta\delta\rho})e^{-2\beta y_1^0\rho}d\rho=\frac{1}{2\beta y_1^0}-\frac{1}{2\beta(y_1^0+\delta)}\to\infty\text{ as }y_1^0\to 0.$$

On the other hand, for $n=2$, we have that

$$\begin{aligned}&\int_0^\infty\rho^{-1}(1-e^{-2\beta\delta\rho})e^{-2\beta y_1^0\rho}d\rho\\&=\int_0^\epsilon\rho^{-1}(1-e^{-2\beta\delta\rho})e^{-2\beta y_1^0\rho}d\rho+\int_\epsilon^\infty\rho^{-1}(1-e^{-2\beta\delta\rho})e^{-2\beta y_1^0\rho}d\rho\\&=c'+\int_\epsilon^\infty\rho^{-1}(1-e^{-2\beta\delta\rho})e^{-2\beta y_1^0\rho}d\rho\\&=c'+\int_{2\beta y_1^0\epsilon}^\infty\rho^{-1}(1-e^{-(\delta/y_1^0)\rho})e^{-\rho}d\rho\to\infty\quad\text{as }y_1^0\to 0.\end{aligned}$$

Therefore we conclude that $\int_{\mathbb{R}^{n-1}}\int_0^\delta e^{-2\beta y_1|\eta'|}e^{-2\beta y_1^0|\eta'|}dy_1d\eta'\to\infty$ as $y_1^0\to 0$ and the proof of the lemma is complete. □

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