Local Uniqueness for the Fixed Energy Fixed Angle Inverse Problem in Obstacle Scattering

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We consider the inverse problem of determining a sound-soft obstacle in $\mathbf{R}^n$, $n \geq 2$, from its scattering amplitude at a fixed incident direction $\theta \in S^{n-1}$ and a fixed energy $k > 0$. This is a formally determined inverse problem since the data depend on the same number of variables, $n-1$, as does the object we want to recover. The purpose of this note to show that the well known Schiffer's proof presented in [LP] of uniqueness when given all incident directions implies also local uniqueness at a fixed θ as well.

By obstacles, we mean compact subsets of $\mathbf{R}^n$ with smooth boundary and connected complement. The scattering amplitude $A_{\mathcal{O}}(k,\theta,\omega)$ related to an obstacle $\mathcal{O}$ is defined as follows. For $k > 0$, $\theta \in S^{n-1}$, we define the scattering solution $u(x,\theta,k)$ as the solution to the boundary value problem (see e.g., [CK])

$$\begin{cases} (-\Delta - k^2)u &= 0, \quad \text{in } \mathbf{R}^n \setminus \mathcal{O}, \\ u|_{\partial\mathcal{O}} &= 0, \end{cases}$$

such that $u = e^{ik\theta\cdot x} + v$, with v satisfying the Sommerfeld outgoing condition at infinity: $(\partial/\partial r - ik)v = O(r^{-(n+1)/2})$, as $r = |x| \to \infty$. Then

$$v(x,\theta,k) = e^{ik\theta\cdot x} + \frac{e^{ikr}}{r^{(n-1)/2}} A_{\mathcal{O}}\left(k,\theta,\frac{x}{r}\right) + O\left(\frac{1}{r^{(n+1)/2}}\right), \quad \text{as } r = |x| \to \infty.$$

The function $A_{\mathcal{O}}(k,\omega,\theta)$ is the scattering amplitude related to $\mathcal{O}$.

It is known that Schiffer's proof implies uniqueness if $A_{\mathcal{O}}$ is known for all ω, fixed $k_0 > 0$, and N incident directions θ's; or for all ω, fixed θ_0, and N frequencies k's; where N is greater than the number of the Dirichlet eigenvalues of the Laplacian in a ball containing the obstacles. In particular, this implies uniqueness at a fixed θ_0 and a fixed k_0 for all obstacles contained in a ball with sufficiently small radius R. In the 3D case, the condition is given by $k_0 R < \pi$. We refer to [CK], [I], [KK], [CS] for details and references.

Let Ω_{ext} be the connected unbounded component of $\mathbf{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$. Set $\Omega_{\text{int}} = \mathbf{R}^n \setminus \overline{\Omega}_{\text{ext}}$. Then $\overline{\Omega}_{\text{int}} \supset \mathcal{O}_1 \cup \mathcal{O}_2$. Note that Ω_{int} is an open set that contains the interior of $\mathcal{O}_1 \cup \mathcal{O}_2$ as well as all components of $\mathbf{R}^n \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$ disconnected from infinity. In what follows, ω_n is the volume of the unit ball in $\mathbf{R}^n$ (not to be mistaken with the outgoing direction ω), in particular, $\omega_3 = 4\pi/3$.

Theorem 1 *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two obstacles. Assume that for the corresponding scattering amplitudes we have*

$$A_{\mathcal{O}_1}(k_0,\theta_0,\omega) = A_{\mathcal{O}_2}(k_0,\theta_0,\omega)$$

for a fixed $\theta_0 \in S^{n-1}$, fixed $k_0 > 0$ and all $\omega \in S^{n-1}$. If

$$\text{Vol}(\Omega_{\text{int}} \setminus \mathcal{O}_i) < \omega_n k_0^{-n}, \quad i = 1, 2,$$

then $\mathcal{O}_1 = \mathcal{O}_2$.

*Partly supported by NSF Grant DMS-0196440 and MSRI
†Partly supported by NSF Grant DMS-007048

Our argument is based on an estimate of the first eigenvalue of the Dirichlet Laplacian in a bounded domain.

Lemma 1 *Let k^2 be a Dirichlet eigenvalue of $-\Delta$ in the bounded domain G. Then*

$$\omega_n \leq k^n \mathrm{Vol}(G).$$

Proof. We use the Poincaré inequality in the form presented in [GT]:

$$\|u\| \leq \left(\frac{\mathrm{Vol}(G)}{\omega_n}\right)^{1/n} \|\nabla u\|, \quad \text{for any } u \in H_0^1(G). \tag{1}$$

Let u be a normalized eigenfunction corresponding to k^2. Then $\|\nabla u\| = k$ and $u \in H_0^1(G)$. Applying (1), we get

$$1 \leq k \left(\frac{\mathrm{Vol}(G)}{\omega_n}\right)^{1/n},$$

which implies the lemma. □

Proof of Theorem 1. Let $u_j(x, \theta, k)$ be the scattering solution related to $\mathcal{O}_j$, $j = 1, 2$. By a well known argument based on Rellich's lemma, $A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_2}(k_0, \theta_0, \omega)$ implies that $u_1(x, \theta_0, k_0) = u_2(x, \theta_0, k_0)$ for all x outside a ball containing $\mathcal{O}_1 \cup \mathcal{O}_2$. We know that u_1 and u_2 solve

$$\begin{cases} (-\Delta - k_0^2)u_j &= 0, \quad \text{in } \mathbf{R}^n \setminus \mathcal{O}_j, \\ u_j|_{\partial \mathcal{O}_j} &= 0. \end{cases}$$

Then by analytic continuation, $u_1 = u_2$ on $\partial\Omega$. Let $G_j = \Omega_{\mathrm{int}} \setminus \mathcal{O}_j$, $j = 1, 2$. Suppose that $\mathcal{O}_1 \neq \mathcal{O}_2$. Then for $j = 1$ or $j = 2$, G_j is an open non-empty set. Suppose that this is G_1. Then u_1 solves the problem

$$\begin{cases} (-\Delta - k_0^2)u_1 &= 0, \quad \text{in } G_1, \\ u_1|_{\partial G_1} &= 0. \end{cases}$$

Moreover, u_1 is not identically equal to zero in G_1 because it is a real analytic function not vanishing for large x. Thus k_0^2 is a Dirichlet eigenvalue of the Laplacian in G_1. By Lemma 1, $\omega_n k_0^{-n} \leq \mathrm{Vol}(G_1)$. This contradicts our assumption, which proves the theorem. □

Corollary 1 *Fix $k_0 > 0$, $\theta_0 \in S^{n-1}$. Let $\mathcal{O}_- \subset \mathcal{O}_+$ be two domains and assume that $\mathrm{Vol}(\mathcal{O}_+ \setminus \mathcal{O}_-) < \omega_n k_0^{-n}$. Let $\mathcal{O}_- \subset \mathcal{O}_j \subset \mathcal{O}_+$, $j = 1, 2$, and assume that $A_{\mathcal{O}_1}(k_0, \theta_0, \omega) = A_{\mathcal{O}_2}(k_0, \theta_0, \omega)$. Then $\mathcal{O}_1 = \mathcal{O}_2$.*

Proof. The domains $G_j = \Omega_{\mathrm{int}} \setminus \mathcal{O}_i$ are included in $\mathcal{O}_+ \setminus \mathcal{O}_-$. □

In particular, in a small neighborhood of the boundary of any obstacle $\mathcal{O}$, there is no other obstacle with the same scattering amplitude at the fixed $k_0 > 0$, θ_0, with boundary in that neighborhood. We would like to emphasize that this is different than the argument based on an a priori knowledge of the diameter of the domain where the obstacles are located mentioned above. In Theorem 1 and Corollary 1, we do not impose smallness assumptions on k_0 or on the diameters of the obstacles. We prove unconditional local uniqueness at fixed k_0, θ_0 near any obstacle.

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