PARTIAL CAUCHY DATA FOR GENERAL SECOND ORDER ELLIPTIC OPERATORS IN TWO DIMENSIONS

OLEG YU. IMANUVILOV, GUNTHER UHLMANN, AND MASAHIRO YAMAMOTO

Abstract. We consider the problem of determining the coefficients of a first-order perturbation of the Laplacian in two dimensions by measuring the corresponding Cauchy data on an arbitrary open subset of the boundary. From this information we obtain a coupled system of $\partial_\bar{z}$ and $\partial_z$ which the coefficients satisfy. As a corollary we show that for a simply connected domain we can determine uniquely the coefficients up to the natural obstruction. Another consequence of our result is that the magnetic field and the electric potential are uniquely determined by measuring the partial Cauchy data associated to the magnetic Schrödinger equation measured on an arbitrary open subset of the boundary. We also show that the coefficients of any real vector field perturbation of the Laplacian, the convection terms, are uniquely determined by their partial Cauchy data.

1. Introduction

We consider the problem of determining a complex-valued potential $q$ and complex-valued coefficients $A$ and $B$ in a bounded two dimensional domain from the Cauchy data measured on an arbitrary open subset of the boundary for the associated second order elliptic operator $\Delta + 2A \frac{\partial}{\partial z} + 2B \frac{\partial}{\partial \bar{z}} + q$. Specific cases of interest are the magnetic Schrödinger operator and the Laplacian with convection terms. We remark that general second order elliptic operators can be reduced to this form by using isothermal coordinates (e.g., [21]). The case of the conductivity equation has been considered in [13]. For global uniqueness results in the two dimensional case for the conductivity equation with full data measurements under different regularity assumptions see [1], [4], [17]. Such a problem originates in [7].

Below we more precisely formulate our inverse problem under consideration. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial \Omega = \bigcup_{k=1}^{N} \gamma_k$, where $\gamma_k$, $1 \leq k \leq N$, are connected components and smooth closed contours and let $\nu$ be the unit outward normal vector to $\partial \Omega$. We denote $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$. Henceforth we set $i = \sqrt{-1}$, $x_1, x_2 \in \mathbb{R}$, $z = x_1 + ix_2$, $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$, and we identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$. We also denote $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2})$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2})$.

From now on we assume that $(A, B, q), (A_j, B_j, q_j) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega}) \times C^{4+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ are complex-valued functions.

Let $u \in H^1(\Omega)$ be a solution of the Dirichlet problem

$$L(x, D)u = \Delta u + 2A \frac{\partial u}{\partial z} + 2B \frac{\partial u}{\partial \bar{z}} + qu = 0 \text{ in } \Omega, \quad u|_{\partial \Omega} = f,$$

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where \( f \in H_{\frac{1}{2}}(\partial \Omega) \) is a given boundary input. The Dirichlet-to-Neumann (DN) map, assuming that 0 is not a Dirichlet eigenvalue, is defined by
\[
\Lambda_{A,B,q}(f) = \frac{\partial u}{\partial \nu}|_\Gamma.
\]
More generally we define the set of Cauchy data by:
\[
\widetilde{C}_{A,B,q} = \left\{ \left( u|_{\partial \Omega}, \frac{\partial u}{\partial \nu}|_{\partial \Omega} \right) \mid (\Delta + 2A \frac{\partial}{\partial z} + 2B \frac{\partial}{\partial \overline{z}} + q)u = 0 \text{ in } \Omega, \ u \in H^1(\Omega) \right\}.
\]
We have \( \widetilde{C}_{A,B,q} \subset H^{\frac{1}{2}}(\partial \Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \).

Let \( \tilde{\Gamma} \subset \partial \Omega \) be a fixed non-empty open subset of the boundary and \( \Gamma_0 = \partial \Omega \setminus \tilde{\Gamma} \).

Our main theorem states that the coefficients must satisfy a system of \( q \) operators
\[
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\]
that there exists a function $\eta \in C^{5+\alpha}(\Omega)$ such that

$$A_1 - A_2 = 2\frac{\partial \eta}{\partial z}, \quad B_1 - B_2 = 2\frac{\partial \eta}{\partial \bar{z}}.$$ 

The existence of such an $\eta$ is proved as follows. We define $f = A_1 - A_2$ and $g = B_1 - B_2$. Then $\frac{\partial f}{\partial z} = \frac{\partial g}{\partial \bar{z}}$. Now we a one-form $a$ by $a = f d\bar{z} + gdz$. Then we have

$$da = \frac{\partial f}{\partial z} dz \wedge d\bar{z} + \frac{\partial g}{\partial \bar{z}} dz \wedge dz = \frac{\partial f}{\partial z} dz \wedge d\bar{z} = \frac{\partial g}{\partial \bar{z}} dz \wedge dz = 0$$

Therefore in a simply connected domain $\Omega$, we can choose $\eta$ such that there exists a function $q = \frac{\partial \eta}{\partial z}$ and therefore $f = \frac{\partial q}{\partial \bar{z}}$ and $g = \frac{\partial q}{\partial z}$, proving proving the existence of $\eta$.

Therefore by (1.6)

$$(1.9) \quad q_1 = q_2 + \Delta \eta + 4\frac{\partial q}{\partial z} \frac{\partial \eta}{\partial \bar{z}} + 2\frac{\partial \eta}{\partial \bar{z}} A_2 + 2\frac{\partial \eta}{\partial \bar{z}} B_2.$$ 

The operator $L_1(x, D)$ given by (1.8) has the Laplace operator as the principal part, the coefficients of $\frac{\partial}{\partial z}$ is $2A_2 + 4\frac{\partial q}{\partial \bar{z}}$, the coefficient of $\frac{\partial}{\partial \bar{z}}$ is $2B_2 + 4\frac{\partial q}{\partial z}$, and the coefficient of the zero order term is given by the right-hand side of (1.9). By (1.5) we have that $\frac{\partial q}{\partial z} |_{\Gamma} = 0$ and $\eta |_{\Gamma} = 0$. The proof of the corollary is completed. \hfill \Box

We now apply our result to the case of the magnetic Schrödinger operator. Denote $\tilde{A} = (\tilde{A}_1, \tilde{A}_2)$, where $\tilde{A}_j$ are real-valued, $\tilde{A} = \tilde{A}_1 - i\tilde{A}_2$, rot $\tilde{A} = \frac{\partial \tilde{A}_2}{\partial x_1} - \frac{\partial \tilde{A}_1}{\partial x_2}$, $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$. Consider the magnetic Schrödinger operator

$$(1.10) \quad L_{\tilde{A},\tilde{q}}(x, D) = \sum_{k=1}^{2} (D_k + \tilde{A}_k)^2 + \tilde{q}.$$ 

Let us define the following set of partial Cauchy data

$$\tilde{C}_{\tilde{A}^{(1)}\tilde{q}^{(1)}} = \left\{(u|_{\Gamma}, \frac{\partial u}{\partial \nu}|_{\Gamma})| L_{\tilde{A}^{(1)}\tilde{q}^{(1)}}(x, D)u = 0 \text{ in } \Omega, u|_{\Gamma_0} = 0, u \in H^1(\Omega) \right\}.$$ 

For the case of full data it is known that there is a gauge invariance in the problem and we can recover at best the magnetic field [20]. The same is valid for the case of partial Cauchy data [10]. We prove here that the converse holds in two dimensions.

**Corollary 1.2.** Let $\alpha > 0$, real-valued vector fields $\tilde{A}^{(1)}, \tilde{A}^{(2)} \in C^{5+\alpha}(\Omega)$ and complex-valued potentials $\tilde{q}^{(1)}, \tilde{q}^{(2)} \in C^{4+\alpha}(\Omega)$ be such that $\tilde{C}_{\tilde{A}^{(1)}\tilde{q}^{(1)}} = \tilde{C}_{\tilde{A}^{(2)}\tilde{q}^{(2)}}$. Then $\tilde{q}^{(1)} = \tilde{q}^{(2)}$ and rot $\tilde{A}^{(1)} = \text{rot} \tilde{A}^{(2)}$.

**Proof.** A straightforward calculation gives

$$(1.11) \quad L_{\tilde{A},\tilde{q}}(x, D) = -\Delta + 2i\tilde{A}_1 \frac{\partial}{\partial x_1} + 2i\tilde{A}_2 \frac{\partial}{\partial x_2} + |\tilde{A}|^2 + \frac{1}{i} \frac{\partial \tilde{A}_1}{\partial x_1} + \frac{1}{i} \frac{\partial \tilde{A}_2}{\partial x_2} + \tilde{q}.$$

Then the operator $L_{\tilde{A},\tilde{q}}(x, D)$ is a particular case of (1.1). Suppose that the vector fields $\tilde{A}^{(1)}, \tilde{A}^{(2)}$ and the potentials $\tilde{q}^{(1)}, \tilde{q}^{(2)}$ have the same Dirichlet-to-Neumann map. Taking into
account that $A_j = -\frac{1}{i} \tilde{A}^{(j)}$, $B_j = -\frac{1}{i} \tilde{A}^{(j)}$, $q_j = -(\frac{2}{i} \frac{\partial \tilde{A}^{(j)}}{\partial z} - \text{rot} \tilde{A}^{(j)} + |\tilde{A}^{(j)}|^2 + \tilde{q}^{(j)})$, we see that (1.6) gives
\[
\text{rot} \tilde{A}^{(1)} - \text{rot} \tilde{A}^{(2)} + \tilde{q}^{(2)} - \tilde{q}^{(1)} \equiv 0
\]
and (1.7) gives
\[
(1.12) \quad \frac{2}{i} \frac{\partial \tilde{A}^{(1)}}{\partial z} - \frac{2}{i} \frac{\partial \tilde{A}^{(2)}}{\partial z} - \frac{2}{i} \frac{\partial \tilde{A}^{(1)}}{\partial z} + \frac{2}{i} \frac{\partial \tilde{A}^{(2)}}{\partial z} + \text{rot} \tilde{A}^{(1)} - \text{rot} \tilde{A}^{(2)} + \tilde{q}^{(2)} - \tilde{q}^{(1)} \equiv 0.
\]
Using the identity $\frac{2}{i} \frac{\partial A}{\partial z} - \frac{2}{i} \frac{\partial A}{\partial z} = -2 \text{rot} A$ we transform (1.12) to the form
\[-(\text{rot} \tilde{A}^{(1)} - \text{rot} \tilde{A}^{(2)}) + \tilde{q}^{(2)} - \tilde{q}^{(1)} \equiv 0.
\]
The proof of the corollary is completed. \hfill \Box

Corollary 1.2 is new even in the case when the data is measured on the whole boundary. In two dimensions, Sun proved in [20] that for measurements on the whole boundary uniqueness holds assuming that both the magnetic potential and the electric potential are small. Kang and Uhlmann proved global uniqueness for the case of measurements on the whole boundary for a special case of the magnetic Schrödinger equation, namely the Pauli Hamiltonian [14]. In dimension $n \geq 3$ global uniqueness was shown in [18] for the case of full data. The regularity assumptions in the result were improved by Salo in [19]. The case of partial data was considered in [10], based on the methods of [15] and [6], with an improvement on the regularity of the coefficients in [16].

Our main theorem implies that the Dirichlet-to-Neumann map can uniquely determine any two of $(A, B, q)$. First we can prove that $A$ and $B$ are uniquely determined if $q$ is known. We discuss the uniqueness for Laplace operators with convection terms.

\[
(1.13) \quad L(x, D)u = \Delta u + a(x) \frac{\partial u}{\partial x_1} + b(x) \frac{\partial u}{\partial x_2} + q(x)u.
\]
Here $a, b, q$ are complex-valued functions. Let us define the following set of partial Cauchy data
\[\tilde{C}_{a^{(j)}, b^{(j)}} = \left\{(u|_{\Gamma}, \frac{\partial u}{\partial n}|_{\Gamma})| \Delta u + a^{(j)}(x) \frac{\partial u}{\partial x_1} + b^{(j)}(x) \frac{\partial u}{\partial x_2} + q(x)u = 0 \text{ in } \Omega, u|_{\Gamma_0} = 0, u \in H^1(\Omega) \right\}.\]

We have

**Corollary 1.3.** Let $\alpha > 0$ and two pairs of complex-valued coefficients $(a^{(1)}, b^{(1)}) \in C^{5+\alpha}(\Omega) \times C^{5+\alpha}(\Omega)$ and $(a^{(2)}, b^{(2)}) \in C^{5+\alpha}(\Omega) \times C^{5+\alpha}(\Omega)$ be such that $\tilde{C}_{a^{(1)}, b^{(1)}} = \tilde{C}_{a^{(2)}, b^{(2)}}$. Then $(a^{(1)}, b^{(1)}) \equiv (a^{(2)}, b^{(2)})$.

**Proof.** Taking into account that $\frac{\partial}{\partial x_1} = (\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}})$ and $\frac{\partial}{\partial x_2} = i(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}})$, we can rewrite the operator (1.13) in the form
\[
L(x, D)u = \Delta u + (a(x) + ib(x)) \frac{\partial u}{\partial z} + (a(x) - ib(x)) \frac{\partial u}{\partial \bar{z}} + q(x)u.
\]
The pairs \((a^{(1)}, b^{(1)})\) and \((a^{(2)}, b^{(2)})\) be such that corresponding operators defined by (1.13) have the same Dirichlet-to-Neumann map. Denote \(2A_k(x) = a^{(k)}(x) + ib^{(k)}(x)\) and \(2B_k(x) = a^{(k)}(x) - ib^{(k)}(x)\). By (1.6), we have
\[
\begin{align*}
-2\frac{\partial}{\partial z}(A_1 - A_2) - (B_1 - B_2)A_1 - (A_1 - A_2)B_2 &= 0 \quad \text{in } \Omega, \\
-2\frac{\partial}{\partial z}(B_1 - B_2) - (A_1 - A_2)B_1 - (B_1 - B_2)A_2 &= 0 \quad \text{in } \Omega.
\end{align*}
\]
By (1.5)
\[
(A_1 - A_2)|_{\Gamma} = (B_1 - B_2)|_{\Gamma} = 0.
\]
Using these identities and equations (1.14), (1.15) we obtain
\[
\frac{\partial(A_1 - A_2)}{\partial \nu}|_{\Gamma} = \frac{\partial(B_1 - B_2)}{\partial \nu}|_{\Gamma} = 0.
\]
The uniqueness of the Cauchy problem for the system (1.14)-(1.15) can be proved in the standard way by using a Carleman estimate (e.g., [12]). Therefore we have \(A_1 = A_2\) and \(B_1 = B_2\) in \(\Omega\).

We remark that Corollary generalizes the result of [9] who proved this result assuming that the measurements are made on the whole boundary. In dimension \(n \geq 3\) global uniqueness was shown in [8] for the case of full data.

Similarly to Corollary 1.2, we can prove that the Dirichlet-to-Neumann map can uniquely determine a potential \(q\) and one of \(A\) and \(B\) in (1.1).

**Corollary 1.4.** For \(j = 1, 2\), let \((A_j, B_j, q_j) \in C^{5+\alpha}(\Omega) \times C^{5+\alpha}(\Omega) \times C^{4+\alpha}(\Omega)\) for some \(\alpha > 0\) and be complex-valued. We assume either \(A_1 = A_2\) or \(B_1 = B_2\) in \(\Omega\). Then \(C(A_1, B_1, q_1) = C(A_2, B_2, q_2)\) implies \((A_1, B_1, q_1) = (A_2, B_2, q_2)\).

The proof of Theorem 1.1 follows the general method of [13]. In this case we need to prove a new Carleman estimate with degenerate harmonic weights to construct appropriate complex geometrical optics solutions. These solutions have a different form to take into account the first order terms. The new form of these solutions complicates considerably the arguments, especially the asymptotic expansions needed to analyze the behavior of the solutions. In Section 2 we prove the Carleman estimate which we need. In Section 3 we state the estimates and asymptotics which we will use in the construction of the complex geometrical optics solutions. This construction is done in Section 4. The proof of Theorem 1.1 is completed in Section 5. In section 6 and 7 we discuss some technical lemmas needed in the previous sections.
2. Carleman estimate

**Notations** We use throughout the paper the following notations. \( i = \sqrt{-1}, x_1, x_2, \xi_1, \xi_2 \in \mathbb{R}, z = x_1 + ix_2, \zeta = \xi_1 + i\xi_2, \overline{z} \) denotes the complex conjugate of \( z \in \mathbb{C}, D_k = \frac{1}{i} \frac{\partial}{\partial x_k}. \) We identify \( x = (x_1, x_2) \in \mathbb{R}^2 \) with \( z = x_1 + ix_2 \in \mathbb{C}. \) We set \( \partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}), \)
\[
\partial_k = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}), \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}, \quad \partial_{\bar{\omega}} = \frac{\partial}{\partial \bar{\omega}}.
\]
Denote by \( k, \tau \in \mathbb{C}^{n \times n} \) the Hessian matrix with entries \( \frac{\partial^2 f}{\partial x_i \partial x_j}. \) \( \| \cdot \|^2_{H^k(\Omega)} = \| \cdot \|^2_{H^k(\Omega)} + |\tau|^{2k}\| \cdot \|_{L^2(\Omega)} \) is the standard semiclassical Sobolev space with inner product given by \( \langle \cdot, \cdot \rangle_{H^k(\Omega)} = \langle \cdot, \cdot \rangle_{H^k(\Omega)} + |\tau|^{2k}\langle \cdot, \cdot \rangle_{L^2(\Omega)}. \) \( \mathcal{L}(X, Y) \) denotes the Banach space of all bounded linear operators from a Banach space \( X \) to another Banach space \( Y. \)

Let \( \Phi (z) = \varphi(x_1, x_2) + i\psi(x_1, x_2) \in C^2(\overline{\Omega}) \) with real-valued \( \varphi \) and \( \psi \) satisfy
\[
(2.1) \quad \frac{\partial \Phi}{\partial \bar{z}}(z) = 0 \quad \text{in } \Omega, \quad \text{Im } \Phi|_{\Gamma_0} = 0.
\]

Denote by \( \mathcal{H} \) the set of all the critical points of the function \( \Phi \)
\[
\mathcal{H} = \{ z \in \overline{\Omega} \mid \frac{\partial \Phi}{\partial \bar{z}}(z) = 0 \}.
\]
Assume that \( \Phi \) has no critical points on \( \overline{\Gamma}, \) and that all critical points on the boundary are nondegenerate:
\[
(2.2) \quad \mathcal{H} \cap \partial \Omega \subset \Gamma_0, \quad \frac{\partial^2 \Phi}{\partial z^2}(z) \neq 0, \quad \forall z \in \mathcal{H}.
\]
Then \( \Phi \) has only a finite number of critical points and we can set:
\[
(2.3) \quad \mathcal{H} \setminus \Gamma_0 = \{ \bar{x}_1, ..., \bar{x}_\ell \}, \quad \mathcal{H} \cap \Gamma_0 = \{ \bar{x}_{\ell+1}, ..., \bar{x}_{\ell+\ell'} \}.
\]
The following proposition was proved in [13].

**Proposition 2.1.** Let \( \bar{x} \) be an arbitrary point in \( \Omega. \) There exists a sequence of functions \( \{ \Phi_\epsilon \}_{\epsilon \in (0,1)} \) satisfying (2.1) such that all the critical points of \( \Phi_\epsilon \) are nondegenerate and there exists a sequence \( \{ \bar{x}_\epsilon \}, \epsilon \in (0,1) \) such that
\[
\bar{x}_\epsilon \in \mathcal{H}_\epsilon = \{ z \in \overline{\Omega} \mid \frac{\partial \Phi_\epsilon}{\partial \bar{z}}(z) = 0 \}, \quad \bar{x}_\epsilon \rightarrow \bar{x} \quad \text{as } \epsilon \rightarrow +0.
\]
Moreover for any \( j \) from \( \{1, ..., N \} \) we have
\[
\mathcal{H}_\epsilon \cap \gamma_j = \emptyset \quad \text{if } \gamma_j \cap \bar{\Gamma} \neq \emptyset, \quad \mathcal{H}_\epsilon \cap \gamma_j \subset \Gamma_0 \quad \text{if } \gamma_j \cap \bar{\Gamma} = \emptyset,
\]
\[
\text{Im } \Phi_\epsilon(\bar{x}_\epsilon) \notin \{ \text{Im } \Phi_\epsilon(x) \mid x \in \mathcal{H}_\epsilon \setminus \{ \bar{x}_\epsilon \} \} \quad \text{and } \text{Im } \Phi_\epsilon(\bar{x}_\epsilon) \neq 0.
\]
In order to prove (1.5) we need the following proposition.
**Proposition 2.2.** Let $\Gamma_* \subset \subset \bar{\Gamma}$ be an arc with the left endpoint $x_-$ and the right endpoint $x_+$ oriented clockwise. For any $\hat{x} \in \text{Int}\Gamma_*$ there exists a function $\Phi(z)$ which satisfies (2.1), (2.2), $\text{Im}\Phi|_{\partial\Omega \setminus \Gamma_*=0}$ and

\begin{equation}
\hat{x} \in \mathcal{G} = \{x \in \Gamma_* | \frac{\partial \text{Im}\Phi(x)}{\partial \mathbf{r}}(x) = 0\}, \text{ card}\mathcal{G} < \infty,
\end{equation}

\begin{equation}
(\frac{\partial}{\partial \mathbf{r}})^2 \text{Im}\Phi(x) \neq 0 \quad \forall x \in \mathcal{G} \setminus \{\hat{x}\},
\end{equation}

Moreover

\begin{equation}
\text{Im}\Phi(\hat{x}) \neq \text{Im}\Phi(x) \quad \forall x \in \mathcal{G} \setminus \{\hat{x}\} \quad \text{and} \quad \text{Im}\Phi(\hat{x}) \neq 0.
\end{equation}

\begin{equation}
(\frac{\partial}{\partial \mathbf{r}+0})^6 \text{Im}\Phi(x_-) \neq 0, \quad (\frac{\partial}{\partial \mathbf{r}-0})^6 \text{Im}\Phi(x_+) \neq 0.
\end{equation}

**Proof.** Denote $\Gamma_0^\ast = \partial\Omega \setminus \Gamma_*$. Let $\hat{x}_-, \hat{x}_+ \in \partial\Omega$ be points such that the arc $[\hat{x}_-, \hat{x}_+] \subset (x_-, x_+)$ and $\hat{x} \in (\hat{x}_-, \hat{x}_+)$ be an arbitrary point and $x_0$ be another fixed point from the interval $(\hat{x}, \hat{x}_+)$. We claim that there exists a pair $(\varphi, \psi) \in C^5(\bar{\Omega}) \times C^5(\bar{\Omega})$ which solves the system of Cauchy-Riemann equations in $\Omega$ such that

- A) $\psi|\Gamma_0^\ast = 0, [\frac{\partial \varphi}{\partial \mathbf{r}}|_{\gamma_j} \neq 0$ if $\gamma_j \cap \Gamma_* \neq \emptyset, \frac{\partial \psi}{\partial \mathbf{r}}(\hat{x}) = 0, (\frac{\partial}{\partial \mathbf{r}})^2 \psi(\hat{x}) \neq 0$,

- A') $(\frac{\partial}{\partial \mathbf{r}+0})^6 \psi(x_-) \neq 0, (\frac{\partial}{\partial \mathbf{r}-0})^6 \psi(x_+) \neq 0$,

- B) The restriction of the function $\psi$ to the arc $[\hat{x}_-, \hat{x}_+]$ is a Morse function,

- C) $\frac{\partial \psi}{\partial \mathbf{r}} > 0$ on $(x_-, \hat{x}_-], \frac{\partial \psi}{\partial \mathbf{r}} < 0$ on $[\hat{x}_+, x_+]$,

- D) $\psi(\hat{x}) \notin \{\psi(x)|x \in \partial\Omega \setminus \{\hat{x}\}, \frac{\partial \psi}{\partial \mathbf{r}}(x) = 0\},$

- E) if $\gamma_j \cap \Gamma_* = \emptyset$, then the restriction of the function $\varphi$ on $\gamma_j$ has only two nondegenerate critical points.

Such a pair of functions may be constructed in the following way. Let $\gamma_1 \cap \Gamma_* \neq \emptyset$ and $\gamma_j \cap \Gamma_* = \emptyset$ for all $j \in \{2, \ldots, N\}$. First by Corollary 6.1 in the Appendix, for some $\alpha \in (0, 1)$, there exists a solution $(\tilde{\varphi}, \tilde{\psi}) \in C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega})$ to the Cauchy-Riemann equations with the following boundary data

$$\tilde{\psi}|_{\partial\Omega \setminus [x_0, \hat{x}_+]} = \psi_*, \quad \frac{\partial \tilde{\varphi}}{\partial \mathbf{r}}|_{\gamma_0 \setminus [x_0, \hat{x}_+]} < \beta < 0$$

and such that if $\gamma_j \cap \Gamma_* = \emptyset$ the function $\varphi$ has only two nondegenerate critical points located on the contour $\gamma_j$. The function $\psi_*$ has the following properties: $\psi_*|\Gamma_0^\ast = 0, \frac{\partial \psi_*}{\partial \mathbf{r}} > 0$ on $(x_-, \hat{x}_-], \frac{\partial \psi_*}{\partial \mathbf{r}} < 0$ on $[\hat{x}_+, x_+]$. The function $\psi_*$ on the set $[\hat{x}_-, x_0]$ has only one critical point $\hat{x}$ and $\psi_*(\hat{x}) \neq 0$. On the set $(x_0, \hat{x}_+)$ the Cauchy data is not fixed. The restriction of
the function \( \tilde{\psi} \) on \([x_0, \hat{x}_+]\) can be approximated in the space \( C^{5+\alpha}(\overline{[x_0, \hat{x}_+]}) \) by a sequence of Morse functions \( \{g_\epsilon\}_{\epsilon \in (0,1)} \) such that

\[
\left( \frac{\partial}{\partial \Omega} \right)^k \tilde{\psi}(x) = \left( \frac{\partial}{\partial \Omega} \right)^k g_\epsilon(x) \quad x \in \{\hat{x}_+, x_0\}, \quad k \in \{0, 1, \ldots, 5\},
\]

and

\[
\psi_*(\hat{x}) \notin \{g_\epsilon(x)\} \left| \frac{\partial g_\epsilon(x)}{\partial \tau} = 0 \right. \}
\]

Let us consider some arc \( J \subset \subset (x_-, \hat{x}_-) \). On this arc we have \( \frac{\partial \tilde{\psi}}{\partial \Omega} > 0 \), say,

\[
(2.8) \quad \frac{\partial \tilde{\psi}}{\partial \Omega} > \beta' > 0 \quad \text{on} \quad J \quad \text{for some positive} \quad \beta'.
\]

Let \( (\varphi_\epsilon, \psi_\epsilon) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega}) \) be a solution to the Cauchy-Riemann equations with boundary data \( \psi_\epsilon = 0 \) on \( \partial \Omega \setminus (J \cup [x_0, \hat{x}_+]) \) and \( \psi_\epsilon = g_\epsilon - \tilde{\psi} \) on \([x_0, \hat{x}_+]\) and on \( J \) the Cauchy data is chosen in such a way that

\[
(2.9) \quad \|\varphi_\epsilon\|_{C^{5+\alpha}(\partial \Omega)} + \|\psi_\epsilon\|_{C^{5+\alpha}(\partial \Omega)} \to 0 \quad \text{as} \quad \|g_\epsilon - \tilde{\psi}\|_{C^{5+\alpha}(\overline{[x_0, \hat{x}_+]})} \to 0.
\]

By (2.8), (2.9) for all small positive \( \epsilon \), the restriction of the function \( \tilde{\psi} + \psi_\epsilon \) to \( \partial \Omega \) satisfies

\[
(\tilde{\psi} + \psi_\epsilon)|_{\Gamma^*_0} = 0, \quad \frac{\partial (\tilde{\psi} + \psi_\epsilon)}{\partial \nu}|_{\gamma_0 \setminus [x_0, \hat{x}_+]} < 0, \quad \frac{\partial (\tilde{\psi} + \psi_\epsilon)}{\partial \tau} > 0 \quad \text{on} \quad [x_-, \hat{x}_-],
\]

\[
\frac{\partial (\tilde{\psi} + \psi_\epsilon)}{\partial \tau} < 0 \quad \text{on} \quad [\hat{x}_+, x_+], \quad (\tilde{\psi} + \psi_\epsilon)|_{[x_0, \hat{x}_+]} = g_\epsilon, \quad (\tilde{\psi} + \psi_\epsilon)|_{[x_-, x_0]} = \psi_*.
\]

If \( j \geq 2 \) then the restriction of the function \( \varphi_\epsilon + \tilde{\varphi} \) on \( \gamma_j \) has only two critical points located on the contour \( \gamma_j \subset \Gamma^*_0 \). These critical points are nondegenerate if \( \epsilon \) is sufficiently small.

Therefore the restriction of the function \( (\tilde{\psi} + \psi_\epsilon) \) on \( \Gamma_* \) has a finite number of critical points. Some of these points may be the critical points of \( (\tilde{\psi} + \psi_\epsilon) \) considered as the function on \( \overline{\Omega} \). We change slightly the function \( (\tilde{\psi} + \psi_\epsilon) \) such that all of its critical points are in \( \Omega \). Suppose that function \( \tilde{\psi} + \psi_\epsilon \) has critical points on \( \Gamma_* \). Then these critical points should be among the set of critical points of the function \( g_\epsilon \), otherwise it would be the point \( \hat{x} \). We denote these points by \( \hat{x}_1, \ldots, \hat{x}_m \). Let \( (\tilde{\varphi}, \tilde{\psi}) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega}) \) be a solution to the Cauchy-Riemann problem (6.1) with the following boundary data

\[
\tilde{\psi}|_{\Gamma^*_0} = 0, \quad \tilde{\psi}(\hat{x}) = 1, \quad \tilde{\psi}|_{\partial \Omega \setminus [\hat{x}_+, \hat{x}_-]} = 0, \quad \left| \frac{\partial \tilde{\psi}}{\partial \nu} \right|_{\gamma_0 \setminus J} > 0.
\]

For all small positive \( \epsilon_1 \) the function \( \tilde{\psi} + \psi_\epsilon + \epsilon_1 \tilde{\psi} \) does not have a critical point on \( \partial \Omega \) and the restriction of this function on \( \Gamma \) has a finite number of nondegenerate critical points. Therefore we take \( (\varphi_\epsilon + \varphi_\epsilon, \psi_\epsilon + \psi_\epsilon) \) as the pairs of functions satisfying A) - E).

The function \( \varphi + i\psi \) with pair \( (\varphi, \psi) \) satisfying conditions (A)-E) satisfies all the hypotheses of Proposition 2.2 except that some of its critical points might possibly be degenerate. In order to fix this problem we consider a perturbation of the function \( \varphi + i\psi \) which is
constructed in the following way. By Proposition 6.2, there exists a function \( w \) holomorphic in \( \Omega \), such that

\[
\text{Im } w|_{\Gamma_0} = 0, \quad w|_{\mathcal{H}_0} = \frac{\partial w}{\partial z}|_{\mathcal{H}_0} = 0, \quad \frac{\partial^2 w}{\partial z^2}|_{\mathcal{H}_0} \neq 0.
\]

Denote \( \Phi_\delta = \varphi + i\psi + \delta w \). For all sufficiently small positive \( \delta \), we have

\[
\mathcal{H}_0 \subset \mathcal{H}_\delta \equiv \{ x \in \Omega | \frac{\partial}{\partial z} \Phi_\delta(x) = 0 \}.
\]

We now show that for all sufficiently small positive \( \delta \), all critical points of the function \( \Phi_\delta \) are nondegenerate. Let \( \tilde{x} \) be a critical point of the function \( \varphi + i\psi \). If \( \tilde{x} \) is a nondegenerate critical point, by the implicit function theorem, there exists a ball \( B(\tilde{x}, \delta_1) \) such that the function \( \Phi_\delta \) in this ball has only one nondegenerate critical point for all small \( \delta \). Let \( \tilde{x} \) be a degenerate critical point of \( \varphi + i\psi \). Without loss of generality we may assume that \( \tilde{x} = 0 \).

In some neighborhood of 0, we have

\[
\frac{\partial}{\partial z} \text{degenerate critical point of } \Phi.
\]

Let \( \delta \) be a solution to

\[
\partial x = x, \delta + i x_2, \delta \to 0. \quad \text{Then either}
\]

\[
\begin{align*}
\partial \Phi_\delta(z) = 0 & \text{ or } z_1 = c_1 + o(\delta) \quad \text{as } \delta \to 0. \\
\end{align*}
\]

Therefore \( \partial^2 \Phi_\delta(z_\delta) \neq 0 \) for all sufficiently small \( \delta \).

The following proposition was proven in [13]:

**Proposition 2.3.** Let \( \Phi \) satisfy (2.1) and (2.2). Let \( \tilde{f} \in L^2(\Omega) \), and \( \tilde{v} \in H^1(\Omega) \) be a solution to

\[
2 \frac{\partial}{\partial z} \tilde{v} - \tau \frac{\partial \Phi}{\partial z} \tilde{v} = \tilde{f} \quad \text{in } \Omega
\]

or \( \tilde{v} \) be a solution to

\[
2 \frac{\partial}{\partial \bar{z}} \tilde{v} - \tau \frac{\partial \Phi}{\partial \bar{z}} \tilde{v} = \tilde{f} \quad \text{in } \Omega.
\]

In the case (2.12) we have

\[
\begin{align*}
\| \frac{\partial}{\partial x_1} (e^{-i\tau \psi} \tilde{v}) \|_{L^2(\Omega)}^2 & - \tau \int_{\partial \Omega} |(\nabla \varphi, \nu)| \bar{\tilde{v}}^2 d\sigma \\
+ & \text{Re} \int_{\partial \Omega} i \left( \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \bar{\tilde{v}} \right) \bar{v} d\sigma + \| \frac{\partial}{\partial x_2} (e^{-i\tau \psi} \tilde{v}) \|_{L^2(\Omega)}^2 = \| \tilde{f} \|_{L^2(\Omega)}^2.
\end{align*}
\]

In the case (2.13) we have

\[
\begin{align*}
\| \frac{\partial}{\partial x_1} (e^{i\tau \psi} \tilde{v}) \|_{L^2(\Omega)} & - \tau \int_{\partial \Omega} |(\nabla \varphi, \nu)| \bar{\tilde{v}}^2 d\sigma + \text{Re} \int_{\partial \Omega} i \left( \left( -\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2} \right) \bar{\tilde{v}} \right) \bar{v} d\sigma \\
+ & \| \frac{\partial}{\partial x_2} (e^{i\tau \psi} \tilde{v}) \|_{L^2(\Omega)}^2 = \| \tilde{f} \|_{L^2(\Omega)}^2.
\end{align*}
\]

Let \( \alpha \in (0, 1) \) and \( \mathcal{A}, \mathcal{B} \in C^{6+\alpha}(\bar{\Omega}) \) be two complex-valued solutions to the boundary value problem

\[
2 \frac{\partial \mathcal{A}}{\partial z} = -A \quad \text{in } \Omega, \quad \text{Im } \mathcal{A}|_{\Gamma_0} = 0, \quad 2 \frac{\partial \mathcal{B}}{\partial z} = -B \quad \text{in } \Omega, \quad \text{Im } \mathcal{B}|_{\Gamma_0} = 0.
\]
Consider the boundary value problem
\[
\begin{aligned}
\left\{
\begin{array}{l}
\mathcal{K}(x,D)u = (4 \frac{\partial}{\partial x} \frac{\partial}{\partial z} + 2A \frac{\partial}{\partial z} + 2B \frac{\partial}{\partial \bar{z}})u = f \quad \text{in} \quad \Omega, \\
u|_{\partial \Omega} = 0.
\end{array}
\right.
\end{aligned}
\]

For this problem we have the following Carleman estimate with boundary terms.

**Proposition 2.4.** Suppose that \( \Phi \) satisfies (2.1), (2.2), \( u \in H^1_0(\Omega) \) and the coefficients \( A, B \in \{ D \in C^1(\Omega) \| D \|_{C^1(\overline{\Omega})} \leq K \} \). Then there exist \( \tau_0 = \tau_0(K, \Phi) \) and \( C = C(K, \Phi) \) independent of \( u \) and \( \tau \) such that for all \( |\tau| > \tau_0 \)

\[
|\tau|\|ue^{\tau \varphi}\|_{L^2(\Omega)}^2 + |\tau u|_{L^2(\Omega)}^2 + \|\frac{\partial u}{\partial \nu} e^{\tau \varphi}\|_{L^2(\Omega)}^2 + \tau^2 \|\frac{\partial \Phi}{\partial \nu} u e^{\tau \varphi}\|_{L^2(\Omega)}^2 \leq C_1(\|\mathcal{K}(x,D)u\|_{L^2(\Omega)}^2 + |\tau| \int_{\Gamma} |\frac{\partial u}{\partial \nu}|^2 e^{2\tau \varphi} d\sigma).
\]

**(2.17)**

**Proof.** Denote \( \widetilde{v} = u e^{\tau \varphi} \), \( \mathcal{K}(x,D)u = f \). Observe that \( \varphi(x_1, x_2) = \frac{1}{2}(\Phi(z) + \Phi(z)) \). Therefore

\[
e^{\tau \varphi} \mathcal{K}(x,D)(e^{-\tau \varphi} \widetilde{v}) = (2 \frac{\partial}{\partial z} - (\tau \frac{\partial \Phi}{\partial z} - B))(2 \frac{\partial}{\partial \bar{z}} - (\tau \frac{\partial \Phi}{\partial \bar{z}} - A))\widetilde{v} + (-2 \frac{\partial A}{\partial z} - AB)\widetilde{v} = \]

\[
(2 \frac{\partial}{\partial z} - (\tau \frac{\partial \Phi}{\partial z} - A))(2 \frac{\partial}{\partial \bar{z}} - (\tau \frac{\partial \Phi}{\partial \bar{z}} - B))\widetilde{v} + (-2 \frac{\partial B}{\partial z} - AB)\widetilde{v} = f e^{\tau \varphi}.
\]

Denote \( \widetilde{w}_1 = Q(z)(\frac{\partial}{\partial z} - \tau \frac{\partial \Phi}{\partial z} + A)\widetilde{v}, \) \( \widetilde{w}_2 = Q(z)(\frac{\partial}{\partial \bar{z}} - \tau \frac{\partial \Phi}{\partial \bar{z}} + B)\widetilde{v} \), where \( Q(z), \widetilde{Q}(z) \in C^2(\overline{\Omega}) \) are some holomorphic functions in \( \Omega \) that will be specified below. Thanks to the zero Dirichlet boundary condition for \( u \) we have

\[
\widetilde{w}_1|_{\partial \Omega} = 2Q(z)\frac{\partial \widetilde{v}}{\partial z}|_{\partial \Omega} = (\nu_1 + i\nu_2)Q(z)\frac{\partial \widetilde{v}}{\partial \nu}|_{\partial \Omega}, \quad \widetilde{w}_2|_{\partial \Omega} = 2Q(z)\frac{\partial \widetilde{v}}{\partial \nu}|_{\partial \Omega} = (\nu_1 - i\nu_2)Q(z)\frac{\partial \widetilde{v}}{\partial \nu}|_{\partial \Omega}.
\]

By Proposition 2.3 we have the following integral equalities:

\[
\|(\frac{\partial}{\partial x_1} - i\tau \frac{\partial \psi}{\partial x_1})(\widetilde{w}_1 e^{B})\|^2_{L^2(\Omega)} - \tau \int_{\partial \Omega} (\nabla \varphi, \nu)|Q|^2|\frac{\partial \widetilde{v}}{\partial \nu}|^2 e^{B\tau} d\sigma
\]

\[
+ \text{Re} \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2})(\widetilde{w}_1 e^{B}))\widetilde{w}_1 e^{B} d\sigma +
\]

\[
+ \|(\frac{\partial}{\partial x_2} - i\tau \frac{\partial \psi}{\partial x_2})(\widetilde{w}_1 e^{B})\|^2_{L^2(\Omega)} = \|Q(f e^{\tau \varphi} + (2 \frac{\partial A}{\partial z} + AB)\widetilde{v})e^B\|^2_{L^2(\Omega)}
\]

\[
(2.19)
\]

and

\[
\|(\frac{\partial}{\partial x_1} + i\tau \frac{\partial \psi}{\partial x_1})(\widetilde{w}_2 e^{A})\|^2_{L^2(\Omega)} - \tau \int_{\partial \Omega} (\nabla \varphi, \nu)|Q|^2|\frac{\partial \widetilde{v}}{\partial \nu}|^2 e^{A\tau} d\sigma
\]

\[
+ \text{Re} \int_{\partial \Omega} i((-\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2})(\widetilde{w}_2 e^{A}))\widetilde{w}_2 e^{A} d\sigma +
\]

\[
+ \|(\frac{\partial}{\partial x_2} + i\tau \frac{\partial \psi}{\partial x_2})(\widetilde{w}_2 e^{A})\|^2_{L^2(\Omega)} = \|Q(f e^{\tau \varphi} + (2 \frac{\partial B}{\partial \bar{z}} + AB)\widetilde{v})e^A\|^2_{L^2(\Omega)}.
\]

\[
(2.20)
\]
We now simplify the integral \( \text{Re} i \int_{\partial \Omega} ((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}) (\tilde{w}_1 e^B)) \tilde{w}_1 e^B d\sigma \). We recall that \( \tilde{v} = u e^{\tau \varphi} \) and \( \tilde{w}_1 = Q(z)(\nu_1 + i\nu_2) \frac{\partial}{\partial \nu} = Q(z)(\nu_1 + i\nu_2) \frac{\partial}{\partial \nu} e^{\tau \varphi} \). Denote \( R + iP = Q(z)(\nu_1 + i\nu_2) e^A \). Therefore

\[
(2.21) \quad \text{Re} \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}) (\tilde{w}_1 e^B)) \tilde{w}_1 e^B d\sigma = \]

Let us simplify the integral \( \text{Re} \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2})(\tilde{w}_2 e^A)) \tilde{w}_2 e^A d\sigma \). We recall that \( \tilde{v} = u e^{\tau \varphi} \) and \( \tilde{w}_2 = (\nu_1 - i\nu_2)Q(z) \frac{\partial}{\partial \nu} = (\nu_1 - i\nu_2)Q(z) \frac{\partial}{\partial \nu} e^{\tau \varphi} \). Denote \( \tilde{R} + i\tilde{P} = Q(z)(\nu_1 - i\nu_2) e^A \). Thus

\[
(2.22) \quad \text{Re} \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2})(\tilde{w}_2 e^A)) \tilde{w}_2 e^A d\sigma = \]

Using the above formula we obtain

\[
\|(\frac{\partial}{\partial x_1} + i\tau \frac{\partial}{\partial x_1})(\tilde{w}_2 e^A)\|^2_{L^2(\Omega)} + \|(\frac{\partial}{\partial x_2} + i\tau \frac{\partial}{\partial x_2})(\tilde{w}_2 e^A)\|^2_{L^2(\Omega)} - \tau \int_{\partial \Omega} (\nu, \nabla \varphi) |Q|^2 |\frac{\partial \tilde{\nu}}{\partial \nu}| |e^B|^2 d\sigma - \tau \int_{\partial \Omega} (\nu, \nabla \varphi) |Q|^2 |\tilde{\nu}| |e^A|^2 d\sigma + \|(\frac{\partial}{\partial x_1} - i\tau \frac{\partial}{\partial x_1})(\tilde{w}_1 e^B)\|^2_{L^2(\Omega)} + \|(\frac{\partial}{\partial x_2} - i\tau \frac{\partial}{\partial x_2})(\tilde{w}_1 e^B)\|^2_{L^2(\Omega)} + \int_{\partial \Omega} (\frac{\partial \tilde{R}}{\partial \tilde{\tau}} P - \frac{\partial \tilde{P}}{\partial \tilde{\tau}} R) |\frac{\partial \tilde{\nu}}{\partial \nu}| |\tilde{w}_1 e^B|^2 d\sigma + \int_{\partial \Omega} (\frac{\partial \tilde{R}}{\partial \tilde{\tau}} P - \frac{\partial \tilde{P}}{\partial \tilde{\tau}} R) |\frac{\partial \tilde{\nu}}{\partial \nu}| |\tilde{w}_1 e^A|^2 d\sigma = \]

\[
(2.23) \quad \|Q(f e^{\tau \varphi} + (2 \frac{\partial A}{\partial z} + AB) \tilde{v}) e^B\|^2_{L^2(\Omega)} + \|Q(f e^{\tau \varphi} + (2 \frac{\partial B}{\partial z} + AB) \tilde{v}) e^A\|^2_{L^2(\Omega)}.
\]
We can rewrite (2.23) in the form
\[
\left\| \frac{\partial}{\partial x_1} \left( e^{i\psi} \bar{w}_2 e^A \right) \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial}{\partial x_2} \left( e^{i\psi} \bar{w}_2 e^A \right) \right\|_{L^2(\Omega)}^2
\]
\[
- \tau \int_{\partial \Omega} (\nu, \nabla \phi) \left| \bar{Q} \right|^2 |\bar{\psi}_\nu|^2 |e^B|^2 d\sigma - \tau \int_{\partial \Omega} (\nu, \nabla \phi) |Q|^2 |\psi_\nu|^2 |e^A|^2 d\sigma
\]
\[
+ \left\| \frac{\partial}{\partial x_1} \left( e^{-i\psi} \bar{w}_1 e^B \right) \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial}{\partial x_2} \left( e^{-i\psi} \bar{w}_1 e^B \right) \right\|_{L^2(\Omega)}^2
\]
\[
\quad + 2 \int_{\partial \Omega} (\frac{\partial R}{\partial \tau} P - \frac{\partial P}{\partial \tau} R) |\bar{\psi}_\nu|^2 |e^B|^2 d\sigma + \int_{\partial \Omega} \left( \frac{\partial R}{\partial \tau} \bar{P} - \frac{\partial P}{\partial \tau} \bar{R} \right) |\bar{\psi}_\nu|^2 |e^A|^2 d\sigma =
\]
(2.24) \[\left\| \bar{Q} (f e^{i\tau} + 2 \frac{\partial A}{\partial z} + AB) \bar{v} e^B \right\|_{L^2(\Omega)}^2 + \left\| Q (f e^{i\tau} + 2 \frac{\partial B}{\partial z} + AB) \bar{v} e^A \right\|_{L^2(\Omega)}^2.
\]
Next we show that it is possible to make a choice of functions \( \bar{Q} \) and \( Q \) such that
\[
(\frac{\partial R}{\partial \tau} P - \frac{\partial P}{\partial \tau} R) > 0 \text{ on } \Gamma_0, \quad (\frac{\partial \bar{R}}{\partial \tau} \bar{P} - \frac{\partial \bar{P}}{\partial \tau} \bar{R}) > 0 \text{ on } \Gamma_0.
\]
Let \( \gamma_j \) be a contour from \( \partial \Omega \). We parametrize the curve \( \gamma_j \) by arc length \( s \) starting from one fixed point on \( \gamma_j \): \( x(s) : [0, \ell_j] \to \gamma_j \). Here \( \ell_j \) denotes the total length of \( \gamma_j \). We note that \( \frac{\partial R}{\partial \tau} = \frac{d}{ds} R(x(s)). \) If there exists a holomorphic function \( \bar{Q} \) such that \( R(x(s)) = \ell_j \sin(s/\ell_j), P(x(s)) = \ell_j \cos(s/\ell_j). \) Then
\[
(\frac{\partial R}{\partial \tau} P - \frac{\partial P}{\partial \tau} R) = \ell_j \text{ on } \gamma_j \quad \forall j \in \{1, \ldots, N\}.
\]
Taking into account that \( R + iP = Q(z)(\nu_1 + i\nu_2)e^B \) we set
\[
(2.27) \quad b_1 = \text{Re} \left\{ \frac{\ell_j \sin(s/\ell_j) - i \ell_j \cos(s/\ell_j)}{(\nu_1 - i\nu_2)} e^{-B} \right\}, \quad b_2 = \text{Im} \left\{ \frac{\ell_j \sin(s/\ell_j) - i \ell_j \cos(s/\ell_j)}{(\nu_1 - i\nu_2)} e^{-B} \right\}.
\]
Using Proposition 5.1 in [13] we choose \( \bar{Q}(z) \) such that on \( \Gamma_0 \) the function \( \bar{Q}(z) \) is close to \( b_1 + ib_2 \) in the norm of the space \( C^1(\Gamma_0) \). Then by (2.27) we have the first inequality in (2.25). The choice of the function \( Q \) may be done in a similar fashion.

By (2.25) we obtain from (2.24) that
\[
(2.28) \quad \left\| \frac{\partial \bar{\psi}}{\partial \nu} \right\|_{L^2(\Gamma_0)} \leq C_2 \left( \left\| fe^{i\tau} \right\|_{L^2(\Omega)} + \left\| \bar{v} \right\|_{L^2(\Omega)} \right).
\]
From now on we assume that \( \bar{Q} = Q = 1 \). Observe that there exists a positive constant \( C_3, \) independent of \( \tau, \) such that
\[
(2.29) \quad \frac{1}{C_3} \left( \left\| \bar{w}_1 \right\|_{L^2(\Omega)}^2 + \left\| \bar{w}_2 \right\|_{L^2(\Omega)}^2 \right) \leq \frac{1}{2} \left\| \frac{\partial}{\partial x_1} (e^{i\psi} \bar{w}_2 e^A) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{\partial}{\partial x_2} (e^{i\psi} \bar{w}_2 e^A) \right\|_{L^2(\Omega)}^2
\]
\[
- \tau \int_{\partial \Omega_+} (\nu, \nabla \phi) \left| \bar{Q} \right|^2 |\bar{\psi}_\nu|^2 |e^B|^2 d\sigma - \tau \int_{\partial \Omega_-} (\nu, \nabla \phi) |Q|^2 |\psi_\nu|^2 |e^A|^2 d\sigma
\]
\[
+ \frac{1}{2} \left\| \frac{\partial}{\partial x_1} (e^{-i\psi} \bar{w}_1 e^B) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{\partial}{\partial x_2} (e^{-i\psi} \bar{w}_1 e^B) \right\|_{L^2(\Omega)}^2.
\]
By (2.33) and (2.28) we obtain from (2.24), (2.29)

\[ (2.31) \]

Therefore

\[ (2.30) \]

By (2.29), (2.30) and the definitions of \( \tilde{w}_1, w_1^* \) and \( \tilde{w}_2, w_2^* \), we have

\[
\| \text{Re} \tilde{v} - \tau \text{Im} \tilde{v} \|^2_{L^2(\Omega)} + \| 2 \frac{\partial \text{Re} \tilde{v}}{\partial z} \|^2_{L^2(\Omega)} + \| \frac{\partial \text{Im} \tilde{v}}{\partial z} \|^2_{L^2(\Omega)} \\
\leq C_5(\| \tilde{w}_1 \|^2_{L^2(\Omega)} + \| \tilde{w}_2 \|^2_{L^2(\Omega)} + \| \tilde{v} \|^2_{L^2(\Omega)}).
\]

Therefore

\[ (2.31) \]

Now since by assumption (2.2) the function \( \Phi \) has zeros of at most second order, there exists a constant \( C_7 > 0 \) independent of \( \tau \) such that

\[ (2.32) \]

By (2.31) and (2.32)

\[ (2.33) \]

By (2.33) and (2.28) we obtain from (2.24), (2.29)

\[ (2.34) \]

This concludes the proof of the proposition. \( \square \)

As a corollary we derive a Carleman inequality for the function \( u \) which satisfies the integral equality

\[ (2.35) \]

for all \( w \in \mathcal{X} = \{ w \in H^1(\Omega) | w|_{\Gamma_0} = 0, \mathcal{K}(x, D)w \in L^2(\Omega) \} \). We have
Corollary 2.1. Suppose that $\Phi$ satisfies (2.1), (2.2), $f \in H^1(\Omega), g \in H^\frac{1}{2}(\Gamma)$, $u \in L^2(\Omega)$ and the coefficients $A, B \in \{C \in C^1(\Omega) \mid ||C||_{C^1(\Gamma)} \leq K\}$. Then there exist $\tau_0 = \tau_0(K, \Phi)$ and $C = C(K, \Phi)$, independent of $u$ and $\tau$, such that for solutions of (2.35):

$$\|ue^{-\tau \varphi}\|^2_{L^2(\Omega)} \leq C_1 \|	au\| (\|f e^{-\tau \varphi}\|_{H^1(\Omega)}^2 + \|g e^{-\tau \varphi}\|_{H^\frac{1}{2}(\Gamma)}^2) \quad \forall \tau \geq \tau_0. $$

Proof. Let $\epsilon$ be some positive number. Consider the extremal problem

$$J_{\epsilon}(w) = \frac{1}{2} \|ue^{-\tau \varphi}\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon} \|\mathcal{K}(x, D)^* w - ue^{2\tau \varphi}\|^2_{L^2(\Omega)} + \frac{1}{2\|\tau\|} \|ue^{-\tau \varphi}\|^2_{L^2(\Gamma)} \to \inf,$$

(2.37)

$$w \in \hat{X} = \{w \in H^\frac{1}{2}(\Omega) | \mathcal{K}(x, D)^* w \in L^2(\Omega), w|_{\Gamma_0} = 0\}.$$

There exists a unique solution to (2.37), (2.38) which we denote by $\hat{w}_{\epsilon}$. By Fermat’s theorem

$$J_{\epsilon}'(\hat{w}_{\epsilon})[\delta] = 0 \quad \forall \delta \in \hat{X}.$$

Using the notation $p_{\epsilon} = \frac{1}{\epsilon} (\mathcal{K}(x, D)^* \hat{w}_{\epsilon} - ue^{2\tau \varphi})$ this implies

$$\mathcal{K}(x, D)p_{\epsilon} + \hat{w}_{\epsilon} e^{-\tau \varphi} = 0 \quad \text{in } \Omega, \quad p_{\epsilon}|_{\partial \Omega} = 0, \quad \frac{\partial p_{\epsilon}}{\partial \nu}|_{\Gamma} = \frac{\hat{w}_{\epsilon}}{|\tau|} e^{-2\tau \varphi}. $$

(2.39)

By Proposition 2.4 we have

$$|\tau| \|p_{\epsilon} e^{\tau \varphi}\|^2_{L^2(\Omega)} + \|p_{\epsilon} e^{\tau \varphi}\|^2_{H^1(\Omega)} + \|\frac{\partial p_{\epsilon}}{\partial \nu} e^{\tau \varphi}\|^2_{L^2(\Gamma_0)} + \tau^2 \|\frac{\partial \Phi}{\partial z} p_{\epsilon} e^{\tau \varphi}\|^2_{L^2(\Omega)}$$

(2.40)

$$\leq C_{11} (\|\hat{w}_{\epsilon} e^{-\tau \varphi}\|^2_{L^2(\Omega)} + \frac{1}{\|\tau\|} \int_{\Gamma} |\hat{w}_{\epsilon}|^2 e^{-2\tau \varphi} d\sigma) \leq 2C_{11} J_{\epsilon}(\hat{w}_{\epsilon}).$$

Taking the scalar product of equation (2.39) with $\hat{w}_{\epsilon}$ we obtain

$$2J_{\epsilon}(\hat{w}_{\epsilon}) + \langle ue^{2\tau \varphi}, p_{\epsilon} \rangle_{L^2(\Omega)} = 0.$$

Applying to the second term of the above equality estimate (2.40) we have

$$|\tau| J_{\epsilon}(\hat{w}_{\epsilon}) \leq C_{12} \|ue^{-\tau \varphi}\|^2_{L^2(\Omega)}.$$

Using this estimate we pass to the limit in (2.39) as $\epsilon$ goes to zero. We obtain

$$\mathcal{K}(x, D)p + \hat{w}_{\epsilon} e^{-2\tau \varphi} = 0 \quad \text{in } \Omega, \quad p|_{\partial \Omega} = 0, \quad \frac{\partial p}{\partial \nu}|_{\Gamma} = \frac{\hat{w}_{\epsilon}}{|\tau|} e^{-2\tau \varphi},$$

(2.41)

$$\mathcal{K}(x, D)^* \hat{w} - ue^{2\tau \varphi} = 0 \quad \text{in } \Omega, \quad \hat{w}|_{\Gamma_0} = 0,$$

(2.42)

and

$$|\tau| \|\hat{w}_{\epsilon} e^{-\tau \varphi}\|^2_{L^2(\Omega)} + \|\hat{w}_{\epsilon} e^{-\tau \varphi}\|^2_{L^2(\Gamma)} \leq C_{13} \|ue^{-\tau \varphi}\|^2_{L^2(\Omega)}.$$

(2.43)

Since $\hat{w} \in L^2(\Omega)$ we have $p \in H^2(\Omega), \frac{\partial p}{\partial \nu} \in H^\frac{1}{2}(\partial \Omega)$ and therefore $\hat{w} \in H^\frac{1}{2}(\partial \Omega)$. By (2.40)-(2.43) we get

$$\|\hat{w}_{\epsilon} e^{-\tau \varphi}\|_{H^\frac{1}{2}, r(\partial \Omega)} \leq C_{14} |\tau|^\frac{1}{2} \|ue^{-\tau \varphi}\|_{L^2(\Omega)}.$$
Taking the scalar product of (2.42) with $\hat{w}e^{-2\tau\varphi}$ and using the estimates (2.44), (2.43) we get

$$
\frac{1}{|\tau|} \|
abla \hat{w}e^{-2\tau\varphi}\|_{L^2(\Omega)}^2 + |\tau| \|\hat{w}e^{-2\tau\varphi}\|_{L^2(\Omega)}^2 + \frac{1}{|\tau|} \|\hat{w}e^{-2\tau\varphi}\|_{H^{2,\gamma}(\Gamma)}^2 \leq C_{15} \|ue^{\tau\varphi}\|_{L^2(\Omega)}^2.
$$

From this estimate and a standard duality argument, the statement of Corollary 2.1 follows immediately. □

Consider the following boundary value problem

$$
\frac{\partial a}{\partial \tilde{z}} = 0 \text{ in } \Omega, \quad \frac{\partial d}{\partial \tilde{z}} = 0 \text{ in } \Omega, \quad (a(z)e^A + d(\tilde{z})e^B)|_{\Gamma_0} = \beta.
$$

The existence of such functions $a(z)$ and $d(\tilde{z})$ is given by the following proposition.

**Proposition 2.5.** Let $\alpha \in (0, 1)$, $A$ and $B$ be as in (2.16). If $\beta \in C^{5+\alpha}(\bar{\Gamma}_0)$ the problem (2.46) has at least one solution $(a, d) \in C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega})$ such that

$$
\|(a, d)\|_{C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega})} \leq C_{17}\|eta\|_{C^{5+\alpha}(\bar{\Gamma}_0)}.
$$

If $\beta \in H^{\frac{5}{2}}(\Gamma_0)$, then the problem (2.46) has at least one solution $(a, d) \in H^1(\Omega) \times H^1(\Omega)$ such that

$$
\|(a, d)\|_{H^1(\Omega) \times H^1(\Omega)} \leq C_{18}\|eta\|_{H^{\frac{5}{2}}(\Gamma_0)}.
$$

**Proof.** Let $\tilde{\Omega}$ be a domain in $\mathbb{R}^2$ with smooth boundary such that $\Omega \subset \tilde{\Omega}$ and there exists an open subdomain $\tilde{\Gamma}_0 \subset \partial \tilde{\Omega}$ satisfying $\Gamma_0 \subset \tilde{\Gamma}_0$. Denote $\Gamma^* = \partial \tilde{\Omega} \setminus \tilde{\Gamma}_0$. We extend $A, B$ to $\tilde{\Gamma}_0$ keeping the regularity and we extend $\beta$ to $\tilde{\Gamma}_0$ in such a way that $\|eta\|_{H^{\frac{5}{2}}(\tilde{\Gamma}_0)} \leq C_{19}\|eta\|_{H^{\frac{5}{2}}(\Gamma_0)}$ or $\|eta\|_{C^{5+\alpha}(\tilde{\Gamma}_0)} \leq C_{19}\|eta\|_{C^{5+\alpha}(\Gamma_0)}$ where the constant $C_{19}$ is independent of $\beta$. By the trace theorem there exist a constant $C_{20}$ independent of $\beta$, and a pair $(r, \tilde{r})$ such that $(re^A + \tilde{r}e^B)|_{\tilde{\Gamma}_0} = \beta$ and if $\beta \in H^{\frac{5}{2}}(\tilde{\Gamma}_0)$ then $(r, \tilde{r}) \in H^1(\Omega) \times H^1(\Omega)$ and

$$
\|(r, \tilde{r})\|_{H^1(\Omega) \times H^1(\Omega)} \leq C_{20}\|eta\|_{H^{\frac{5}{2}}(\tilde{\Gamma}_0)}.
$$

Similarly if $\beta \in C^{5+\alpha}(\tilde{\Gamma}_0)$ then $(r, \tilde{r}) \in C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega})$ and

$$
\|(r, \tilde{r})\|_{C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega})} \leq C_{21}\|eta\|_{C^{5+\alpha}(\Gamma_0)}.
$$

Let $f = \frac{\partial r}{\partial \tilde{z}}$ and $\tilde{f} = \frac{\partial \tilde{r}}{\partial \tilde{z}}$. Consider the extremal problem

$$
J_\epsilon(p, \tilde{p}) = \|(p, \tilde{p})\|_{L^2(\bar{\Omega})}^2 + \frac{1}{\epsilon} \|rac{\partial p}{\partial \tilde{z}} - f\|_{L^2(\bar{\Omega})}^2 + \frac{1}{\epsilon} \|rac{\partial \tilde{p}}{\partial \tilde{z}} - \tilde{f}\|_{L^2(\bar{\Omega})}^2 \rightarrow \inf, \quad (p, \tilde{p}) \in \mathcal{K},
$$

where $\mathcal{K} = \{(h_1, h_2) \in L^2(\bar{\Omega}) \times L^2(\bar{\Omega}) | (h_1e^A + h_2e^B)|_{\bar{\Gamma}_0} = 0\}$. Denote the solution to this extremal problem as $(p_\epsilon, \tilde{p}_\epsilon)$. Then

$$
J'_\epsilon(p_\epsilon, \tilde{p}_\epsilon)(\delta, \tilde{\delta}) = 0 \quad \forall (\delta, \tilde{\delta}) \in \mathcal{K}.
$$

Hence

$$
(\delta, \tilde{\delta})_{L^2(\bar{\Omega})} + \frac{1}{\epsilon} \left( \frac{\partial p_\epsilon}{\partial \tilde{z}} - f \right)_{L^2(\bar{\Omega})} + \frac{1}{\epsilon} \left( \frac{\partial \tilde{p}_\epsilon}{\partial \tilde{z}} - \tilde{f} \right)_{L^2(\bar{\Omega})} = 0 \quad \forall (\delta, \tilde{\delta}) \in \mathcal{K}.
$$
Denote $P_\epsilon = -\frac{1}{\epsilon}(\frac{\partial p}{\partial z} - f), \tilde{P}_\epsilon = -\frac{1}{\epsilon}(\frac{\partial \tilde{p}}{\partial z} - \tilde{f})$. From (2.49) we obtain

$$\frac{\partial P_\epsilon}{\partial \bar{z}} = \frac{\partial \tilde{P}_\epsilon}{\partial z} = \bar{p}_\epsilon, \quad \frac{\partial \tilde{P}_\epsilon}{\partial z} = \bar{p}_\epsilon, \quad P_\epsilon|_{\Gamma_\epsilon} = \tilde{P}_\epsilon|_{\Gamma_\epsilon} = 0, (\nu_1 + i\nu_2)P_\epsilon e^B - (\nu_1 - i\nu_2)\tilde{P}_\epsilon e^A)|_{\Gamma_\epsilon} = 0.$$

We claim that there exists a constant $C_{22}$ independent of $\epsilon$ such that

$$\|(P_\epsilon, \tilde{P}_\epsilon)\|_{H^1(\tilde{\Omega})} \leq C_{22}(\|(p_\epsilon, \tilde{p}_\epsilon)\|_{L^2(\tilde{\Omega})} + \|(P_\epsilon, \tilde{P}_\epsilon)\|_{L^2(\tilde{\Omega})}).$$

It clearly suffices to prove the estimate (2.51) locally assuming that $\text{supp } (p_\epsilon, \tilde{p}_\epsilon)$ is in a small neighborhood of zero and the vector $(0, 1)$ is orthogonal to $\partial \Omega$ on the intersection of this neighborhood with the boundary. Using a conformal transformation we may assume that $\partial \Omega \cap \text{supp } P_\epsilon, \partial \Omega \cap \text{supp } \tilde{P}_\epsilon \subset \{x_1 = 0\}$. In order to prove this fact we consider the system of equations

$$\frac{\partial \mathbf{u}}{\partial \bar{z}} + \hat{B} \frac{\partial \mathbf{u}}{\partial x_1} = \mathbf{F}, \quad \text{supp } \mathbf{u} \subset B(0, \delta) \cap \{x_2 \geq 0\}.$$

Here $\mathbf{u} = (u_1, u_2, u_3, u_4) = (\text{Re } P_\epsilon, \text{Im } P_\epsilon, \text{Re } \tilde{P}_\epsilon, \text{Im } \tilde{P}_\epsilon)$, $\mathbf{F} = 2(\text{Re } p_\epsilon, \text{Im } p_\epsilon, \text{Re } \tilde{p}_\epsilon, \text{Im } \tilde{p}_\epsilon)$, $\hat{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. The matrix $\hat{B}$ has two eigenvalues $\pm i$ and four linearly independent eigenvectors:

$$\mathbf{q}_3 = (0, 0, 1, i), \quad \mathbf{q}_4 = (-1, -i, 0, 0) \quad \text{corresponding to the eigenvalue } -i,$$

$$\mathbf{q}_1 = (1, i, 0, 0), \quad \mathbf{q}_2 = (0, 0, 1, -i) \quad \text{corresponding to the eigenvalue } i.$$

We set $\mathbf{r}_1 = (\nu_1 e^B, -\nu_2 e^B, -\nu_1 e^A, -\nu_2 e^A)$, $\mathbf{r}_2 = (\nu_1 e^B, \nu_1 e^B, \nu_2 e^A, -\nu_1 e^A)$. Consider the matrix $D = \{d_{ij}\}$ where $d_{ij} = r_j \cdot q_i$. We have

$$D = \begin{pmatrix} (\nu_1 - i\nu_2)e^B & -(\nu_1 - i\nu_2)e^A \\ (\nu_2 + i\nu_1)e^B & (\nu_2 + i\nu_1)e^A \end{pmatrix}.$$

Since the Lopatinski determinant $\text{det } D \neq 0$ we obtain (2.51) (see e.g. [22]).

Suppose that for any $C$ one can find $\epsilon$ such that the estimate

$$\|(P_\epsilon, \tilde{P}_\epsilon)\|_{H^1(\tilde{\Omega})} \leq \tilde{C}(\|(p_\epsilon, \tilde{p}_\epsilon)\|_{L^2(\tilde{\Omega})})$$

fails. That is, for all $\epsilon \in (0, 1)$, there exist $p_\epsilon, \tilde{p}_\epsilon, P_\epsilon, \tilde{P}_\epsilon, C_\epsilon > 0$ such that $\lim_{\epsilon \to 0} C_\epsilon = \infty$ and

$$\|(P_\epsilon, \tilde{P}_\epsilon)\|_{H^1(\tilde{\Omega})} \leq \tilde{C}_\epsilon(\|(p_\epsilon, \tilde{p}_\epsilon)\|_{L^2(\tilde{\Omega})}).$$

We set $(Q_\epsilon, \tilde{Q}_\epsilon) = (P_\epsilon, \tilde{P}_\epsilon)/\|(P_\epsilon, \tilde{P}_\epsilon)\|_{H^1(\tilde{\Omega})}$ and $(q_\epsilon, \tilde{q}_\epsilon) = (p_\epsilon, \tilde{p}_\epsilon)/\|(P_\epsilon, \tilde{P}_\epsilon)\|_{H^1(\tilde{\Omega})}$. Then $\|(q_\epsilon, \tilde{q}_\epsilon)\|_{L^2(\tilde{\Omega})} \to 0$ as $\epsilon \to 0$. Passing to the limit in (2.50) we have

$$\frac{\partial Q}{\partial \bar{z}} = 0 \quad \text{in } \tilde{\Omega}, \quad \frac{\partial \tilde{Q}}{\partial z} = 0 \quad \text{in } \tilde{\Omega}, \quad Q|_{\Gamma_\epsilon} = \tilde{Q}|_{\Gamma_\epsilon} = 0.$$

By the uniqueness for the Cauchy problem for the operator $\partial_\bar{z}$ we have $Q = \tilde{Q} = 0$. On the other hand, since $\|(Q_\epsilon, \tilde{Q}_\epsilon)\|_{H^1(\tilde{\Omega})} = 1$, we can extract a subsequence, denoted the same, which
is convergent in $L^2(\tilde{\Omega})$. Therefore the sequence $(Q_\epsilon, \tilde{Q}_\epsilon)$ converges to zero in $L^2(\tilde{\Omega})$. By (2.51), we have $1/C_{23} \leq \| (Q_\epsilon, \tilde{Q}_\epsilon) \|_{L^2(\tilde{\Omega})} + \| (Q_\epsilon, \tilde{Q}_\epsilon) \|_{L^2(\tilde{\Omega})}$. Therefore $\liminf_{\epsilon \to 0} \| (Q_\epsilon, \tilde{Q}_\epsilon) \|_{L^2(\tilde{\Omega})} \neq 0$, and this is a contradiction. Hence

$$\| (P_\epsilon, \tilde{P}_\epsilon) \|_{H^1(\tilde{\Omega})} \leq C_{24} \| (P_\epsilon, \tilde{P}_\epsilon) \|_{L^2(\tilde{\Omega})}, \quad \forall \epsilon > 0.$$ 

Let us plug in (2.49) the function $(p_\epsilon, \tilde{p}_\epsilon)$ instead of $(\delta, \hat{\delta})$. Then, by the above inequality, in view of the definitions of $P_\epsilon$ and $\tilde{P}_\epsilon$, we have

$$\| (p_\epsilon, \tilde{p}_\epsilon) \|_{L^2(\tilde{\Omega})} \leq C_{25} \| (f, \tilde{f}) \|_{L^2(\tilde{\Omega})} \leq C_{26} \| (f, \tilde{f}) \|_{L^2(\tilde{\Omega})} \| (P_\epsilon, \tilde{P}_\epsilon) \|_{L^2(\tilde{\Omega})},$$

$$\leq C_{27} \| (f, \tilde{f}) \|_{L^2(\tilde{\Omega})} \| (p_\epsilon, \tilde{p}_\epsilon) \|_{L^2(\tilde{\Omega})}.$$ 

This inequality implies that the sequence $(p_\epsilon, \tilde{p}_\epsilon)$ is bounded in $L^2(\tilde{\Omega})$ and

$$\left( \frac{\partial p_\epsilon}{\partial z}, \frac{\partial \tilde{p}_\epsilon}{\partial \bar{z}} \right) \to (f, \tilde{f}) \quad \text{in } L^2(\tilde{\Omega}) \times L^2(\tilde{\Omega}).$$ 

Then we construct a solution to (2.46) such that

$$\| (p, \tilde{p}) \|_{L^2(\tilde{\Omega})} \leq C_{28} \| (f, \tilde{f}) \|_{L^2(\tilde{\Omega})}.$$ 

Observe that we can write the boundary value problem

$$\frac{\partial p}{\partial z} = f \quad \text{in } \Omega, \quad \frac{\partial \tilde{p}}{\partial \bar{z}} = \tilde{f} \quad \text{in } \Omega, \quad (p e^A + \tilde{p} e^B)|_{\tilde{\Gamma}_0} = 0$$

in the form of (2.52) with $u = (\text{Re } p, \text{Im } p, \text{Re } \tilde{p}, \text{Im } \tilde{p}), F = 2(\text{Re } f, \text{Im } f, \text{Re } \tilde{f}, \text{Im } \tilde{f})$. We set $r_1 = (e^A, -e^A, -e^B, -e^B), r_2 = (e^A, e^A, e^B, -e^B)$. Consider the matrix $D = \{ d_{j\ell} \}$ where $d_{j\ell} = r_j \cdot q_\ell$. We have

$$D = \begin{pmatrix} e^B & -e^A \\ e^B & e^A \end{pmatrix}.$$ 

Since the Lopatinski determinant $\det D \neq 0$ the estimate (2.53) imply (2.47) and (2.48) (see e.g., [22] Theorem 4.1.2.). This completes the proof of the proposition.

Consider the following problem

$$L(x, D)u = f e^{\tau \varphi} \quad \text{in } \Omega, \quad u|_{\Gamma_0} = g e^{\tau \varphi}.$$ 

We have

**Proposition 2.6.** Let $A, B \in C^{5+\alpha}(\overline{\Omega}), q \in L^\infty(\Omega)$ and $\epsilon, \alpha$ be a small positive numbers. There exists $\tau_0 > 0$ such that for all $|\tau| > \tau_0$ there exists a solution to the boundary value problem (2.54) such that

$$\frac{1}{\sqrt{|\tau|}} \| \nabla u e^{-\tau \varphi} \|_{L^2(\Omega)} + \sqrt{|\tau|} \| u e^{-\tau \varphi} \|_{L^2(\Omega)} \leq C_{20}(\| f \|_{L^2(\Omega)} + \| g \|_{H^{\frac{1}{2}}(\Gamma_0)}).$$ 

Let $\epsilon$ be a sufficiently small positive number. If $\text{supp } f \subset G_{\epsilon} = \{ x \in \Omega | \text{dist}(x, H \setminus \Gamma_0) > \epsilon \}$ and $g = 0$ then there exists $\tau_0 > 0$ such that for all $|\tau| > \tau_0$ there exists a solution to the boundary value problem (2.54) such that

$$\| \nabla u e^{-\tau \varphi} \|_{L^2(\Omega)} + |\tau| \| u e^{-\tau \varphi} \|_{L^2(\Omega)} \leq C_{30}(\epsilon) \| f \|_{L^2(\Omega)}.$$
Proof. First we reduce the problem (2.54) to the case $g = 0$. Let $r(z)$ be a holomorphic function and $ar{r}(\bar{z})$ be an antiholomorphic function such that $(e^{A_r} + e^{B_r})|_{\Gamma_0} = g$ where $A, B \in C^{6+\alpha}(\bar{\Omega})$ are defined as in (2.16). The existence of such functions $r, \bar{r}$ follows from Proposition 2.5, and these functions can be chosen in such a way that
\[
\|r\|_{H^1(\Omega)} + \|\bar{r}\|_{H^1(\Omega)} \leq C_{31}\|g\|_{H^{\frac{1}{2}}(\Gamma_0)}.
\]

We look for a solution $u$ in the form
\[
u = (e^{A+r\Phi} + e^{B+r\bar{\Phi}}) + \tilde{u},
\]
where
\[
\begin{align}
L(x, D)\tilde{u} &= \tilde{f}e^{r\varphi} \quad \text{in } \Omega, \quad \tilde{u}|_{\Gamma_0} = 0
\end{align}
\]
and \(\tilde{f} = f - (q - 2\frac{\partial A}{\partial r} - AB)e^{A_r}e^{tr\psi} - (q - 2\frac{\partial B}{\partial r} - AB)e^{B_r}e^{-r\psi} \).

In order to prove (2.55) we consider the following extremal problem:
\[
\begin{align}
\tilde{I}(u) &= \frac{1}{2}||ue^{-r\varphi}||^2_{H^1(\Omega)} + \frac{1}{2}||L(x, D)u - \tilde{f}e^{r\varphi}||^2_{L^2(\Omega)} + \frac{1}{2}||ue^{-r\varphi}||^2_{H^{\frac{1}{2}}(\tilde{\Gamma})} \to \inf,
\end{align}
\]
\[
\begin{align}
u &\in \mathcal{Y} = \{w \in H^1(\Omega)|w|_{\Gamma_0} = 0, L(x, D)w \in L^2(\Omega)\}.
\end{align}
\]
There exists a unique solution to problem (2.58), (2.59) which we denote as \(\hat{u}_e\). By Fermat’s theorem
\[
\tilde{I}'(\hat{u}_e)[\delta] = 0 \quad \forall \delta \in \mathcal{Y}.
\]
Let \(p_e = \frac{1}{\epsilon}(L(x, D)\hat{u}_e - \tilde{f}e^{r\varphi})\). Applying Corollary 2.1 we obtain from (2.60)
\[
\frac{1}{|\tau|}||p_e e^{-r\varphi}||^2_{L^2(\Omega)} \leq C_{32}(||\hat{u}_e e^{-r\varphi}||^2_{H^1(\Omega)} + ||\hat{u}_e e^{-r\varphi}||^2_{H^{\frac{1}{2}}(\tilde{\Gamma})}) \leq 2C_{32}\tilde{I}(\hat{u}_e).
\]
Substituting in (2.60) with \(\delta = \hat{u}_e \) we obtain
\[
2\tilde{I}(\hat{u}_e) + (\tilde{f}e^{r\varphi}, p_e)_{L^2(\Omega)} = 0.
\]
Applying to this equality estimate (2.61) we have
\[
\tilde{I}(\hat{u}_e) \leq C_4||\tilde{f}||^2_{L^2(\Omega)}.
\]
Using this estimate we pass to the limit as $\epsilon \to +0$. We obtain
\[
L(x, D)u - \tilde{f}e^{r\varphi} = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0,
\]
and
\[
||ue^{-r\varphi}||^2_{H^1(\Omega)} + ||ue^{-r\varphi}||^2_{L^2(\Omega)} \leq C_{33}||\tilde{f}||^2_{L^2(\Omega)}.
\]
Since \(\|\tilde{f}\|_{L^2(\Omega)} \leq C_{34}(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma_0)})\), inequality (2.63) implies (2.55). In order to prove (2.56) we consider the following extremal problem
\[
\begin{align}
\tilde{J}(u) &= \frac{1}{2}||ue^{-r\varphi}||^2_{L^2(\Omega)} + \frac{1}{2}||L(x, D)u - f e^{r\varphi}||^2_{L^2(\Omega)} + \frac{1}{2||\tau||}||ue^{-r\varphi}||^2_{L^2(\tilde{\Gamma})} \to \inf,
\end{align}
\]
\[
\begin{align}u &\in \bar{\mathcal{X}} = \{w \in H^{\frac{1}{2}}(\Omega)|w|_{\Gamma_0} = 0, L(x, D)w \in L^2(\Omega)\}.
\end{align}
\]
There exists a unique solution to problem (2.64), (2.65) which we denote as \( \hat{u}_\epsilon \). By Fermat’s theorem
\[
\tilde{J}_\epsilon(\hat{u}_\epsilon)[\delta] = 0 \quad \forall \delta \in \tilde{X}.
\]
This equality implies
\[
L(x, D)^*p_\epsilon + \hat{u}_\epsilon e^{-2\tau \varphi} = 0 \quad \text{in } \Omega, \quad \tilde{p}_\epsilon|_{\partial \Omega} = 0, \quad \frac{\partial p_\epsilon}{\partial \nu}|_{\Gamma} = \frac{\hat{u}_\epsilon}{|\tau|} e^{-2\tau \varphi}.
\]
By Proposition 2.4
\[
\frac{1}{|\tau|} \| p_\epsilon e^{\gamma \varphi} \|^2_{H^{1,\gamma}(\Omega)} + \| \frac{\partial p_\epsilon}{\partial \nu} e^{\gamma \varphi} \|^2_{L^2(\Gamma_0)} + \tau^2 \| \frac{\partial \Phi}{\partial z} p_\epsilon e^{\gamma \varphi} \|^2_{L^2(\Omega)} \leq 2 \tilde{J}_\epsilon(\hat{u}_\epsilon).
\]
(2.67)
Taking the scalar product of equation (2.66) with \( \hat{u}_\epsilon \) we obtain
\[
2 \tilde{J}_\epsilon(\hat{u}_\epsilon) + (fe^{\gamma \varphi}, p_\epsilon)_{L^2(\Omega)} = 0.
\]
(2.68)
Applying to this equality estimate (2.67) we have
\[
|\tau|^2 \tilde{J}_\epsilon(\hat{u}_\epsilon) \leq C_\epsilon \| f \|^2_{L^2(\Omega)}.
\]
(2.69)
Using this estimate we pass to the limit in (2.66). We obtain that
\[
L(x, D)^*p + ue^{-2\tau \varphi} = 0 \quad \text{in } \Omega, \quad p|_{\partial \Omega} = 0, \quad \frac{\partial p}{\partial \nu}|_{\Gamma} = \frac{u}{|\tau|} e^{-2\tau \varphi},
\]
(2.70)
Moreover (2.68) implies
\[
|\tau|^2 \| u e^{-\gamma \varphi} \|^2_{L^2(\Omega)} + \| u e^{-\gamma \varphi} \|^2_{L^2(\Gamma)} \leq C_\epsilon \| f \|^2_{L^2(\Omega)}.
\]
(2.71)
This finishes the proof of the proposition. \( \square \)

### 3. Estimates and Asymptotics

In this section we prove some estimates and obtain asymptotic expansions needed in the construction of the complex geometrical optics solutions in Section 4.

Consider the operator
\[
L_1(x, D) = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} + 2A_1 \frac{\partial}{\partial z} + 2B_1 \frac{\partial}{\partial \bar{z}} + q_1 =
\]
\[
(2 \frac{\partial}{\partial z} + B_1)(2 \frac{\partial}{\partial \bar{z}} + A_1) + q_1 - 2 \frac{\partial A_1}{\partial z} - A_1B_1 =
\]
(3.1)
Let \( A_1, B_1, A_2, B_2 \in C^{6+\alpha}(\overline{\Omega}) \) with some \( \alpha \in (0, 1) \) satisfy
\[
2 \frac{\partial A_1}{\partial \bar{z}} = -A_1 \quad \text{in } \Omega, \quad \text{Im} A_1|_{\Gamma_0} = 0, \quad 2 \frac{\partial B_1}{\partial z} = -B_1 \quad \text{in } \Omega, \quad \text{Im} B_1|_{\Gamma_0} = 0
\]
(3.2)
Let \( \tau, A, B \) be holomorphic functions and \( \Phi(z) \) be an antiholomorphic function, we have

\[
L_1(x, D)(e^{A_1}e^{r\Phi}) = (q_1 - 2\partial_{\xi_1}^A - A_1B_1)e^{A_1}e^{r\Phi},
\]

\[
L_1(x, D)(e^{B_1}e^{r\Phi}) = (q_1 - 2\partial_{\xi_1}^B - A_1B_1)e^{B_1}e^{r\Phi}.
\]

Let us introduce the operators:

\[
\partial_{\xi}^{-1}g = \frac{1}{2\pi i} \int_{\Omega} g(\xi_1, \xi_2) d\xi_1 d\xi_2,
\]

\[
\partial_{\zeta}^{-1}g = \frac{1}{2\pi i} \int_{\Omega} \frac{g(\xi_1, \xi_2)}{\zeta - z} d\xi_1 d\xi_2.
\]

We have (e.g., p.47, 56, 72 in [21]):

**Proposition 3.1. A)** Let \( m \geq 0 \) be an integer number and \( \alpha \in (0, 1) \). The operators \( \partial_{\xi}^{-1}, \partial_{\zeta}^{-1} \in L(C^{m+\alpha}(\Omega)) \).

**B)** Let \( 1 \leq p \leq 2 \) and \( 1 < \gamma < \frac{2p}{2-p} \). Then \( \partial_{\xi}^{-1}, \partial_{\zeta}^{-1} \in L(L^p(\Omega), L^\gamma(\Omega)) \).

**C)** Let \( 1 < p < \infty \). Then \( \partial_{\xi}^{-1}, \partial_{\zeta}^{-1} \in L(L^p(\Omega), W^1_p(\Omega)) \).

Assume that \( \mathcal{A}, \mathcal{B} \) satisfy (2.16). Setting \( T_Bg = e^{B}\partial_{\zeta}^{-1}(e^{-B}g) \) and \( P_Ag = e^{A}\partial_{\zeta}^{-1}(e^{-A}g) \), we have

\[
(2\partial_{\xi} + B)T_Bg = g \quad \text{in} \quad \Omega, \quad (2\partial_{\xi} + A)P_Ag = g \quad \text{in} \quad \Omega.
\]

We define two other operators:

\[
(3.3) \quad \mathcal{R}_{r,A}g = \frac{1}{2}e^{A}e^{-r(\Phi - \Phi)}\partial_{\zeta}^{-1}(ge^{-A}e^{r(\Phi - \Phi)}), \quad \mathcal{R}_{r,B}g = \frac{1}{2}e^{B}e^{-r(\Phi - \Phi)}\partial_{\zeta}^{-1}(ge^{-B}e^{r(\Phi - \Phi)}).
\]

The following proposition follows from straightforward calculations.

**Proposition 3.2.** Let \( g \in C^\alpha(\overline{\Omega}) \) for some positive \( \alpha \). The function \( \mathcal{R}_{r,A}g \) is a solution to

\[
(3.4) \quad 2\partial_{\zeta} \mathcal{R}_{r,A}g - 2r\partial_{\zeta}^\Phi \mathcal{R}_{r,A}g + AR_{r,A}g = g \quad \text{in} \quad \Omega.
\]

The function \( \mathcal{R}_{r,B}g \) solves

\[
(3.5) \quad 2\partial_{\zeta} \mathcal{R}_{r,B}g + 2r\partial_{\zeta}^\Phi \mathcal{R}_{r,B}g + BR_{r,B}g = g \quad \text{in} \quad \Omega.
\]

We have

**Proposition 3.3.** Let \( g \in C^2(\Omega) \), \( g|_{\partial} = 0 \) and \( g|_{\mathcal{N}} = 0 \). Then for any \( 1 \leq p < \infty \)

\[
(3.6) \quad \left\| \mathcal{R}_{r,A}g + \frac{g}{2r\partial_{\zeta}^\Phi} \right\|_{L^p(\Omega)} + \left\| \mathcal{R}_{r,B}g - \frac{g}{2r\partial_{\zeta}^\Phi} \right\|_{L^p(\Omega)} = o\left(\frac{1}{|r|}\right) \quad \text{as} \quad |r| \rightarrow +\infty.
\]
Proof. We give a proof of the asymptotic formula for \( \tilde{R}_{\tau,B}g \). The proof for the \( R_{\tau,A}g \) is similar. Let \( \tilde{g}(\zeta,\bar{\zeta}) = ge^{-B} \). Then

\[
e^{-B}R_{\tau,B}g = -\frac{e^{\tau(\bar{\Phi}-\Phi)}}{\pi} \int_{\Omega} \frac{g(\zeta,\bar{\zeta})}{\bar{\zeta} - z} e^{\tau(\Phi(\zeta) - \Phi(z))} d\xi_1d\xi_2
\]

\[
= -\frac{e^{\tau(\bar{\Phi}-\Phi)}}{\pi} \lim_{\delta \to +0} \int_{\Omega \setminus B(z,\delta)} \frac{g(\zeta,\bar{\zeta})}{\bar{\zeta} - z} e^{\tau(\Phi(\zeta) - \Phi(z))} d\xi_1d\xi_2.
\]

Let \( z = x_1 + ix_2 \) and \((x_1, x_2)\) be not a critical point of the function \( \Phi \). Then

\[
e^{-B}R_{\tau,B}g = -\frac{e^{\tau(\bar{\Phi}-\Phi)}}{\pi \tau} \lim_{\delta \to +0} \int_{\Omega \setminus B(z,\delta)} \frac{1}{\bar{\zeta} - z} \frac{\partial}{\partial \zeta} \left( \frac{\tilde{g}(\zeta,\bar{\zeta})}{\partial \Phi(\zeta)} \right) e^{\tau(\Phi(\zeta) - \Phi(z))} d\xi_1d\xi_2
\]

\[
- \frac{e^{\tau(\bar{\Phi}-\Phi)}}{\pi \tau} \lim_{\delta \to +0} \int_{S(z,\delta)} \frac{\tilde{g}(\zeta,\bar{\zeta})(\overline{\nu_1 - i\nu_2})}{\bar{\zeta} - z} 2\partial \Phi(\zeta) e^{\tau(\Phi(\zeta) - \Phi(z))} d\xi_1d\xi_2.
\]

Since \( \tilde{g}|_{\mathcal{H}} = 0 \), we have

\[
(3.7) \quad \left| \frac{\partial}{\partial \zeta} \left( \frac{\tilde{g}(\zeta,\bar{\zeta})}{\partial \Phi(\zeta)} \right) \right| 
\leq C \sum_{k=1}^{\ell} \| \tilde{g} \|_{C^1(B)} \quad \in L^p(\Omega) \quad \forall p \in (1,2).
\]

Hence

\[
e^{-B}R_{\tau,B}g = \frac{e^{\tau(\bar{\Phi}-\Phi)}}{\pi \tau} \int_{\Omega} \frac{1}{\bar{\zeta} - z} \frac{\partial}{\partial \zeta} \left( \frac{\tilde{g}(\zeta,\bar{\zeta})}{\partial \Phi(\zeta)} \right) e^{\tau(\Phi(\zeta) - \Phi(z))} d\xi_1d\xi_2 - \frac{\tilde{g}(z,\bar{z})}{\tau \partial \Phi(z)}.
\]

Denote \( G_\tau(x) = \int_{\Omega} \frac{1}{\bar{\zeta} - z} \frac{\partial}{\partial \zeta} \left( \frac{\tilde{g}(\zeta,\bar{\zeta})}{\partial \Phi(\zeta)} \right) e^{\tau(\Phi(\zeta) - \Phi(z))} d\xi_1d\xi_2 \). By the stationary phase argument, we see that

\[
(3.8) \quad G_\tau(x) \to 0 \quad as \quad |\tau| \to +\infty \quad \forall x \in \overline{\Omega}.
\]

Denote

\[
F(\xi_1, \xi_2) = \left| \frac{\partial}{\partial \zeta} \tilde{g}(\zeta, \bar{\zeta}) \right|.
\]

Clearly

\[
(3.9) \quad |G_\tau(x)| \leq \int_{\Omega} \frac{|F(\xi_1, \xi_2)|}{|z - \zeta|} d\xi_1d\xi_2 \quad a.e. \ in \ \Omega \ \forall \tau.
\]

By (3.7) \( F \) belongs to \( L^p(\Omega) \) for any \( p \in (1,2) \). For \( f \in L^p(\mathbb{R}^2) \), we set

\[
I_r f(z) = \int_{\mathbb{R}^2} |z - \zeta|^{-\frac{2}{r}} f(\zeta, \bar{\zeta}) d\xi_1d\xi_2.
\]

Then, by the Hardy-Littlewood-Sobolev inequality, if \( r > 1 \) and \( \frac{1}{r} = 1 - \left( \frac{1}{p} - \frac{1}{q} \right) \) for \( 1 < p < q < \infty \), then

\[
\| I_r f \|_{L^q(\mathbb{R}^2)} \leq C_{p,q} \| f \|_{L^p(\mathbb{R}^2)}.
\]
Set \( r = 2 \). Then we have to choose \( \frac{1}{p} - \frac{1}{q} = \frac{1}{2} \), that is, we can arbitrarily choose \( p > 2 \) close to 2, so that \( q \) is arbitrarily large. Hence \( \int_\Omega \frac{F}{|z|^q} \, d\xi_1 d\xi_2 \) belongs to \( L^q(\Omega) \) with positive \( q \). By (3.8), (3.9) and the dominated convergence theorem

\[
G_\tau \to 0 \quad \text{in} \quad L^q(\Omega) \quad \forall q \in (1, \infty).
\]
The proof of the proposition is finished. \( \Box \)

Using the stationary phase argument (e.g., Bleistein and Handelsman [2]), we will show

**Proposition 3.4.** Let \( g \in L^1(\Omega) \) and a function \( \Phi \) satisfy (2.1), (2.2). Then

\[
\lim_{|\tau| \to +\infty} \int_\Omega g e^{\tau(\Phi(z) - \overline{\Phi(z)})} \, dx = 0.
\]

**Proof.** Let \( \{g_k\}_{k=1}^\infty \subset C_0^\infty(\Omega) \) be a sequence of functions such that \( g_k \to g \) in \( L^1(\Omega) \). Let \( \epsilon > 0 \) be an arbitrary number. Suppose that \( j \) is large enough such that \( \|g - g_j\|_{L^1(\Omega)} \leq \frac{\epsilon}{2} \). Then

\[
|\int_\Omega g e^{\tau(\Phi(z) - \overline{\Phi(z)})} \, dx| \leq |\int_\Omega (g - g_j) e^{\tau(\Phi(z) - \overline{\Phi(z)})} \, dx| + |\int_\Omega g_j e^{\tau(\Phi(z) - \overline{\Phi(z)})} \, dx|.
\]
The first term on the right-hand side of this inequality is less then \( \epsilon/2 \) and the second goes to zero as \( |\tau| \) approaches to infinity by the stationary phase argument (see e.g. [2]). \( \Box \)

We now consider the contribution from the critical points.

**Proposition 3.5.** Let \( \Phi \) satisfy (2.1) and (2.2). Let \( g \in C^{1+\alpha}(\overline{\Omega}) \) for some \( \alpha > 0 \), \( g|_{\partial \Omega} = 0 \) and \( g|_{\mathcal{H}} = 0 \). Then there exist constants \( p_k \) such that

\[
(3.10) \quad \int_\Omega g e^{\tau(\Phi(z) - \overline{\Phi(z)})} \, dx = \frac{1}{\tau^2} \sum_{k=1}^\ell p_k e^{2i\tau \psi(\bar{x}_k)} + o\left( \frac{1}{\tau^2} \right) \quad \text{as} \quad |\tau| \to +\infty.
\]

**Proof.** Let \( \delta > 0 \) be a sufficiently small number and \( \bar{e}_k \in C^\infty_0(B(\bar{x}_k, \delta)), \bar{e}_k|_{B(\bar{x}_k, \delta/2)} \equiv 1 \). By the stationary phase argument

\[
I(\tau) = \int_\Omega g e^{\tau(\Phi - \overline{\Phi})} \, dx = \sum_{k=1}^\ell \int_{B(\bar{x}_k, \delta)} \bar{e}_k g e^{\tau(\Phi - \overline{\Phi})} \, dx + o\left( \frac{1}{\tau^2} \right) =
\]

\[
\sum_{k=1}^\ell e^{2i\tau \psi(\bar{x}_k)} \int_{B(\bar{x}_k, \delta)} \bar{e}_k g e^{\tau(\Phi - \overline{\Phi}) - 2i\tau \psi(\bar{x}_k)} \, dx + o\left( \frac{1}{\tau^2} \right) \quad \text{as} \quad |\tau| \to +\infty.
\]

Since all the critical points of \( \Phi \) are nondegenerate, in some neighborhood of \( \bar{x}_k \) one can take local coordinates such that \( \Phi - \overline{\Phi} - 2i \tau \psi(\bar{x}_k) = z^2 - \bar{z}^2 \). Therefore

\[
I(\tau) = \sum_{k=1}^\ell e^{2i\tau \psi(\bar{x}_k)} \int_{B(0, \delta')} q_k e^{(z^2 - \bar{z}^2)} \, dx + o\left( \frac{1}{\tau^2} \right) \quad \text{as} \quad |\tau| \to +\infty,
\]
where \( q_k \in C^1_0(B(0, \delta')) \) and \( q_k(0) = 0 \). Hence there exist functions \( r_{1,k}, r_{2,k} \in C^3_0(B(0, \delta')) \) such that \( q_k = 2zr_{1,k} + 2\pi r_{2,k} \). Integrating by parts, one can decompose \( I(\tau) \) as

\[
I(\tau) = -\frac{1}{\tau} \sum_{k=1}^\ell e^{2\tau\psi(\overline{x}_k)} \int_{B(0, \delta')} \left( \frac{\partial r_{1,k}}{\partial z} - \frac{\partial r_{2,k}}{\partial z} \right) e^{\tau(z^2 - \overline{\tau}^2)} dx + o\left( \frac{1}{\tau^2} \right) =
\]

\[
-\frac{1}{\tau} \sum_{k=1}^\ell e^{2\tau\psi(\overline{x}_k)} \int_{B(0, \delta')} \left( \frac{\partial r_{1,k}}{\partial z} - \frac{\partial r_{2,k}}{\partial z} \right) (0) \chi(x) e^{\tau(z^2 - \overline{\tau}^2)} dx + o\left( \frac{1}{\tau^2} \right) =
\]

\[
-\frac{1}{\tau} \sum_{k=1}^\ell e^{2\tau\psi(\overline{x}_k)} \int_{B(0, \delta')} \overline{q}_k e^{\tau(z^2 - \overline{\tau}^2)} dx + o\left( \frac{1}{\tau^2} \right) \quad \text{as} \quad |\tau| \to +\infty,
\]

where \( \chi, \overline{q}_k \in C^2_0(B(0, \delta')) \), \( \chi|_{B(0, \delta'/2)} \equiv 1 \) and \( \overline{q}_k(0) = 0 \). Hence there exist functions \( \overline{r}_{1,k}, \overline{r}_{2,k} \in C^3_0(B(0, \delta')) \) such that \( \overline{q}_k = 2z\overline{r}_{1,k} + 2\pi \overline{r}_{2,k} \). Integrating by parts and applying Proposition 3.3 we obtain

\[
\lim_{|\tau| \to +\infty} \tau \int_{B(0, \delta')} \overline{q}_k e^{\tau(z^2 - \overline{\tau}^2)} dx = -\frac{1}{\tau} \sum_{k=1}^\ell e^{2\tau\psi(\overline{x}_k)} \int_{B(0, \delta')} \left( \frac{\partial \overline{r}_{1,k}}{\partial z} - \frac{\partial \overline{r}_{2,k}}{\partial z} \right) e^{\tau(z^2 - \overline{\tau}^2)} dx = 0.
\]

Therefore (3.10) follows from a standard application of stationary phase. The proof of the proposition is completed. \( \square \)

**Proposition 3.6.** Let \( 0 < \epsilon' < \epsilon \), a function \( \Phi \) satisfy (2.1), (2.2) and \( \overline{\sigma}_\epsilon \cap (\mathcal{H} \setminus \Gamma_0) = \emptyset \). Suppose that \( g \in C^\alpha(\overline{\Omega}) \) for some \( \alpha > 0 \), \( g|_{\partial \Omega} = 0 \) and \( g|_{\mathcal{H}} = 0 \). Then

\[
(3.11) \quad \left| \tau \right| \left\| \nabla \overline{R}_{\tau,B} g \right\|_{L^\infty(\mathcal{O}_{\epsilon'})} + \left\| \nabla \nabla \overline{R}_{\tau,B} g \right\|_{L^\infty(\mathcal{O}_{\epsilon'})} \leq C_1(\epsilon', \alpha) \left\| g \right\|_{C^\alpha(\overline{\Omega}) \cap H^1(\Omega)}.
\]

Moreover

\[
(3.12) \quad \left\| \nabla \nabla \overline{R}_{\tau,B} g \right\|_{L^2(\Omega)} \leq C_2(\epsilon', \alpha) \left\| g \right\|_{C^\alpha(\overline{\Omega}) \cap H^1(\Omega)}.
\]

**Proof.** Denote \( \overline{g} = \overline{g} e^{-K} \). Let \( x = (x_1, x_2) \) be an arbitrary point from \( \mathcal{O}_{\epsilon'} \) and \( z = x_1 + ix_2 \). Then

\[
-\pi \partial_z^{-1}(e^{\tau(\overline{\Phi} - \overline{\Phi})} \overline{g}) = \int_{\Omega} \frac{\overline{g} e^{\tau(\overline{\Phi} - \overline{\Phi})}}{\zeta - z} d\xi_1 d\xi_2 = \lim_{\delta \to 0} \sum_{k=1}^\ell \int_{\Omega \setminus B(\overline{x}_k, \delta)} \overline{g} e^{\tau(\overline{\Phi} - \overline{\Phi})} \frac{\partial^2 \Phi}{\partial z^2} d\xi_1 d\xi_2.
\]

Integrating by parts and taking \( \delta \) sufficiently small we have

\[
-\pi \partial_z^{-1}(e^{\tau(\overline{\Phi} - \overline{\Phi})} \overline{g}) = -\frac{1}{\tau} \lim_{\delta \to 0} \int_{\Omega \setminus \bigcup_{k=1}^\ell B(\overline{x}_k, \delta)} \frac{\partial^2 \Phi}{\partial z^2} e^{\tau(\overline{\Phi} - \overline{\Phi})} d\xi_1 d\xi_2
\]

\[
+\frac{1}{\tau} \lim_{\delta \to 0} \int_{\Omega \setminus \bigcup_{k=1}^\ell B(\overline{x}_k, \delta)} \frac{\overline{g} e^{\tau(\overline{\Phi} - \overline{\Phi})}}{(\zeta - z)^2} d\xi_1 d\xi_2
\]

\[
+ \frac{1}{2\tau} \lim_{\delta \to 0} \int_{\bigcup_{k=1}^\ell \mathcal{S}(\overline{x}_k, \delta)} (\overline{v}_1 - \overline{v}_2) \frac{\overline{g} e^{\tau(\overline{\Phi} - \overline{\Phi})}}{(\zeta - z)^2} d\sigma.
\]
Since \( g|_\mathcal{H} = 0 \) and all the critical points of \( \Phi \) are nondegenerate we have that \( \| g \|_{C^0(S(\bar{x}_k, \delta))} \leq \delta^n \| g \|_{C^n(\bar{\Omega})} \). Therefore

\[
\frac{1}{2\tau} \lim_{\delta \to 0} \int_{U_{\delta} \setminus S(\bar{x}_k, \delta)} (\bar{\nu}_1 - i\bar{\nu}_2) \frac{\tilde{g}}{(\zeta - \bar{\tau})^{\frac{n}{2}}} e^{r(\Phi - \bar{\tau})} d\sigma = 0.
\]

Since \( \left| \frac{\partial^2 g}{\partial \bar{\tau}^2} (\zeta, \bar{\tau}) \right| \leq C_3 \| \tilde{g} \|_{C^\alpha(\bar{\Omega})} \sum_{k=1}^6 \frac{1}{|\zeta - x_k|^{1-\alpha}} \) we see that \( \frac{\partial^2 g}{\partial \bar{\tau}^2} (\zeta, \bar{\tau}) \in L^1(\Omega) \) and

\[
-\pi \partial_z^{-1}(e^{r(\Phi - \bar{\tau})}\tilde{g}) = -\frac{1}{\tau} \int_{\Omega} \frac{\partial \tilde{g}}{\partial \bar{\tau}} e^{r(\Phi - \bar{\tau})} d\xi_1 d\xi_2 + \frac{1}{\tau} \int_{\Omega} \frac{\tilde{g}}{(\zeta - \bar{\tau})^{\frac{n}{2}}} e^{r(\Phi - \bar{\tau})} d\xi_1 d\xi_2.
\]

(3.14)

From this equality and definition (3.3) of the operator \( \tilde{\mathcal{R}}_{\tau,B} \), the estimate (3.11) follows immediately. To prove (3.12) we observe

\[
\frac{\partial \tilde{\mathcal{R}}_{\tau,B} g}{\partial \bar{\tau}} = \frac{\partial \mathcal{B}}{\partial \bar{\tau}} \tilde{\mathcal{R}}_{\tau,B} g + \tilde{\mathcal{R}}_{\tau,B} \left\{ \frac{\partial g}{\partial \bar{\tau}} - \frac{\partial \mathcal{B}}{\partial \bar{\tau}} g \right\} + \frac{\tau}{2\pi} e^{r(\Phi - \bar{\tau})} + \mathcal{B} \int_{\Omega} \frac{\partial \Phi}{\partial \bar{\tau}} - \frac{\partial \bar{\tau}}{\partial \bar{\tau}} \tilde{g} e^{r(\Phi - \bar{\tau})} d\xi_1 d\xi_2.
\]

By Proposition 3.1

\[
\| \frac{\partial \mathcal{B}}{\partial \bar{\tau}} \tilde{\mathcal{R}}_{\tau,B} g + \tilde{\mathcal{R}}_{\tau,B} \left\{ \frac{\partial g}{\partial \bar{\tau}} - \frac{\partial \mathcal{B}}{\partial \bar{\tau}} g \right\} \|_{L^2(\Omega)} \leq C_4 \| g \|_{H^1(\Omega)}.
\]

Using arguments similar to (3.13), (3.14) we obtain

\[
\| \frac{\tau}{2\pi} \int_{\Omega} \frac{\partial \Phi}{\partial \bar{\tau}} - \frac{\partial \bar{\tau}}{\partial \bar{\tau}} \tilde{g} e^{r(\Phi - \bar{\tau})} d\xi_1 d\xi_2 \|_{L^2(\Omega)} \leq C_5 \| g \|_{C^n(\bar{\Omega}) \cap H^1(\Omega)}.
\]

Hence

\[
\| \frac{\partial \tilde{\mathcal{R}}_{\tau,B} g}{\partial \bar{\tau}} \|_{L^2(\Omega)} \leq C_6 \| g \|_{C^n(\bar{\Omega}) \cap H^1(\Omega)}.
\]

Combining this estimate with (3.11) we conclude

\[
\| \nabla \tilde{\mathcal{R}}_{\tau,B} g \|_{L^2(\Omega)} \leq C_7 \| g \|_{C^n(\bar{\Omega}) \cap H^1(\Omega)}.
\]

Using this estimate and equation (3.4) we have

\[
|\tau| \left\| \frac{\partial \Phi}{\partial \bar{\tau}} \tilde{\mathcal{R}}_{\tau,B} g \right\|_{L^2(\Omega)} \leq C_8 \| g \|_{C^n(\bar{\Omega}) \cap H^1(\Omega)}
\]

finishing the proof of the proposition. \(\square\)

Let \( e_1, e_2 \in C^\infty(\bar{\Omega}) \) be functions such that

(3.15)

\[
e_1 + e_2 = 1 \quad \text{in} \ \Omega,
\]

\( e_2 \) vanishes in some neighborhood of \( \mathcal{H} \setminus \Gamma_0 \) and \( e_1 \) vanishes in a neighborhood of \( \partial \Omega \).
**Proposition 3.7.** Let for some $\alpha \in (0, 1)$ $A, B \in C^{5+\alpha}(\Omega)$, and the functions $A, B \in C^{6+\alpha}(\Omega)$ satisfy (2.16). Let $e_1, e_2$ be defined as in (3.15). Let $g \in L^p(\Omega)$ for some $p > 2$, $\text{supp} \ g \subset \text{supp} \ e_1$. We define $u$ by

$$u = \tilde{R}_{r, B}(e_1(P_A g - \tilde{M} e^A)) + \frac{e_2(P_A g - \tilde{M} e^A)}{2\tau \partial_z \Phi},$$

where $\tilde{M} = \tilde{M}(z)$ is a polynomial such that $(P_A g - \tilde{M} e^A)|_{H} = 0$ and $\frac{\partial^k}{\partial z^k}(P_A g - \tilde{M} e^A)|_{H} = 0$ for any $k$ from $\{1, \ldots, 4\}$. Then we have

$$(3.16) \quad \mathcal{P}(x, D)(ue^{\tau \Phi}) \triangleq (2 \frac{\partial}{\partial z} + A)(2 \frac{\partial}{\partial z} + B)(ue^{\tau \Phi}) = ge^{\tau \Phi} + \frac{e^{\tau \Phi}}{\tau} |\tau| h_\tau \quad \text{as} \quad |\tau| \to +\infty,$$

where

$$\|h_\tau\|_{L^\infty(\Omega)} \leq C_9(p) \|g\|_{L^p(\Omega)}$$

and

$$(3.17) \quad \frac{1}{|\tau|^\frac{1}{2}} \|\nabla u\|_{L^2(\Omega)} + |\tau|^\frac{1}{2} \|u\|_{L^2(\Omega)} + \|u\|_{H^{1, \tau}(\Omega, \tau)} \leq C_{10} \|g\|_{L^p(\Omega)}.$$

**Proof.** By Proposition 3.1 $P_A g$ belongs to $W^1_p(\Omega)$. Since $p > 2$, by the Sobolev embedding theorem there exists $\alpha > 0$ such that $P_A g \in C^\alpha(\Omega)$. By properties of elliptic operators and the fact that $\text{supp} \ e_2 \cap \text{supp} \ g = \emptyset$ we have that $P_A g \in C^5(\text{supp} \ e_2)$. The estimate (3.17) follows from Proposition 3.6. Short calculations give

$$(3.18) \quad \mathcal{P}(x, D)(ue^{\tau \Phi}) = ge^{\tau \Phi} + \frac{e^{\tau \Phi}}{\tau} \mathcal{P}(x, D) \left( \frac{e_2(P_A g - \tilde{M} e^A)}{2\partial_z \Phi} \right).$$

This formula implies (3.16) with $h_\tau = e^{\tau \psi} \mathcal{P}(x, D) \left( \frac{e_2(P_A g - \tilde{M} e^A)}{2\partial_z \Phi} \right) / \text{sign} \tau$. \hfill \Box

The following proposition will play a critically important role in the construction of the complex geometric optic solutions.

**Proposition 3.8.** Let $f \in L^p(\Omega)$ for some $p > 2$, $\epsilon'$ be a small positive number such that $O_{\epsilon'} \cap (\mathcal{H} \setminus \Gamma_0) = \emptyset$. Then there exists $\tau_0$ such that for all $|\tau| > \tau_0$ there exists a solution to the boundary value problem

$$(3.19) \quad L(x, D)w = f e^{\tau \Phi} \quad \text{in} \quad \Omega, \quad w|_{\Gamma_0} = q e^{\tau \psi}/\tau$$

such that

$$\sqrt{|\tau|} \|we^{-\tau \psi}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}} \|\nabla we^{-\tau \psi}\|_{L^2(\Omega)} + \|we^{-\tau \psi}\|_{H^{1, \tau}(\Omega, \tau)} \leq C_{11}(\|f\|_{L^p(\Omega)} + \|q\|_{H^{\frac{1}{2}}(\Gamma_0)}).$$

**Proof.** Let $\chi \in C_0^\infty(\Omega)$ be equal to one in some neighborhood of the set $\mathcal{H} \setminus \Gamma_0$. By Proposition 2.6 there exists a solution to the problem (3.19) with inhomogeneous term $(1 - \chi)f$ and boundary data $q/\tau$ such that

$$(3.20) \quad \|w_1 e^{-\tau \psi}\|_{H^{1, \tau}(\Omega)} \leq C_{12}(\|f\|_{L^2(\Omega)} + \|q\|_{H^{\frac{1}{2}}(\Gamma_0)}).$$

Denote $w_2 = \tilde{R}_{r, B}(e_1(\chi f) - \tilde{M} e^A)) + \frac{e_2(\chi f - \tilde{M} e^A)}{2\tau \partial_z \Phi}$ where $\tilde{M} = \tilde{M}(z)$ is a polynomial such that $(P_A g - \tilde{M} e^A)|_{H} = 0$ and $\frac{\partial^k}{\partial z^k}(P_A g - \tilde{M} e^A)|_{H} = 0$ for any $k$ from $\{1, \ldots, 4\}$. Let $q_\tau$
be the restriction of $w_2$ to $\Gamma_0$. By (3.12) there exists a constant $C_{13}$ independent of $\tau$ such that

$$|\tau||q_\tau|_{C^1(\Gamma_0)} \leq C_{13} ||f||_{L^p(\Omega)}.$$  

By Proposition 3.7 there exists a constant $C_{14}$ independent of $\tau$ such that

$$\sqrt{|\tau|} ||w_2 e^{-\tau\varphi}||_{L^2(\Omega)} + \frac{1}{|\tau|^2} |\nabla w_2 e^{-\tau\varphi}|_{L^2(\Omega)} + ||w_2 e^{-\tau\varphi}||_{H^{1,\tau}(\Omega, \gamma)} \leq C_{14} ||f||_{L^p(\Omega)}.$$  

Let $\tilde{a}_\tau, \tilde{b}_\tau \in H^1(\Omega)$ be holomorphic and an antiholomorphic functions respectively such that $(\tilde{a}_\tau e^A + \tilde{b}_\tau e^B)|_{\Gamma_0} = -q_\tau$. By (3.21) and Proposition 2.5 there exist constants $C_{15}, C_{16}$ independent of $\tau$ such that

$$\left|\tilde{a}_\tau\right|_{H^1(\Omega)} + \left|\tilde{b}_\tau\right|_{H^1(\Omega)} \leq C_{15} ||q_\tau||_{C^1(\Omega)} \leq C_{16} \frac{||f||_{L^p(\Omega)}}{|\tau|}.$$  

The function $W = (w_2 + \tilde{a}_\tau e^A) e^{r\varphi} + \tilde{b}_\tau e^{B + r\varphi}$ satisfies

$$L(x, D)W = \chi f e^{r\varphi} + e^{r\varphi} \frac{h_\tau}{\sqrt{|\tau|}} \text{ in } \Omega, \quad W|_{\Gamma_0} = 0,$$

where

$$\left|\tilde{h}_\tau\right|_{L^2(\Omega)} \leq C_{17} ||f||_{L^2(\Omega)}$$

with some constant $C_{17}$ independent of $\tau$. By (3.22), (3.23)

$$\sqrt{|\tau|} ||W e^{-\tau\varphi}||_{L^2(\Omega)} + \frac{1}{|\tau|^2} |\nabla W e^{-\tau\varphi}|_{L^2(\Omega)} + ||W e^{-\tau\varphi}||_{H^{1,\tau}(\Omega, \gamma)} \leq C_{18} ||f||_{L^p(\Omega)}.$$  

Let $\tilde{W}$ be a solution to problem (2.54) with inhomogeneous term and boundary data $f = -\frac{h_\tau}{\sqrt{|\tau|}}, \quad g = 0$ respectively given by Proposition 2.6. The estimate (2.55) has the form

$$||\tilde{W} e^{-\tau\varphi}||_{H^{1,\tau}(\Omega)} \leq C_{19} ||\tilde{h}_\tau||_{L^2(\Omega)} \leq C_{20} ||f||_{L^2(\Omega)}.$$  

Then the function $w_1 + W + \tilde{W}$ solves (3.19). The estimate (3.18) follows form (3.20), (3.25) and (3.26). The proof of the proposition is completed. \hfill \square

4. Complex Geometrical Optics Solutions

For a complex-valued vector field $(A_1, B_1)$ and complex-valued potential $q_1$ we will construct solutions to the boundary value problem

$$L_1(x, D) u_1 = 0 \text{ in } \Omega, \quad u_1|_{\Gamma_0} = 0$$

of the form

$$u_1(x) = a_\tau(z) e^{A_1 + r\varphi} + d_\tau(\bar{z}) e^{B_1 + r\varphi} + u_{11} e^{r\varphi} + u_{12} e^{r\varphi}.$$  

Here $A_1$ and $B_1$ are defined by (2.16) respectively for $A_1$ and $B_1$, $a_\tau(z) = a(z) + \frac{a_1(z)}{r} + \frac{a_2(z)}{r^2}$, $d_\tau(\bar{z}) = d(\bar{z}) + \frac{d_1(\bar{z})}{r} + \frac{d_2(\bar{z})}{r^2}$,

$$a, d \in C^{5+\alpha}(\overline{\Omega}), \quad \frac{\partial a}{\partial \bar{z}} = 0 \text{ in } \Omega, \quad \frac{\partial d}{\partial z} = 0 \text{ in } \Omega.$$
The function $u$ is given by

\[(a(z)e^{A_1} + d(\bar{z})e^{B_1})|_{\gamma_0} = 0.\]

Let $\tilde{x}$ be some fixed point from $\mathcal{H}$. Suppose in addition that

\[\partial^k a|_{\gamma \cap \partial \Omega} = 0, \quad \partial^k d|_{\gamma \cap \partial \Omega} = 0 \quad \forall k \in \{0, \ldots, 5\}, \quad a|_{\gamma \setminus \{\tilde{x}\}} = d|_{\gamma \setminus \{\tilde{x}\}} = 0, \quad a(\tilde{x}) \neq 0, d(\tilde{x}) \neq 0.

Such functions exists by Proposition 6.2.

Denote

\[g_1 = T_{B_1}((q_1-2\partial B_1/\partial \bar{z}) - A_1B_1)e^{B_1}, \quad g_2 = P_{A_1}((q_1-2\partial A_1/\partial z) - A_1B_1)a_1e^{A_1} - M_1(z)e^{A_1},\]

where $M_1(z)$ and $M_2(\bar{z})$ are polynomials such that

\[\partial^k g_1|_{\gamma} = \partial^k g_2|_{\gamma} = 0 \quad \forall k \in \{0, \ldots, 5\}.

Thanks to our assumptions on the regularity of $A_1, B_1$ and $q$, $g_1, g_2$ belong to $C^{6+\alpha}(\overline{\Omega})$.

Note that by (4.6), (4.5)

\[\partial^{k+j} g_1|_{\gamma \cap \partial \Omega} = \partial^{k+j} g_2|_{\gamma \cap \partial \Omega} = 0 \quad \text{if } k + j \leq 5.

The function $a_1(z)$ is holomorphic in $\Omega$ and $d_1(\bar{z})$ is antiholomorphic in $\Omega$ and

\[a_1(z)e^{A_1} + d_1(\bar{z})e^{B_1} = \frac{g_1}{\partial \Phi} + \frac{g_2}{\partial \Phi} \quad \text{on } \Gamma_0.

The existence of such functions is given again by Proposition 2.5. Observe that by (4.7) the functions $\frac{\partial^{k+i} g_1}{\partial z^k \partial \bar{z}}$, $\frac{\partial^{k+i} g_2}{\partial z^k \partial \bar{z}} \in C^4(\overline{\Omega})$. Let

\[\tilde{g}_1 = T_{B_1}((q_1-2\partial B_1/\partial \bar{z}) - A_1B_1)e^{B_1}, \quad \tilde{g}_2 = P_{A_1}((q_1-2\partial A_1/\partial z) - A_1B_1)a_1e^{A_1} - \hat{M}_1(z)e^{A_1},\]

where $\hat{M}_1(z)$ and $\hat{M}_2(\bar{z})$ are polynomials such that

\[\partial^k \tilde{g}_1|_{\gamma} = \partial^k \tilde{g}_2|_{\gamma} = 0 \quad \forall k \in \{0, \ldots, 3\}.

The function $u_{11}$ is given by

\[u_{11} = -e^{-ir\psi}R_{-\tau, A_1}\{e_1(g_1 + \tilde{g}_1/\bar{\tau})\} - e^{-ir\psi}e_2\frac{g_1 + \hat{g}_1}{2\tau \partial \Phi} + e^{-ir\psi}e_2\frac{g_1 + \tilde{g}_1}{2\tau \partial \Phi}L_1(x, D)\left(\frac{e_2g_1}{\partial \Phi}\right)\]

\[+ e^{-ir\psi}\tilde{R}_{\tau, B_1}\{e_1(g_2 + \tilde{g}_2/\bar{\tau})\} - e^{-ir\psi}e_2\frac{g_2 + \hat{g}_2}{2\tau \partial \Phi} + e^{-ir\psi}e_2\frac{g_2 + \tilde{g}_2}{2\tau \partial \Phi}L_1(x, D)\left(\frac{e_2g_2}{\partial \Phi}\right)\]

Now let us determine the functions $u_{12}$, $a_2(z)$ and $d_2(\bar{z})$.

First we can obtain the following asymptotic formulae for any point on the boundary of $\Omega$.

\[\mathcal{R}_{-\tau, A_1}\{e_1g_1\} = \frac{1}{2\tau^2} e^{A_1+2ir\psi}L_1(x, D)\left(\frac{e^{-2ir\psi}(\bar{\tau}g_1)}{\tau} \right) + W_{\tau, 1},\]

\[\mathcal{R}_{\tau, B_1}\{e_1g_2\} = \frac{1}{2\tau^2} e^{B_1-2ir\psi}L_1(x, D)\left(\frac{e^{2ir\psi}(\bar{\tau}g_2)}{\tau} \right) + W_{\tau, 1},\]
where
\[ q_1 = \frac{\partial_\tau \tilde{g}_1(\tilde{x})}{4 \partial \Phi(\tilde{x})}, \quad m_1 = \frac{1}{8} \left( \frac{\partial_\tau \tilde{g}_1(\tilde{x}) \partial \Phi(\tilde{x}) + \partial \Phi(\tilde{x})}{\partial_\tau \Phi(\tilde{x})} \right), \]
\[ \tilde{g}_1 = \frac{\partial_\tau \tilde{g}_2(\tilde{x})}{4 \partial \Phi(\tilde{x})}, \quad \tilde{m}_1 = \frac{1}{8} \left( \frac{\partial_\tau \tilde{g}_2(\tilde{x}) \partial \Phi(\tilde{x}) - \partial \Phi(\tilde{x})}{\partial_\tau \Phi(\tilde{x})} \right), \]
\[ \tilde{g}_1 = e^{-A_1 g_1}, \tilde{g}_2 = e^{-B_1 g_2} \text{ and } \mathcal{W}_{r,1}, \mathcal{W}_{r,2} \in H^\frac{1}{2}(\Gamma_0) \text{ satisfy} \]
\[ (4.12) \quad \|\mathcal{W}_{r,1}\|_{H^\frac{1}{2}(\Gamma_0)} + \|\mathcal{W}_{r,2}\|_{H^\frac{1}{2}(\Gamma_0)} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty. \]

The proof of (4.10) and (4.11) is given in Section 7.

Denote
\[ p_+ = e^{A_1} \left( \frac{q_1(\tilde{x})}{(\tilde{z} - \tilde{x})^2} + \frac{m_1(\tilde{x})}{(\tilde{z} - \tilde{x})^2} \right), \]
\[ p_- = e^{B_1} \left( \frac{\tilde{q}_1(\tilde{x})}{(\tilde{z} - \tilde{x})^2} + \frac{\tilde{m}_1(\tilde{x})}{(\tilde{z} - \tilde{x})^2} \right). \]

Thanks to Proposition 2.5 we can define functions \( a_{2,\pm}(z) \in C^2(\Omega) \) and \( d_{2,\pm}(\tau) \in C^2(\Omega) \) satisfying
\[ a_{2,\pm}(z) e^{A_1} + d_{2,\pm}(\tau) e^{B_1} = p_{\pm} \quad \text{on } \Gamma_0. \]

Straightforward computations give
\[ L_1(x, D)((a(z) + \frac{a_1(z)}{\tau}) e^{A_1+r\Phi} + (d(\tilde{z}) + \frac{d_1(\tilde{z})}{\tau}) e^{B_1+r\overline{\Phi}} + e^{r\varphi} u_{11}) \]
\[ = (q_1 - 2 \frac{2A_1}{d_2} - A_1 B_1) e^{r\Phi} \left( -\nabla_{\tau, B_1} \{ e_1(g_2 + \tilde{g}_2/\tau) \} - \frac{e_2(g_1 + \tilde{g}_1/\tau)}{2\tau d_\Phi} \right) \]
\[ + (q_1 - 2 \frac{2B_1}{d_2} - A_1 B_1) e^{r\overline{\Phi}} \left( -\nabla_{-\tau, A_1} \{ e_1(g_1 + \tilde{g}_1/\tau) \} - \frac{e_2(g_1 + \tilde{g}_1/\tau)}{2\tau d_\Phi} \right) \]
\[ + e^{r\varphi} \left( L_1(x, D) \left( \frac{1}{\partial A_1} \partial_\Phi L_1(x, D) \left( \frac{\partial_\Phi g_{11}}{\partial_\Phi} \right) \right) + e^{r\overline{\varphi}} L_1(x, D) \left( \frac{1}{\partial A_1} \partial_\Phi L_1(x, D) \left( \frac{\partial_\Phi g_{11}}{\partial_\Phi} \right) \right) \right). \]

Using Proposition 3.3 we transform the right-hand side of (4.14) as follows.
\[ L_1(x, D)((a(z) + \frac{a_1(z)}{\tau}) e^{A_1+r\Phi} + (d(\tilde{z}) + \frac{d_1(\tilde{z})}{\tau}) e^{B_1+r\overline{\Phi}} + u_{11} e^{r\varphi}) \]
\[ = -(q_1 - 2 \frac{2A_1}{d_2} - A_1 B_1) e^{r\Phi} \cdot \frac{g_{11}}{2\tau d_\Phi} \]
\[ - (q_1 - 2 \frac{2B_1}{d_2} - A_1 B_1) e^{r\overline{\Phi}} \cdot \frac{g_{11}}{2\tau d_\Phi} + o_{L^1(\Omega)} \left( \frac{1}{\tau} \right) \quad \text{as } |\tau| \to +\infty. \]

We are looking for \( u_{12} \) in the form \( u_{12} = u_0 + u_{-1} \). The function \( u_{-1} \) is given by
\[ u_{-1} = \frac{e^{ir\varphi}}{\tau} \nabla_{\tau, B_1} \{ e_1 g_3 \} + \frac{e^{-ir\overline{\varphi}}}{\tau} \nabla_{-\tau, A_1} \{ e_1 g_6 \} + \frac{e_2 g_{11} e^{r\varphi}}{2\tau d_\Phi} + \frac{e_2 g_{11} e^{-r\overline{\varphi}}}{2\tau d_\Phi}, \]
where
\[ g_5 = P_{A_1}((q_1 - 2 \frac{2A_1}{d_2} - A_1 B_1) g_1) - M_5(z) e^{A_1} \]
\[ g_6 = P_{B_1}((q_1 - 2 \frac{2B_1}{d_2} - A_1 B_1) g_2) - M_6(\tau) e^{B_1}. \]
Here $M_5(z), M_6(\bar{z})$ are polynomials such that
\[
g_5|_H = g_6|_H = \nabla g_5|_H = \nabla g_6|_H = 0.
\]
Using Proposition 2.5 we introduce functions $a_{2,0}, d_{2,0} \in C^2(\Omega)$ (holomorphic and antiholomorphic respectively) such that
\[
a_{2,0}(z)e^{A_1} + d_{2,0}(\bar{z})e^{B_1} = \frac{g_5}{2\partial \Phi} + \frac{g_6}{2\partial \bar{\Phi}} \quad \text{on } \Gamma_0.
\]
Next we claim that
\[
\mathcal{R}_{-\tau,A_1}\{e_1g_0\}|_{\Gamma_0} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty, \quad \mathcal{R}_{\tau,B_1}\{e_1g_5\}|_{\Gamma_0} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty.
\]
To see this, let us introduce the function $\mathcal{F}$ with domain $\Gamma_0$.
\[
\mathcal{F} = 2e^{-A_1}e^{\tau(\Phi-\bar{\Phi})}\mathcal{R}_{-\tau,A_1}\{e_1g_0\} = \partial_\tau^{-1}(e_1e^{-A_1+\tau(\Phi-\bar{\Phi})}T_{B_1}((q_1 - 2\frac{\partial B_1}{\partial \tau} - A_1B_1)g_2) - M_6e^{B_1}).
\]
Denoting $r(x) = e^{A_1}T_{B_1}((q_1 - 2\frac{\partial B_1}{\partial \tau} - A_1B_1)g_2) - M_6e^{B_1}$ we have
\[
\mathcal{F}(x) = -\frac{1}{\pi} \int_\Omega \frac{e_1(x)(r(x)e^{2\tau\psi})}{\zeta - \bar{\zeta}} d\xi_1 d\xi_2 = \frac{1}{2i\pi \tau} \int_\Omega \sum_{k=1}^2 \frac{\partial}{\partial x_k} \left( \frac{\partial \psi}{\partial x_k} \right) e^{2\tau\psi} d\xi_1 d\xi_2.
\]
Since $\sum_{k=1}^2 \frac{\partial}{\partial x_k} \left( \frac{\partial \psi}{\partial x_k} \right) \in L^1(\Omega)$, we have $\mathcal{F} = o(\frac{1}{\tau})$. This proves (4.19).

Now we finish the construction of the functions $a_{2,\tau}(z)$ and $d_{2,\tau}(\bar{z})$ by setting
\[
d_{2,\tau}(\bar{z}) = d_{2,0}(\bar{z}) + \frac{1}{2} \left( \frac{d_{2,+}(\bar{z})e^{2\tau\psi(\bar{z})}}{|\text{det } \text{Im } \Phi^{\prime}(\bar{x})|^{\frac{1}{2}}} + \frac{d_{2,-}(\bar{z})e^{-2\tau\psi(\bar{z})}}{|\text{det } \text{Im } \Phi^{\prime}(\bar{x})|^{\frac{1}{2}}} \right),
\]
\[
a_{2,\tau}(z) = a_{2,0}(z) + \frac{1}{2} \left( \frac{a_{2,+}(z)e^{2\tau\psi(z)}}{|\text{det } \text{Im } \Phi^{\prime}(\bar{x})|^{\frac{1}{2}}} + \frac{a_{2,-}(z)e^{-2\tau\psi(z)}}{|\text{det } \text{Im } \Phi^{\prime}(\bar{x})|^{\frac{1}{2}}} \right),
\]
where $a_{2,\tau}, d_{2,\tau}$ satisfy (4.13). To complete the construction of a solution to (4.1) we define $u_0$ as the solution to the inhomogeneous problem
\[
L_1(x,D)(u_0e^{\tau\varphi}) = h_1e^{\tau\varphi} \quad \text{in } \Omega,
\]
(4.20)
\[
u_0e^{\tau\varphi} = e^{\tau\varphi} \mathbf{m}_1 \quad \text{on } \Gamma_0,
\]
where
\[
h_1 = -e^{-\tau\varphi}L_1(x,D)(a_{\tau}(z)e^{A_1+\tau\Phi} + d_{\tau}(\bar{z})e^{B_1+\tau\bar{\Phi}} + u_{11}e^{\tau\varphi} + u_{-1}e^{-\tau\varphi}),
\]
\[
\mathbf{m}_1 = -e^{-\tau\varphi}(a_{\tau}(z)e^{A_1+\tau\Phi} + d_{\tau}(\bar{z})e^{B_1+\tau\bar{\Phi}} + u_{11}e^{\tau\varphi} + u_{-1}e^{-\tau\varphi})|_{\Gamma_0},
\]
Observe that by (4.15) - (4.17)
\[
\|h_1\|_{L^4(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty
\]
and by (4.8), (4.12), (4.13), (4.18)
\[
\|u_0\|_{H^\frac{1}{2}(\Gamma_0)} = o(\frac{1}{\tau^2}) \quad \text{as } |\tau| \to +\infty.
\]
(4.22)
By Proposition 2.6 and Proposition 3.8 there exists a solution to (4.20), (4.21) such that

\[
\left(4.23\right) \quad \frac{1}{\sqrt{|\tau|}}\|u_0\|_{H^1}(\Omega) + \sqrt{|\tau|}\|u_0\|_{L^2}(\Omega) + \|u_0\|_{H^1(\mathcal{C})} = o\left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \rightarrow +\infty.
\]

4.1. Complex geometrical optics solutions for the adjoint operator. We now construction complex geometrical optics solutions for the adjoint operator. This parallels the previous construction since the adjoint has a similar form.

Consider the operator \(L_2(x, D) = 4\frac{\partial}{\partial z} + A_2\frac{\partial}{\partial \overline{z}} + 2B_2\frac{\partial}{\partial z} + q_2\). Its adjoint has the form

\[
\left(4.28\right) \quad L_2(x, D)^* = 4\frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} - 2\overline{A_2}\frac{\partial}{\partial z} - 2\overline{B_2}\frac{\partial}{\partial \overline{z}} + \overline{q_2} - 2\frac{\partial \overline{A_2}}{\partial z} - \frac{\partial \overline{B_2}}{\partial \overline{z}}.
\]

Next we construct solution to the following boundary value problem:

\[
\left(4.24\right) \quad L_2(x, D)^*v = 0 \quad \text{in} \quad \Omega, \quad v|_{\Gamma_0} = 0.
\]

We construct solutions to (4.24) of the form

\[
\left(4.25\right) \quad v(x) = b_\tau(z)e^{B_2-\tau\Phi} + c_\tau(\overline{z})e^{A_2-\overline{\Phi}} + v_{11}e^{-\tau\varphi} + v_{12}e^{-\tau\varphi}, \quad v|_{\Gamma_0} = 0.
\]

Here \(A_2, B_2 \in C^{6+\alpha}(\overline{\Omega})\) satisfy

\[
\left(4.26\right) \quad 2\frac{\partial A_2}{\partial z} = \overline{A_2} \quad \text{in} \quad \Omega, \quad \text{Im} A_2|_{\Gamma_0} = 0, \quad 2\frac{\partial B_2}{\partial z} = \overline{B_2} \quad \text{in} \quad \Omega, \quad \text{Im} B_2|_{\Gamma_0} = 0,
\]

and \(b_\tau(z) = b(z) + \frac{b_\tau(z)}{\tau} + \frac{b_\tau(z)}{\tau} + c_\tau(\overline{z}) = c(\overline{z}) + \frac{c_\tau(z)}{\tau} + \frac{c_\tau(z)}{\tau}\) and

\[
\left(4.27\right) \quad (b(z)e^{B_2} + c(z)e^{A_2})|_{\Gamma_0} = 0,
\]

\[
\left(4.28\right) \quad \frac{\partial^k b}{\partial z^k}|_{\mathcal{H}\cap \partial \Omega} = 0, \quad \frac{\partial^k c}{\partial \overline{z}^k}|_{\mathcal{H}\cap \partial \Omega} = 0 \quad k \in \{0, \ldots, 5\}, b|_{\mathcal{H}\cap \{\bar{\tau}\}} = c|_{\mathcal{H}\cap \{\bar{\tau}\}} = 0, \quad b(\bar{x}) \neq 0, c(\bar{x}) \neq 0.
\]

The existence of the functions \(b\) and \(c\) is given by Proposition 6.2. Denote

\[
g_3 = P_{-\frac{\partial A_2}{\partial z} - \overline{A_2}\frac{\partial}{\partial \overline{z}}}(\overline{q_2} - 2\frac{\partial \overline{A_2}}{\partial z} - \frac{\partial \overline{B_2}}{\partial \overline{z}}) - M_3(z)e^{B_2}, \quad g_4 = T_{-\frac{\partial B_2}{\partial z} - \overline{B_2}\frac{\partial}{\partial \overline{z}}}(\overline{q_2} - 2\frac{\partial \overline{B_2}}{\partial z} - \frac{\partial \overline{A_2}}{\partial \overline{z}}) - M_4(\overline{z})e^{A_2},
\]

where the polynomials \(M_3(z), M_4(\overline{z})\) are chosen such that

\[
\left(4.29\right) \quad \frac{\partial^k g_3}{\partial z^k}|_{\mathcal{H}\cap \partial \Omega} = 0 \quad \forall k \in \{0, \ldots, 5\}.
\]

By (4.29), (4.28)

\[
\left(4.30\right) \quad \frac{\partial^{k+j} g_3}{\partial z^k \partial \overline{z}^j}|_{\mathcal{H}\cap \partial \Omega} = 0 \quad \forall k + j \leq 5.
\]
Observe that by (4.30) $\frac{\partial^2}{\partial z^2}, \frac{\partial^2}{\partial \overline{z}^2} \in C^{4+\alpha} \Omega)$. Using Proposition 2.5 we introduce a holomorphic function $b_1(z) \in C^2(\Omega)$ and an antiholomorphic function $c_1(z) \in C^2(\Omega)$ such that

$$b_1 e^{B_2} + c_1 e^{A_2} = \frac{e_2 g_3}{\partial^2_z \Phi} + \frac{e_2 g_4}{\partial^2_{\overline{z}} \Phi} \text{ on } \Gamma_0.$$

Let

$$\hat{g}_3 = P_{-2(\overline{\Omega})} ((\overline{q}_2 - 2\overline{\partial A}_2 - A_2 B_2) b_1 e^{B_2}) e^{B_2}, \quad \hat{g}_4 = T_{-2(\overline{\Omega})} ((\overline{q}_2 - 2\overline{\partial B}_2 - A_2 B_2) c_1 e^{A_2}) e^{A_2},$$

where the polynomials $\hat{M}_3(z), \hat{M}_4(z)$ are chosen such that

$$\frac{\partial^k \hat{g}_3}{\partial z^k} |_{\mathcal{H}} = \frac{\partial^k \hat{g}_4}{\partial \overline{z}^k} |_{\mathcal{H}} = 0 \quad \forall k \in \{0, \ldots, 3\}.$$

The function $v_{11}$ is defined by

$$v_{11} = -e^{-ir\psi} \overline{\mathcal{R}}_{-A, -A_2} \{e_1 (g_3 + \hat{g}_3/r)\} + \frac{e^{-ir\psi} e_2 (g_3 + \hat{g}_3/r)}{\partial^2_{\overline{z}} \Phi} e^{-ir\psi} \mathcal{R}_{-A_2, -A_2} \{e_1 g_4 + \hat{g}_4/r\} + \frac{e^{-ir\psi} e_2 (g_4 + \hat{g}_4/r)}{\partial^2_{\overline{z}} \Phi}.$$

Here we set

$$\mathcal{R}_{-A_2, -A_2} \{g\} = \frac{1}{2} e^{A_2} e^{r(\Phi_{\overline{z}})} \frac{\partial^2_{\overline{z}} \Phi}{r}(ge^{-A_2} e^{r(\Phi_{\overline{z}})})$$

provided that $A_2, B_2, A_2, B_2$ satisfy (3.2). By Proposition 7.1 the following asymptotic formulae holds.

$$\mathcal{R}_{-A_2, -A_2} \{e_1 g_3\} |_{\Gamma_0} = \frac{1}{2\tau^2} \left| \text{det } \text{Im } \Phi(\overline{\overline{\Phi}}) \right| \frac{e^{-2r\tau\psi} \hat{r}_1(x)}{(z - \overline{z})^2} + \frac{e^{-2r\tau\psi} \hat{t}_1(x)}{(\overline{z} - \overline{z})^2} + \mathcal{W}_{1, \tau},$$

$$\mathcal{R}_{-A_2, -A_2} \{e_1 g_4\} |_{\Gamma_0} = \frac{1}{2\tau^2} \left| \text{det } \text{Im } \Phi(\overline{\overline{\Phi}}) \right| \frac{e^{-2r\tau\psi} \hat{r}_1(x)}{(z - \overline{z})^2} + \frac{e^{-2r\tau\psi} \hat{t}_1(x)}{(\overline{z} - \overline{z})^2} + \mathcal{W}_{2, \tau},$$

where

$$r_1 = - \overline{\frac{\partial^2_{\overline{z}} \Phi}{\partial^2_{\overline{z}} \Phi}}, \quad t_1 = \frac{1}{8} \left( \frac{\partial^2_{\overline{z}} \Phi}{\overline{\partial^2_{\overline{z}} \Phi}} \right) \left( \frac{\partial^2_{\overline{z}} \Phi}{\partial^2_{\overline{z}} \Phi} - \frac{\partial^2_{\overline{z}} \Phi}{\partial^2_{\overline{z}} \Phi} \right).$$

$$\hat{r}_1 = - \overline{\frac{\partial^2_{\overline{z}} \Phi}{\partial^2_{\overline{z}} \Phi}}, \quad \hat{t}_1 = \frac{1}{8} \left( \frac{\partial^2_{\overline{z}} \Phi}{\overline{\partial^2_{\overline{z}} \Phi}} \right) \left( \frac{\partial^2_{\overline{z}} \Phi}{\partial^2_{\overline{z}} \Phi} + \frac{\partial^2_{\overline{z}} \Phi}{\partial^2_{\overline{z}} \Phi} \right).$$

$$\hat{g}_3 = e^{A_2} g_3, \quad \hat{g}_4 = e^{B_2} g_4.$$ Here the functions $\mathcal{W}_{1,1}, \mathcal{W}_{1,2} \in H^{1/2}(\Gamma_0)$ satisfy

$$\|\mathcal{W}_{1,1}\|_{H^{1/2}(\Gamma_0)} + \|\mathcal{W}_{1,2}\|_{H^{1/2}(\Gamma_0)} = o\left(\frac{1}{\tau^2}\right) \text{ as } |\tau| \to +\infty.$$

Using Proposition 2.5 we define the holomorphic functions $b_{2, \pm}(z) \in C^2(\Omega)$ and antiholomorphic $c_{2, \pm}(z) \in C^2(\Omega)$ such that

$$b_{2, \pm}(z) e^{B_2} + c_{2, \pm}(z) e^{A_2} = \hat{p}_\pm \text{ on } \Gamma_0.$$

Using Proposition 2.5 we introduce functions

\[ \tilde{p}_+(z) = e^{-\frac{i\pi}{2\tau}} \left( \frac{r_1(\bar{z})}{(z - \bar{z})^2} + \frac{i_1(\bar{z})}{(\bar{z} - \bar{z})} \right), \]

\[ \tilde{p}_-(\bar{z}) = e^{\frac{\pi}{2\tau}} \left( \frac{\bar{r}_1(\bar{z})}{(\bar{z} - \bar{z})^2} + \frac{\bar{i}_1(\bar{z})}{(\bar{z} - \bar{z})} \right). \]

Similarly to (4.15) we obtain

\[
L_2(x, D)^* \left( (b(z) + \frac{b_1(z)}{\tau}) e^{B_2 - \tau \Phi(z)} + \left( \frac{b(\bar{z}) + \frac{c(\tau)}{\tau}}{B_2} \right) e^{A_2 - \tau \Phi(\bar{z})} + v_{11} e^{-\tau \varphi} \right)
\]

\[
= \frac{q_1 e^{-\tau \varphi}}{2\tau \partial_{\partial} \Phi}(\bar{q}_2 - 2 \frac{\partial \bar{B}_2}{\partial z} - A_2 B_2) - \frac{q_1 e^{-\tau \varphi}}{2\tau \partial_{\partial} \Phi}(q_2 - 2 \frac{\partial B_2}{\partial z} - A_2 B_2) + a_{L^1(\Omega)} \left( \frac{1}{\tau} \right). \tag{4.38}
\]

We are looking for \( v_{12} \) in the form \( v_{12} = v_0 + v_{-1} \). The function \( v_{-1} \) is given by

\[
v_{-1} = - e^{\frac{i\tau \psi}{\tau}} \partial_{\partial} \Phi (e_1 g_7 - e^{\frac{-i\tau \psi}{\tau}} \partial_{\partial} \Phi (e_1 g_8) + \frac{e_2 g_7}{2{\tau}^2 \partial_{\partial} \Phi} + \frac{e_2 g_8}{2{\tau}^2 \partial_{\partial} \Phi}, \tag{4.39}
\]

where

\[
g_7 = \frac{P - \bar{B}_2((q_2 - 2 \frac{\partial q_2}{\partial z} - A_2 B_2) g_3) - M_7(z)e^{B_2}}{2\partial_{\partial} \Phi}, \quad g_8 = \frac{T - \bar{A}_2((q_2 - 2 \frac{\partial q_2}{\partial z} - A_2 B_2) g_4) - M_8(\bar{z})e^{A_2}}{2\partial_{\partial} \Phi}, \tag{4.40}
\]

and \( M_7(z), M_8(\bar{z}) \) are polynomials such that

\[
g_7|_{\partial\Omega} = g_8|_{\partial\Omega} = \nabla g_7|_{\partial\Omega} = \nabla g_8|_{\partial\Omega} = 0. \tag{4.41}
\]

Using Proposition 2.5 we introduce functions \( b_{2,0}, c_{2,0} \in C^2(\bar{\Omega}) \) such that

\[
b_{2,0}(z) e^{B_2} + c_{2,0}(\tau)e^{A_2} = \frac{g_7}{2\partial_{\partial} \Phi} + \frac{g_8}{2\partial_{\partial} \Phi} \quad \text{on } \Gamma_0. \tag{4.42}
\]

Similarly to (4.19) we have

\[
\left( \frac{1}{\tau} \partial_{\partial} \Phi (e_1 g_7) + \frac{1}{\tau} \partial_{\partial} \Phi (e_1 g_8) \right)|_{\Gamma_0} = o\left( \frac{1}{\tau^2} \right) \quad \text{as } |\tau| \to +\infty.
\]

Now we finish the construction of the functions \( b_{2,\tau}(z) \) and \( c_{2,\tau}(\bar{z}) \) by setting

\[
b_{2,\tau}(z) = b_{2,0}(z) + \frac{1}{2} \left( \frac{b_{2,+}(z) e^{2i\tau \psi(\bar{z})}}{|\det \Phi''(\bar{z})|^{\frac{1}{2}}} + \frac{b_{2,-}(z) e^{-2i\tau \psi(\bar{z})}}{|\det \Phi''(\bar{z})|^{\frac{1}{2}}} \right), \tag{4.43}
\]

and

\[
c_{2,\tau}(z) = c_{2,0}(z) + \frac{1}{2} \left( \frac{c_{2,+(z)} e^{2i\tau \psi(z)}}{|\det \Phi''(z)|^{\frac{1}{2}}} + \frac{c_{2,-}(z) e^{-2i\tau \psi(z)}}{|\det \Phi''(z)|^{\frac{1}{2}}} \right), \tag{4.44}
\]

where \( b_{2,+}, c_{2,-} \) are defined in (4.37).

Consider the following boundary value problem

\[
L_2(x, D)^* (e^{-\tau \varphi} v_0) = h_2 e^{-\tau \varphi} \quad \text{in } \Omega, \tag{4.45}
\]

\[
e^{-\tau \varphi} v_0|_{\Gamma_0} = m_2 e^{-\tau \varphi}, \tag{4.46}
\]
where

\[ h_2 = -e^{\tau \varphi} L_2(x, D)^* (b_\tau(z) e^{B_2 - \tau \Phi} + c_\tau(z) e^{A_2 - \tau \Phi} + v_{11} e^{-\tau \varphi} + v_{-1} e^{-\tau \varphi}) \]
and

\[ m_2 = -e^{\tau \varphi} (b_\tau(z) e^{A_2 - \tau \Phi} + c_\tau(z) e^{B_2 - \tau \Phi} + v_{11} e^{-\tau \varphi} + v_{-1} e^{-\tau \varphi}). \]

By (4.38)-(4.40) we can estimate the norm of the function \( h_2 \) as

\[ \| h_2 \|_{L^4(\Omega)} = o\left( \frac{1}{\tau} \right) \text{ as } |\tau| \to +\infty. \]

By (4.43), (4.44), (4.37), (4.36), (4.26) we have

\[ \| v_0 \|_{H^2(\Gamma_0)} = o\left( \frac{1}{\tau^2} \right) \text{ as } |\tau| \to +\infty. \]

Thanks to (4.47), (4.48), by Proposition 2.6 and Proposition 3.8 for sufficiently small positive \( \epsilon \) there exists a solution to problem (4.45), (4.46) such that

\[ \frac{1}{\sqrt{|\tau|}} \| v_0 \|_{H^1(\Omega)} + \sqrt{|\tau|} \| v_0 \|_{L^2(\Omega)} + \| v_0 \|_{H^1(\partial \Omega)} = o\left( \frac{1}{\tau} \right) \text{ as } |\tau| \to +\infty. \]

5. END OF THE PROOF OF THE MAIN THEOREM

Let \( u_1 \) be a complex geometrical optics solution as in (4.2). Let \( u_2 \) be a solution to the following boundary value problem

\[ L_2(x, D)u_2 = 0 \text{ in } \Omega, \quad u_2|_{\partial \Omega} = u_1|_{\partial \Omega}, \quad \frac{\partial u_2}{\partial \nu}|_{\bar{\Gamma}} = \frac{\partial u_1}{\partial \nu}|_{\bar{\Gamma}}. \]

Setting \( u = u_1 - u_2, q = q_1 - q_2 \) we have

\[ L_2(x, D)u + 2(A_1 - A_2) \frac{\partial u_1}{\partial z} + 2(B_1 - B_2) \frac{\partial u_1}{\partial \bar{z}} + qu_1 = 0 \text{ in } \Omega, \]

\[ u|_{\partial \Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\bar{\Gamma}} = 0. \]

Let \( v \) be a solution to (4.24) in the form (4.25). Taking the scalar product of (5.2) with \( \bar{v} \) in \( L^2(\Omega) \) we get

\[ 0 = \int_{\Omega} (2(A_1 - A_2) \frac{\partial u_1}{\partial z} + 2(B_1 - B_2) \frac{\partial u_1}{\partial \bar{z}} + qu_1) \bar{v} \, dx. \]

Our goal is to get the asymptotic formula for the right hand side of (5.4). We have
Proposition 5.1. The following asymptotic formula is valid as $|\tau| \to +\infty$:

\begin{equation}
(qu, v)_{L^2(\Omega)} = \int_\Omega (qa\phi e^{(A_1 + \Phi)} + qd\phi e^{(B_1 + \Phi)})\,dx \\
+ \int_\Omega \left( \frac{q}{\tau} (a_1 b + a\bar{\tau}) e^{(A_1 - \Phi)} + \frac{q}{\tau} (\bar{b}_1 d + \bar{b}\tau) e^{(A_2 - \Phi)} \right)\,dx \\
+ \frac{1}{\tau} \int_\Omega \left( \frac{a\bar{g}_1 e^{A_1}}{2\partial_2 \Phi} - \frac{\bar{g}_2 e^{A_2}}{2\partial_\Phi} - \frac{\bar{g}_1 e^{B_1}}{2\partial_2 \Phi} + \frac{d\bar{g}_1 e^{B_1}}{2\partial_\Phi} \right)\,dx \\
+ 2\pi \frac{qa\phi}{\tau} e^{(A_1 + \Phi_2 + 2\tau i\text{Re}(\Phi))} + \frac{q\phi}{\tau} e^{(B_1 + \Phi_2 - 2\tau i\text{Re}(\Phi))} + 2\pi \frac{qa\phi}{\tau} e^{(A_1 + \Phi + 2\tau i\text{Re}(\Phi))} + \frac{q\phi}{\tau} e^{(B_1 + \Phi - 2\tau i\text{Re}(\Phi))} \\
+ \frac{1}{2\tau i} \int_{\partial \Omega} qa\phi e^{A_1 + \Phi_2 + 2\tau i\text{Re}(\Phi)} \frac{\nu, \nabla \psi}{|\nabla \psi|^2} \,d\sigma - \frac{1}{2\tau i} \int_{\partial \Omega} q\phi e^{B_1 + \Phi_2 - 2\tau i\text{Re}(\Phi)} \frac{\nu, \nabla \psi}{|\nabla \psi|^2} \,d\sigma + o\left( \frac{1}{\tau} \right).
\end{equation}

Proof. By (4.2), (4.9) and Proposition 3.3 we have

\begin{equation}
u_1(x) = (a(z) + \frac{a_1(z)}{\tau}) e^{A_1 + \tau \Phi} + (d(z) + \frac{d_1(z)}{\tau}) e^{B_1 + \tau \Phi} - \frac{g_1 e^{\tau \Phi}}{2\tau \partial_2 \Phi} - \frac{g_2 e^{\tau \Phi}}{2\tau \partial_\Phi} + o_{L^2(\Omega)}(\frac{1}{\tau}).
\end{equation}

Using (4.25), (4.33) and Proposition 3.3 we obtain

\begin{equation}
u(x) = (b(z) + \frac{b_1(z)}{\tau}) e^{B_2 - \tau \Phi} + (c(z) + \frac{c_1(z)}{\tau}) e^{A_2 - \tau \Phi} + \frac{g_4 e^{\tau \Phi}}{2\tau \partial_2 \Phi} + \frac{g_3 e^{\tau \Phi}}{2\tau \partial_\Phi} + o_{L^2(\Omega)}(\frac{1}{\tau}).
\end{equation}

By (5.6), (5.7) we obtain

\begin{align*}
(qu, v)_{L^2(\Omega)} &= \left( q\left( a + \frac{a_1}{\tau} \right) e^{A_1 + \tau \Phi} + \left( d + \frac{d_1}{\tau} \right) e^{B_1 + \tau \Phi} - \frac{g_1 e^{\tau \Phi}}{2\tau \partial_2 \Phi} - \frac{g_2 e^{\tau \Phi}}{2\tau \partial_\Phi} + o_{L^2(\Omega)}(\frac{1}{\tau}) \right), \\
&= \left( b + \frac{b_1}{\tau} \right) e^{B_2 - \tau \Phi} + \left( c + \frac{c_1}{\tau} \right) e^{A_2 - \tau \Phi} + \frac{g_4 e^{\tau \Phi}}{2\tau \partial_2 \Phi} + \frac{g_3 e^{\tau \Phi}}{2\tau \partial_\Phi} + o_{L^2(\Omega)}(\frac{1}{\tau}) \right)_{L^2(\Omega)} = \\
&= \int_\Omega \left( q(a\beta + \frac{1}{\tau}(d_1 b + d\bar{b}_1)) e^{(A_1 + \Phi_2)} + q(a\bar{c} + \frac{1}{\tau}(a\bar{b}_1 + a\bar{c})) e^{A_1 + \Phi_2} \right)\,dx \\
&+ \frac{1}{\tau} \int_\Omega \left( \frac{a\bar{g}_1 e^{A_1}}{2\partial_2 \Phi} - \frac{\bar{g}_2 e^{A_2}}{2\partial_\Phi} - \frac{\bar{g}_1 e^{B_1}}{2\partial_2 \Phi} + \frac{d\bar{g}_1 e^{B_1}}{2\partial_\Phi} \right)\,dx \\
&+ \int_\Omega \left( a\beta e^{A_1 + \Phi_2 + (\Phi - \Phi)} + d\bar{c} e^{B_1 + \Phi_2 + (\Phi - \Phi)} \right)\,dx + o\left( \frac{1}{\tau} \right).
\end{align*}

Applying the stationary phase argument to the last integral in the right hand side of this formula we finish the proof of Proposition 5.1.

We set

\begin{align}
U(x) &= a_+(z) e^{A_1(x) + \tau \Phi(z)} + d_+(\bar{z}) e^{B_1(x) + \tau \Phi(z)}, \\
V(x) &= b_+(z) e^{B_2(x) - \tau \Phi(z)} + c_+(\bar{z}) e^{A_2(x) - \tau \Phi(z)}.
\end{align}
Short calculations give:

\[
I_1 \equiv 2((A_1 - A_2) \frac{\partial U}{\partial z}, \mathcal{V})_{L^2(\Omega)} = (2(A_1 - A_2)\left(\frac{\partial A_1}{\partial z} + \tau \frac{\partial \Phi}{\partial z}\right) a_r + \frac{\partial a_r}{\partial z})e^{A_1 + \tau \Phi} + d_r \frac{\partial B_1}{\partial z} e^{B_1 + \tau \Phi},
\]

\[
= \sum_{k=1}^3 \tau^{2-k} \kappa_k - \int_\Omega (A_1 - A_2)B_1 d_\tau(z) \frac{\overline{c_r(z)}}{c_r(z)} e^{B_1 + \overline{A_2} - 2i\tau \psi} \, dx
\]

\[
- \left(2 \frac{\partial}{\partial z} (A_1 - A_2) a_r e^{A_1 + \tau \Phi}, b_r e^{B_2 - \tau \Phi}\right)_{L^2(\Omega)}
\]

\[
- (2(A_1 - A_2) a_r e^{A_1 + \tau \Phi}, \frac{\partial B_2}{\partial z} b_r e^{B_2 - \tau \Phi})_{L^2(\Omega)}
\]

\[
+ \int_{\partial \Omega} (A_1 - A_2)(\nu_1 - i\nu_2) d_\tau(z) \frac{\overline{c_r(z)}}{c_r(z)} e^{B_1 + \overline{A_2} - 2i\tau \psi} d\sigma + o\left(\frac{1}{\tau}\right)
\]

\[
= \sum_{k=1}^3 \tau^{2-k} \kappa_k + \int_\Omega \left\{ -(A_1 - A_2)B_1 d_\tau(z) \frac{\overline{c_r(z)}}{c_r(z)} e^{B_1 + \overline{A_2} - 2i\tau \psi} \right. \left. -(A_1 - A_2)B_2 a_r b_\tau(z) e^{A_1 + \overline{B_2} + 2i\tau \psi} - 2 \frac{\partial}{\partial z} (A_1 - A_2) a_r b_\tau(z) e^{A_1 + \overline{B_2} + 2i\tau \psi} \right\} \, dx
\]

\[
+ \int_{\partial \Omega} (A_1 - A_2)(\nu_1 - i\nu_2) a_r b_\tau(z) e^{A_1 + \overline{B_2} + 2i\tau \psi} d\sigma + \frac{1}{\tau} I_1(\partial \Omega) + o\left(\frac{1}{\tau}\right)
\]

(5.9)
and

\[ I_2 \equiv ((B_1 - B_2) \frac{\partial U}{\partial \tau}, V)_{L^2(\Omega)} \]

\[ = (2(B_1 - B_2)(a_\tau e^{A_1 + \tau \Phi} \frac{\partial A_1}{\partial \tau} + \frac{\partial}{\partial \tau} \left( d_\tau e^{B_1 + \tau \Phi} \right)), b_\tau e^{B_2 - \tau \Phi} + c_\tau e^{A_2 - \tau \Phi})_{L^2(\Omega)} \]

\[ = \sum_{k=1}^{3} \tau^{2-k}\kappa_k + \int_{\Omega} \left\{ -(B_1 - B_2) A_1 a_\tau b_\tau e^{A_1 + \Phi_2 - 2i\psi} - (B_1 - B_2) d_\tau e^{B_1 + \Phi_2 - 2i\psi} \right\} dx \]

\[ + \int_{\partial \Omega} (\nu_1 + i \nu_2) d_\tau e^{B_1 + \Phi_2 - 2i\psi} d\sigma + \frac{1}{\tau} I_2(\partial \Omega) + o \left( \frac{1}{\tau} \right) \]

(5.10)

Here \( \kappa_k, \tilde{\kappa}_k \) are some constants independent of \( \tau \) but may be dependent on \( A_j, B_j, \Phi \). The terms \( I_1(\partial \Omega), I_2(\partial \Omega) \) are given by

\[ I_1(\partial \Omega) = \int_{\Omega} \left( A_1 - A_2 \right) e^{A_1 + \Phi_2} \frac{\partial \Phi}{\partial \tau} c(\tilde{z}) \left( \frac{a_{2,+} e^{-2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} + \frac{a_{2,-} e^{2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} \right) dx \]

\[ + \int_{\Omega} \left( A_1 - A_2 \right) e^{A_1 + \Phi_2} \frac{\partial \Phi}{\partial \tau} a(z) \left( \frac{c_{2,+} e^{2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} + \frac{c_{2,-} e^{-2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} \right) dx = \]

\[-2 \int_{\Omega} \frac{\partial}{\partial \tau} e^{A_1 + \Phi_2} \frac{\partial \Phi}{\partial \tau} c(\tilde{z}) \left( \frac{a_{2,+} e^{-2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} + \frac{a_{2,-} e^{2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} \right) dx \]

\[-2 \int_{\Omega} \frac{\partial}{\partial \tau} e^{A_1 + \Phi_2} \frac{\partial \Phi}{\partial \tau} a(z) \left( \frac{c_{2,+} e^{2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} + \frac{c_{2,-} e^{-2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} \right) dx = \]

\[-2 \int_{\partial \Omega} (\nu_1 + i \nu_2) e^{A_1 + \Phi_2} \frac{\partial \Phi}{\partial \tau} c(\tilde{z}) \left( \frac{a_{2,+} e^{-2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} + \frac{a_{2,-} e^{2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} \right) d\sigma \]

(5.11)

\[-2 \int_{\partial \Omega} (\nu_1 + i \nu_2) e^{A_1 + \Phi_2} \frac{\partial \Phi}{\partial \tau} a(z) \left( \frac{c_{2,+} e^{2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} + \frac{c_{2,-} e^{-2i\psi(\tilde{z})}}{|\text{det Im } \Phi(\tilde{z})|^{1/2}} \right) d\sigma \]
and

\[ I_2(\partial\Omega) = \int_{\Omega} (B_1 - B_2) e^{B_1 + B_2} \frac{\partial \Phi}{\partial z} b(z) \left( \frac{d_{2,+} e^{-2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} + \frac{d_{2,-} e^{2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} \right) dx \]

\[ + \int_{\Omega} (B_1 - B_2) e^{B_1 + B_2} \frac{\partial \Phi}{\partial z} d(\bar{z}) \left( \frac{b_{2,+} e^{2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} + \frac{b_{2,-} e^{-2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} \right) dx = \]

\[ -2 \int_{\Omega} \frac{\partial}{\partial z} e^{B_1 + B_2} \frac{\partial \Phi}{\partial z} b(z) \left( \frac{d_{2,+} e^{-2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} + \frac{d_{2,-} e^{2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} \right) dx \]

\[ -2 \int_{\Omega} \frac{\partial}{\partial z} e^{B_1 + B_2} \frac{\partial \Phi}{\partial z} d(\bar{z}) \left( \frac{b_{2,+} e^{2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} + \frac{b_{2,-} e^{-2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} \right) dx = \]

\[ -2 \int_{\sigma} (\nu_1 - i\nu_2) e^{B_1 + B_2} \frac{\partial \Phi}{\partial z} b(z) \left( \frac{d_{2,+} e^{-2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} + \frac{d_{2,-} e^{2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} \right) d\sigma \]

\[ -2 \int_{\sigma} (\nu_1 - i\nu_2) e^{B_1 + B_2} \frac{\partial \Phi}{\partial z} d(\bar{z}) \left( \frac{b_{2,+} e^{2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} + \frac{b_{2,-} e^{-2\tau \psi(\bar{z})}}{|\det \text{Im} \Phi''(\bar{x})|^\frac{1}{2}} \right) d\sigma. \]

(5.12)

Denote

\[ U_1 = -e^{\tau \Phi} \mathcal{R}_{-\tau, A_1} \{ e_1 g_1 \}, \quad U_2 = -e^{\tau \Phi} \mathcal{R}_{\tau, B_1} \{ e_1 g_2 \}. \]

A short calculation gives

\[ 2 \frac{\partial U_1}{\partial \bar{z}} = (-e_1 g_1 + A_1 \mathcal{R}_{-\tau, A_1} \{ e_1 g_1 \}) e^{\tau \Phi} \]

and

\[ 2 \frac{\partial U_2}{\partial \bar{z}} = (-e_1 g_2 + B_1 \mathcal{R}_{\tau, B_1} \{ e_1 g_2 \}) e^{\tau \Phi}. \]

We have

\[ \frac{\partial}{\partial z} \mathcal{R}_{-\tau, A_1} \{ e_1 g_1 \} = \frac{\partial A_1}{\partial z} \mathcal{R}_{-\tau, A_1} \{ e_1 g_1 \} + \tau \frac{\partial \Phi}{\partial z} \mathcal{R}_{-\tau, A_1} \{ e_1 g_1 \} + \mathcal{R}_{-\tau, A_1} \left\{ \frac{\partial (e_1 g_1)}{\partial z} \right\} \]

\[ -\mathcal{R}_{-\tau, A_1} \{ e_1 g_1 \} - \tau \mathcal{R}_{-\tau, A_1} \left\{ \frac{\partial \Phi}{\partial z} e_1 g_1 \right\} = \mathcal{R}_{-\tau, A_1} \left\{ \frac{\partial (e_1 g_1)}{\partial z} \right\} \]

\[ + \tau e^{A_1} e^{-\tau (\Phi - \Phi)} \int_{\Omega} \frac{\partial \Phi (z) - \partial \Phi (z)}{\zeta - z}(e_1 g_1 e^{-A_1})(\xi_1, \xi_2) e^{\tau (\Phi(\bar{x}) - \Phi(\bar{z}))} d\xi_1 d\xi_2 \]

\[ + \frac{e^{A_1}}{2\pi} e^{-\tau (\Phi - \Phi)} \int_{\Omega} \frac{\partial A_1}{\partial \bar{z}} (\zeta, \bar{\zeta}) - \frac{\partial A_1}{\partial z} (\bar{z}, \bar{z}) (e_1 g_1 e^{-A_1})(\xi_1, \xi_2) e^{\tau (\Phi(\bar{x}) - \Phi(\bar{z}))} d\xi_1 d\xi_2. \]

(5.15)

Denote \( P(x, D) = 2(A_1 - A_2) \frac{\partial}{\partial z} + 2(B_1 - B_2) \frac{\partial}{\partial \bar{z}}. \) Let

\[ \mathcal{G}(x, g, A, \tau) = -\frac{1}{2\pi} \int_{\Omega} \left( \tau \frac{\partial \Phi (z)}{\partial \bar{z}} + \frac{\partial A_1}{\partial \bar{z}} (\zeta, \bar{\zeta}) - \frac{\partial A_1}{\partial z} (\bar{z}, \bar{z}) \right) e_1 g e^{-A} e^{\tau (\Phi - \Phi)} d\xi_1 d\xi_2, \]

We set \( \mathcal{G}_1(x, \tau) = \mathcal{G}(x, g_1, A_1, \tau), \mathcal{G}_2(x, \tau) = \mathcal{G}(x, g_2, B_1, \tau), \mathcal{G}_3(x, \tau) = \mathcal{G}(x, g_3, A_2, -\tau), \mathcal{G}_4 = \mathcal{G}(x, g_3, B_2, -\tau). \)
By (3.15), (4.6), (5.15) and Proposition 3.3 we get

\begin{equation}
\frac{\partial}{\partial z} R_{-\tau, A_1} \{ e_1 g_1 \} = \frac{\partial (e_{1g_1})}{\partial z} - e^{A_1 \tau} (\Phi - \Phi) \mathcal{G}_1(\cdot, \tau) + o_{L^2(\Omega)}(\frac{1}{\tau}).
\end{equation}

Simple computations provide the formula

\begin{equation}
\frac{\partial}{\partial z} \tilde{R}_{\tau, B_1} \{ e_1 g_2 \} = \frac{\partial B_1}{\partial z} \tilde{R}_{\tau, B_1} \{ e_1 g_2 \} + \tilde{R}_{\tau, B_1} \{ \partial (e_1 g_2) \}
- \tau \tilde{R}_{\tau, B_1} \{ \partial \Phi \} \frac{\partial B_1}{\partial z} \tilde{R}_{\tau, B_1} \{ e_1 g_2 \} = \tilde{R}_{\tau, B_1} \{ \partial (e_1 g_2) \}
+ \tau \frac{B_1}{2\pi} e^{\tau (\Phi - \Phi)} \int_{\Omega} \frac{d\tau}{\tau} \left( \frac{\partial B_1}{\partial z} \right) \tilde{R}_{\tau, B_1} \{ e_1 g_2 \}.
\end{equation}

By (3.15), (4.6), (5.17), Proposition 3.3 we have

\begin{equation}
\frac{\partial}{\partial z} \tilde{R}_{\tau, B_1} \{ e_1 g_2 \} = \frac{\partial (e_{1g_2})}{\partial z} - B_1 e^{\tau (\Phi - \Phi)} \mathcal{G}_2(\cdot, \tau) + o_{L^2(\Omega)}(\frac{1}{\tau}).
\end{equation}

Denote

\begin{equation*}
V_2 = -e^{-\tau \Phi} R_{-\tau, B_2} \{ e_1 g_3 \}, \quad V_1 = -e^{-\tau \Phi} \tilde{R}_{-\tau, A_2} \{ e_1 g_4 \}.
\end{equation*}

The following proposition is proved in Section 7.

**Proposition 5.2.** There exist two numbers \( \kappa, \kappa_0 \) independent of \( \tau \) such that the following asymptotic formula holds true:

\begin{equation}
( P(x, D)(U_1 + U_2), b_1 e^{B_1 \tau} + c_1 e^{A_2 \tau})_{L^2(\Omega)} + (P(x, D)(a_1 \tau + \tau B_1 + \tau \Phi), V_1 + V_2)_{L^2(\Omega)} = \\
\kappa + \frac{\kappa_0}{\tau} - 2 \int_{\partial \Omega} (\nu_1 + i\nu_2) e^{A_1 + A_2 \tau} \mathcal{G}_1(x, \tau) d\sigma - 2 \int_{\partial \Omega} (\nu_1 - i\nu_2) e^{B_1 + B_2 \tau} \mathcal{G}_2(x, \tau) d\sigma.
\end{equation}

By (5.13), (5.14), (5.16), (5.18) and Proposition 3.4 there exists a constant \( C_0 \) independent of \( \tau \) such that

\begin{align*}
& ( P(x, D)(U_1 + U_2), V_1 + V_2)_{L^2(\Omega)} = ((A_1 - A_2)(-2 \left( \frac{\partial e_{1g_1}}{\partial z} - e^{A_1 \tau} (\Phi - \Phi) \mathcal{G}_1 + o_{L^2(\Omega)}(\frac{1}{\tau}) \right)) e^{\tau \Phi} \\
& + ((B_1 - B_2)(-e_1 g_2 + A_1 \frac{e_1 g_1}{2\tau \partial z \Phi} + o_{L^2(\Omega)}(\frac{1}{\tau})) e^{\tau \Phi}), V_1 + V_2)_{L^2(\Omega)} \\
& + ((B_1 - B_2)(-e_1 g_1 + A_1 \frac{e_1 g_1}{2\tau \partial z \Phi} + o_{L^2(\Omega)}(\frac{1}{\tau})) e^{\tau \Phi}, V_1 + V_2)_{L^2(\Omega)} \\
& - (2(B_1 - B_2)(-\frac{\partial e_{1g_2}}{2\tau \partial z \Phi} - B_1 e^{\tau (\Phi - \Phi)} \mathcal{G}_2 + o_{L^2(\Omega)}(\frac{1}{\tau})) e^{\tau \Phi}, V_1 + V_2)_{L^2(\Omega)} = C_0 \frac{1}{\tau} + o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty.
\end{align*}
Next we claim that
\begin{equation}
(\mathcal{P}(x,D)(u_0 e^{i\tau \varphi}), v)_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to +\infty,
\end{equation}
and
\begin{equation}
(\mathcal{P}(x,D)u, v_0 e^{-i\tau \varphi})_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to +\infty.
\end{equation}

Let us first prove (5.21). By (4.23) and (4.49), we have
\begin{equation}
(\mathcal{P}(x,D)u, v_0 e^{-i\tau \varphi})_{L^2(\Omega)} = \tau \int_\Omega 2\chi(\frac{\partial \Phi}{\partial z} (A_1 - A_2)ae^{A_1 + i\tau \psi} + \frac{\partial \Phi}{\partial z} (B_1 - B_2)be^{B_1 - i\tau \psi})\overline{v_0}dx + o\left(\frac{1}{\tau}\right)
\end{equation}
as $|\tau| \to +\infty$. Here $\chi \in C^\infty_0(\overline{\Omega})$ is a function such that $\chi \equiv 1$ in some neighborhood of $\text{supp} \, e_2$, $\mathcal{H} \setminus \partial \Omega \subset \text{supp} \, e_2$.

The functions $v_{0,+} = e^{i\tau \psi} \overline{v_0}$ and $v_{0,-} = e^{i\tau \psi} \overline{v_0}$ satisfy $e^{i\tau \Phi} \mathcal{L}_2(x,D)(e^{-i\tau \Phi} v_{0,+}) = \mathcal{H}_2 e^{i\tau \psi}$ and $e^{i\tau \Phi} \mathcal{L}_2(x,D)(e^{-i\tau \Phi} v_{0,-}) = \mathcal{H}_2 e^{i\tau \psi}$. More explicitly there exist two first-order operators $\mathcal{P}_k(x,D)$ such that
\begin{equation}
e^{i\tau \Phi} \mathcal{L}_2(x,D)(e^{-i\tau \Phi} v_{0,+}) = \Delta \overline{v_{0,+}} - 2\tau \frac{\partial \Phi}{\partial z} (A_2 \overline{v_{0,+}} + \mathcal{P}_1(x,D) \overline{v_{0,+}}) = o_{L^2(\Omega)}(\frac{1}{\tau}) \quad \text{as} \quad |\tau| \to +\infty
\end{equation}
and
\begin{equation}
e^{i\tau \Phi} \mathcal{L}_2(x,D)(e^{-i\tau \Phi} v_{0,-}) = \Delta \overline{v_{0,-}} - 2\tau \frac{\partial \Phi}{\partial z} (A_2 \overline{v_{0,-}} + \mathcal{P}_2(x,D) \overline{v_{0,-}}) = o_{L^2(\Omega)}(\frac{1}{\tau}) \quad \text{as} \quad |\tau| \to +\infty.
\end{equation}

Let $\chi_1 \in C^\infty_0(\overline{\Omega})$ be a function such that $\chi_1 \equiv 1$ on $\text{supp} \, \chi$. Taking the scalar product of the first equation with $\chi_1 g$ where $g \in C^2(\overline{\Omega})$ we obtain
\begin{equation}
\int_\Omega \tau \frac{\partial \Phi}{\partial z} \overline{v_{0,+}}(2 \frac{\partial}{\partial z} - A_2)gdx = o\left(\frac{1}{\tau}\right) - \int_\Omega (\overline{v_{0,+}}(\Delta + \mathcal{P}_1(x,D)^*)(\chi_1 g) - \tau \overline{v_{0,+}} \frac{\partial \Phi}{\partial z} g(\frac{\partial}{\partial z} - A_2) \chi_1)dx.
\end{equation}

By (4.49) we have
\begin{equation}
\int_\Omega \tau \frac{\partial \Phi}{\partial z} \overline{v_{0,+}}(2 \frac{\partial}{\partial z} - A_2)gdx = o\left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to +\infty.
\end{equation}

Taking the scalar product of the second equation with $\chi_1 g$ where $g \in C^2(\overline{\Omega})$ we have
\begin{equation}
\int_\Omega \tau \frac{\partial \Phi}{\partial z} \overline{v_{0,-}}(2 \frac{\partial}{\partial z} - B_2)gdx = o\left(\frac{1}{\tau}\right) - \int_\Omega (\overline{v_{0,-}}(\Delta + \mathcal{P}_2(x,D)^*)(\chi_1 g) - \tau \overline{v_{0,-}} g(\frac{\partial}{\partial z} - B_2) \chi_1)dx.
\end{equation}

By (4.49) we obtain
\begin{equation}
\int_\Omega \tau \frac{\partial \Phi}{\partial z} \overline{v_{0,-}}(2 \frac{\partial}{\partial z} - B_2)gdx = o\left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to +\infty.
\end{equation}

Taking $g$ such that $(2 \frac{\partial}{\partial z} - A_2)g = (A_1 - A_2)e^{A_1 a(z)}$ in (5.24) and $g$ such that $(2 \frac{\partial}{\partial z} - B_2)g = (B_1 - B_2)b(z)e^{B_1}$ in (5.25) from (5.22) we obtain (5.21).
To prove (5.20) we observe that

\[
(P(x,D)(u_0e^{r\psi}), v) = (P(x,D)(u_0e^{r\psi}), V)_{L^2(\Omega)} + o(\frac{1}{\tau}) = (P(x,D)(u_0e^{r\psi}), \chi V)_{L^2(\Omega)} + o(\frac{1}{\tau})
\]

(5.26)

\[
= (u_0e^{r\psi}, P(x,D)\ast(\chi V))_{L^2(\Omega)} + o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty.
\]

Then we can finish the proof of (5.20) using arguments similar to (5.23)-(5.24).

Denote \(M_1 = \frac{1}{4\delta^2} L_1(x,D)(\frac{e^{2\gamma}}{\delta^2}),\ M_2 = \frac{1}{4\delta^2} L_1(x,D)(\frac{e^{2\gamma}}{\delta^2}),\ M_3 = \frac{1}{4\delta^2} L_2(x,D)\ast(\frac{e^{2\gamma}}{\delta^2}),\ M_4 = \frac{1}{4\delta^2} L_2(x,D)\ast(\frac{e^{2\gamma}}{\delta^2}).\)

\[
M_4 = -\frac{1}{4\delta^2} L_2(x,D)\ast(\frac{e^{2\gamma}}{\delta^2}).
\]

Then there exists a constant \(C\) independent of \(\tau\) such that

\[
(P(x,D)(e^{r\phi}M_1, e^{r\phi}M_2), v)_{L^2(\Omega)} + (P(x,D)u, e^{-r\phi}M_3) = \frac{C}{\tau} + o(\frac{1}{\tau})\quad \text{as } |\tau| \to +\infty.
\]

Denote \(X_1 = \frac{e^{2\gamma}}{\delta^2},\ X_2 = \frac{e^{2\gamma}}{\delta^2},\ X_3 = \frac{e^{2\gamma}}{\delta^2},\ X_4 = \frac{e^{2\gamma}}{\delta^2} .\) Then, using the stationary phase argument we conclude

\[
(P(x,D)(e^{r\phi}X_1, e^{r\phi}X_1), v)_{L^2(\Omega)} + (P(x,D)u, e^{-r\phi}X_3) = \frac{C}{\tau} + o(\frac{1}{\tau})\quad \text{as } |\tau| \to +\infty.
\]

Next we show that

**Proposition 5.3.** Under the conditions of Theorem 1.1

\[
A_1 = A_2, \quad B_1 = B_2 \quad \text{on } \tilde{\Gamma}
\]

and

\[
J(\Phi, a, c, d) = \int_{\tilde{\Gamma}} \left\{ (\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} a(z)e^{A_1} + (\nu_1 - i\nu_2) \frac{\partial \Phi}{\partial z} b(z)e^{B_1} \right\} d\sigma = 0.
\]

**Proof.** Let \(\hat{x}\) be an arbitrary point from \(Int \tilde{\Gamma}\) and \(\Gamma_x \subset \subset \tilde{\Gamma}\). By Proposition 2.2 there exists a weight function \(\Phi\) satisfying (2.4) and (2.6). Then the boundary integrals in (5.9), (5.10) have the following asymptotic:

\[
\int_{\tilde{\Gamma}} (B_1 - B_2) d_x(\zeta)c(\zeta)e^{B_1 + B_2 - 2i\psi} d\sigma + \int_{\tilde{\Gamma}} (A_1 - A_2)(\nu_1 - i\nu_2) a(\zeta) b(\zeta)e^{A_1 + B_2 + 2i\psi} d\sigma =
\]

\[
\sum_{x \in \mathcal{F}\setminus\{x,x_+\}} \left\{ \left( \frac{2\pi}{i \frac{\partial \mu}{\partial z}(x)} \right)^{1/2} (ab(B_1 - B_2))(x)e^{-2i\psi(x)} \right\} + O\left( \frac{1}{\tau} \right) \quad \text{as } |\tau| \to +\infty.
\]
We remind that the set $\mathcal{G}$ is introduced in (2.4). Moreover, in order to avoid the contribution from the points $x_\pm$ we chose functions $a, b$ in such a way that
\begin{equation}
\frac{\partial^{[\beta]}a}{\partial x_1^{[\beta]} x_2^{[\beta]}}(x_\pm) = \frac{\partial^{[\beta]}b}{\partial x_1^{[\beta]} x_2^{[\beta]}}(x_\pm) = \frac{\partial^{[\beta]}c}{\partial x_1^{[\beta]} x_2^{[\beta]}}(x_\pm) = \frac{\partial^{[\beta]}d}{\partial x_1^{[\beta]} x_2^{[\beta]}}(x_\pm) = 0 \quad \forall |\beta| \in \{0, \ldots, 5\}.
\end{equation}

Let $\hat{\chi}_1 \in C^\infty(\partial \Omega)$ be a function such that it is equal 1 near points $x_\pm$ and has support located in a small neighborhood of these points. Then
\begin{align*}
\int_{\Gamma_*} \hat{\chi}_1(B_1 - B_2)d_\tau e^{B_1 + B_2 - 2i\tau \psi}d\sigma + \int_{\Gamma_*} \hat{\chi}_1(A_1 - A_2)(\nu_1 - i\nu_2)a_\tau \bar{b}_\tau e^{A_1 + B_2 + 2i\tau \psi}d\sigma = \\
\int_{\Gamma_*} \hat{\chi}_1(B_1 - B_2)d_\tau e^{B_1 + B_2 - 2i\tau \psi} \frac{\partial}{\partial \tau}d\sigma + \int_{\Gamma_*} \hat{\chi}_1(A_1 - A_2)(\nu_1 - i\nu_2)a_\tau \bar{b}_\tau e^{A_1 + B_2} \frac{\partial}{\partial \tau}e^{2i\tau \psi}d\sigma = \\
\int_{\Gamma_*} \frac{\partial}{\partial \tau} \left( \hat{\chi}_1(B_1 - B_2)d_\tau e^{B_1 + B_2} \right) \frac{\partial}{\partial \tau}d\sigma - \int_{\Gamma_*} \frac{\partial}{\partial \tau} \left( \hat{\chi}_1(A_1 - A_2)(\nu_1 - i\nu_2)a_\tau \bar{b}_\tau e^{A_1 + B_2} \right) \frac{\partial}{\partial \tau}e^{2i\tau \psi}d\sigma = O\left(\frac{1}{\tau}\right).
\end{align*}

In order to obtain the last equality used that by (5.32) and (2.7) the functions
\begin{align*}
\frac{\partial}{\partial \tau} \left( \hat{\chi}_1(B_1 - B_2)d_\tau e^{B_1 + B_2} \right), \quad \frac{\partial}{\partial \tau} \left( \hat{\chi}_1(A_1 - A_2)(\nu_1 - i\nu_2)a_\tau \bar{b}_\tau e^{A_1 + B_2} \right)
\end{align*}
are bounded. By (5.5), (5.9)-(5.12), (5.19)-(5.21), (5.27) - (5.28) and (5.31), we can represent the right-hand side of (5.4) as
\begin{align*}
O\left(\frac{1}{\tau}\right) = \tau F_1 + F_0 + \sum_{x \in \mathcal{G} \setminus \{x_-, x_+\}} \left( \frac{2\pi}{i\partial^2 \psi(x)} \right) \frac{1}{2} (\vec{a}B_1 - B_2))(x) e^{-2i\tau \psi(x)} \frac{2}{\sqrt{\tau}} \\
+ \left( \frac{2\pi}{i\partial^2 \psi(x)} \right) \frac{1}{2} (\vec{a}A_1 - A_2))(x) e^{2i\tau \psi(x)} \frac{2}{\sqrt{\tau}}
\end{align*}
Taking into account that $F_1$ is equal to the left-hand side of (5.30) we obtain the equality (5.30). Using (2.6) and applying Bohr’s theorem (e.g., [3], p.393), we obtain (5.29). $\square$

Denote
\begin{align*}
\mathcal{Q}_- = -(B_1 - B_2)A_1 - (A_1 - A_2)B_2 - 2 \frac{\partial}{\partial z} (A_1 - A_2), \\
\mathcal{Q}_+ = -(A_1 - A_2)B_1 - (B_1 - B_2)A_2 - 2 \frac{\partial}{\partial z} (B_1 - B_2).
\end{align*}
Thanks to (5.5), (5.9)-(5.12), (5.19)-(5.21), (5.27)-(5.29), (5.31), we can write down the right-hand side of (5.4) as

\[ I_1 + I_2 = \sum_{k=1}^{3} \tau^{2-k}(\kappa_k + \tilde{\kappa}_k) + \kappa \]

\[ + \int_{\Gamma_0} (A_1 - A_2)(\nu_1 - i\nu_2)a_\nu b_\nu e^{A_1 + \bar{\nu}z}d\sigma + \int_{\Gamma_0} (B_1 - B_2)(\nu_1 + i\nu_2)d_z(\bar{z})c_\nu(\bar{z})e^{B_1 + \bar{\nu}z}d\sigma \]

\[ - \frac{1}{\tau} \int_{\Gamma_0} Q_+ a_\nu b_\nu e^{A_1 + \bar{\nu}z}(\nabla \psi, \nu) |\nabla \psi|^2 d\sigma - \frac{1}{\tau} \int_{\Gamma_0} Q_- a_\nu b_\nu e^{A_1 + \bar{\nu}z}(\nabla \psi, \nu) |\nabla \psi|^2 d\sigma \]

\[ + 2\pi \frac{(Q_+ a_\nu b_\nu)(\bar{x})e^{(A_1 + \bar{\nu}z + 2i\tau\text{Im}\Phi)(\bar{\nu})}}{\tau|\text{det Im}\Phi'(\bar{x})|^{1/2}} \]

\[ + \frac{1}{\tau}(I_1(\partial\Omega) + I_2(\partial\Omega)) \]

\[ - 2 \int_{\partial\Omega} (\nu_1 + i\nu_2)e^{A_1 + \bar{\nu}z}c_\nu(z)\Theta_1(x, \tau)d\sigma \]

\[ - 2 \int_{\partial\Omega} (\nu_1 - i\nu_2)e^{B_1 + \bar{\nu}z}d_z(\bar{z})\Theta_2(x, \tau)d\sigma \]

\[ - 2 \int_{\partial\Omega} (\nu_1 + i\nu_2)e^{A_1 + \bar{\nu}z}a_\nu(\bar{z})\Theta_3(x, \tau)d\sigma \]

\[ (5.33) \quad -2 \int_{\partial\Omega} (\nu_1 - i\nu_2)e^{B_1 + \bar{\nu}z}b_\nu(z)\Theta_2(x, \tau)d\sigma + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \to +\infty. \]

We note that \( \kappa_k \) and \( \tilde{\kappa}_k \) denote generic constants which are independent of \( \tau \). In order to transform some terms in the above equality, we need the following proposition:

**Proposition 5.4.** There exist a holomorphic function \( \Psi \in H^{1/2}(\Omega) \) and an antiholomorphic function \( \hat{\Psi} \in H^{-1/2}(\Omega) \) such that

\[ \Psi|_\Gamma = e^{A_1 + \bar{\nu}z}, \quad \hat{\Psi}|_\Gamma = e^{B_1 + \bar{\nu}z} \]

and

\[ e^{B_1 + \bar{\nu}z}\Psi = e^{A_1 + \bar{\nu}z}\hat{\Psi} \quad \text{on } \Gamma_0. \]

**Proof.** Consider the extremal problem:

\[ J(\Psi, \hat{\Psi}) = \|e^{A_1 + \bar{\nu}z}\frac{\partial \Phi}{\partial \bar{z}}a\bar{c} - \Psi\|_{L^2(\Gamma)}^2 + \|e^{B_1 + \bar{\nu}z}\frac{\partial \Phi}{\partial \bar{z}}b\bar{d} - \hat{\Psi}\|_{L^2(\Gamma)}^2 \to \inf, \]

\[ \frac{\partial \Psi}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \quad \frac{\partial \hat{\Psi}}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \quad ((\nu_1 - i\nu_2)\Psi(z) + (\nu_1 + i\nu_2)\hat{\Psi}(\bar{z}))|_{\Gamma_0} = 0. \]

Here the functions \( a, b, c, d \) satisfy (4.3), (4.4), (4.26) and (4.27). Denote the unique solution to this extremal problem as \( (\hat{\Psi}, \tilde{\Psi}) \). Applying the Lagrange principle to this extremal problem we obtain

\[ \text{Re}(e^{A_1 + \bar{\nu}z}\frac{\partial \Phi}{\partial \bar{z}}a\bar{c} - \Psi, \delta)_{L^2(\Gamma)} + \text{Re}(e^{B_1 + \bar{\nu}z}\frac{\partial \Phi}{\partial \bar{z}}b\bar{d} - \hat{\Psi} - \tilde{\Psi})_{L^2(\Gamma)} = 0 \]

\[ (5.37) \]
for any $\delta$ from $H^\frac{1}{2}(\Omega)$ such that
\[
\frac{\partial \delta}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \quad \frac{\partial \bar{\delta}}{\partial z} = 0 \quad \text{in } \Omega, \quad (\nu_1 - i\nu_2)\delta|_{\Gamma_0} = -(\nu_1 + i\nu_2)\bar{\delta}|_{\Gamma_0}
\]
and there exist two functions $P, \tilde{P} \in H^\frac{1}{2}(\Omega)$ such that
\[
(5.38) \quad \frac{\partial P}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \quad \frac{\partial \tilde{P}}{\partial z} = 0 \quad \text{in } \Omega,
\]
\[
(5.39) \quad (\nu_1 + i\nu_2)P = e^{A_1 + \bar{A}_2 \frac{\partial \Phi}{\partial z}} a \bar{c} - \hat{\Psi} \quad \text{on } \Gamma, \quad (\nu_1 - i\nu_2)\tilde{P} = e^{B_1 + \bar{B}_2 \frac{\partial \Phi}{\partial z}} b d - \hat{\tilde{\Psi}} \quad \text{on } \Gamma,
\]
\[
(5.40) \quad (P - \tilde{P})|_{\Gamma_0} = 0.
\]
Denote $\Psi_0(z) = \frac{1}{i}(P(z) - \bar{P}(\bar{z})), \Phi_0(z) = \frac{1}{2}(P(z) + \bar{P}(\bar{z}))$. By (5.40)
\[
\text{Im } \Psi|_{\Gamma_0} = \text{Im } \Phi|_{\Gamma_0} = 0.
\]
Hence
\[
(5.41) \quad P = (\Phi_0 + i\Psi_0), \quad \bar{P} = (\Phi_0 - i\Psi_0).
\]
From (5.37)
\[
(5.42) \quad \text{Re}(e^{A_1 + \bar{A}_2 \frac{\partial \Phi}{\partial z}} a \bar{c} - \hat{\Psi}, \hat{\Phi})_{L^2(\bar{\Gamma})} + \text{Re}(e^{B_1 + \bar{B}_2 \frac{\partial \Phi}{\partial z}} b d - \hat{\tilde{\Psi}}, \hat{\tilde{\Phi}})_{L^2(\bar{\Gamma})} = 0.
\]
By (5.38), (5.39) and (5.41), we have
\[
H_1 = \text{Re}(e^{A_1 + \bar{A}_2 \frac{\partial \Phi}{\partial z}} a \bar{c} - \hat{\Psi}, e^{A_1 + \bar{A}_2 \frac{\partial \Phi}{\partial z}} a \bar{c})_{L^2(\bar{\Gamma})} + \text{Re}(e^{B_1 + \bar{B}_2 \frac{\partial \Phi}{\partial z}} \bar{b} d - \hat{\tilde{\Psi}}, e^{B_1 + \bar{B}_2 \frac{\partial \Phi}{\partial z}} \bar{b} d)_{L^2(\bar{\Gamma})}
\]
\[
\quad + \text{Re}((\nu_1 - i\nu_2)P, e^{A_1 + \bar{A}_2 \frac{\partial \Phi}{\partial z}} a \bar{c})_{L^2(\bar{\Gamma})} + \text{Re}((\nu_1 + i\nu_2)\tilde{P}, e^{B_1 + \bar{B}_2 \frac{\partial \Phi}{\partial z}} \bar{b} d - \hat{\tilde{\Psi}}, e^{B_1 + \bar{B}_2 \frac{\partial \Phi}{\partial z}} \bar{b} d)_{L^2(\bar{\Gamma})} =
\]
\[
2\text{Re}((\nu_1 - i\nu_2)(\Phi_0 + i\Psi_0), e^{A_1 + \bar{A}_2 \frac{\partial \Phi}{\partial z}} a \bar{c})_{L^2(\bar{\Gamma})} + 2\text{Re}((\nu_1 + i\nu_2)(\Phi_0 - i\Psi_0), e^{B_1 + \bar{B}_2 \frac{\partial \Phi}{\partial z}} \bar{b} d)_{L^2(\bar{\Gamma})}.
\]
We can rewrite
\[
2\text{Re}((\nu_1 - i\nu_2)\Phi_0, e^{A_1 + \bar{A}_2 \frac{\partial \Phi}{\partial z}} a \bar{c})_{L^2(\bar{\Gamma})} + 2\text{Re}((\nu_1 + i\nu_2)\Phi_0, e^{B_1 + \bar{B}_2 \frac{\partial \Phi}{\partial z}} \bar{b} d)_{L^2(\bar{\Gamma})} =
\]
\[
(5.43) \quad \Im(\Phi, \Phi_0 a, b, c, \Phi_0 d) + \Im(\Phi, \Phi_0 a, b, c, \Phi_0 d)
\]
and

\[
2\text{Re}((\nu_1 - i\nu_2)(i\bar{\Psi}_0), e^{A_1 + \bar{\Phi}_0} \frac{\partial \bar{\Phi}}{\partial \bar{z}} ac)_{L^2(\Gamma)} + 2\text{Re}((\nu_1 + i\nu_2)(-i\bar{\Psi}_0), e^{B_1 + \bar{\Phi}_0} \frac{\partial \bar{\Phi}}{\partial \bar{z}} bd)_{L^2(\Gamma)} =
\]

\[
-2\text{Im}((\nu_1 - i\nu_2)ac\bar{\Psi}_0, e^{A_1 + \bar{\Phi}_0} \frac{\partial \bar{\Phi}}{\partial \bar{z}} e^{Ai - A_2} - (\nu_1 + i\nu_2)ac\bar{\Psi}_0 \frac{\partial \bar{\Phi}}{\partial \bar{z}} e^{A_1 + \bar{A}_2})dx
\]

\[
-\frac{1}{i} \int_{\Omega} ((\nu_1 + i\nu_2)bd\Psi_0, e^{B_1 + \bar{B}_2} e^{\epsilon \bar{\Psi}} (\nu_1 - i\nu_2)bd\bar{\Psi}_0, e^{B_1 + \bar{\Phi}_0} d\bar{z})dx =
\]

\[
\frac{1}{i} (-\tilde{J}(\Psi, a\Psi, b, c, d\Psi_0) + \tilde{J}(\Psi, a\Psi, b, c, d\bar{\Psi}_0)).
\]

Then by (5.43), (5.44) and Proposition 5.3, \(H_1 = 0\). Taking into account (5.42) we obtain that \(J(\hat{\Psi}, \hat{\Psi}) = 0\). Consequently (5.34) is proved. From (4.3), (4.4), (4.26), (4.27) and (5.36) we obtain (5.35). The proof of the proposition is completed.

Thanks to Proposition 5.4, we can rewrite (5.33) as

\[
(\nu_1 - i\nu_2)ac\bar{\Psi}_0, e^{A_1 + \bar{\Phi}_0} \frac{\partial \bar{\Phi}}{\partial \bar{z}} e^{Ai - A_2} - (\nu_1 + i\nu_2)ac\bar{\Psi}_0 \frac{\partial \bar{\Phi}}{\partial \bar{z}} e^{A_1 + \bar{A}_2}) dx
\]

\[
\frac{1}{i} \int_{\Omega} ((\nu_1 + i\nu_2)bd\Psi_0, e^{B_1 + \bar{B}_2} e^{\epsilon \bar{\Psi}} (\nu_1 - i\nu_2)bd\bar{\Psi}_0, e^{B_1 + \bar{\Phi}_0} d\bar{z})dx =
\]

\[
\frac{1}{i} (-\tilde{J}(\Psi, a\Psi, b, c, d\Psi_0) + \tilde{J}(\Psi, a\Psi, b, c, d\bar{\Psi}_0)).
\]

Here \(F_k\) denote constants which are independent of \(\tau\).
Observe that

\[(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} = - (\nu_1 - i\nu_2) \frac{\partial \Phi}{\partial \bar{z}} \text{ on } \Gamma_0.\]

To see this we argue as follows. We have that \(\frac{\partial}{\partial \nu} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})\). Hence \(\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2} = - \frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial \nu} = - \frac{\partial \psi}{\partial \nu} \) and \(\frac{\partial \psi}{\partial \bar{z}} = \frac{\partial \psi}{\partial \bar{z}}\). Observe that

\[(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} = \frac{1}{2}(\nu_1 \frac{\partial \Phi}{\partial x_1} + \nu_2 \frac{\partial \Phi}{\partial x_2}) + \frac{i}{2}(\nu_2 \frac{\partial \Phi}{\partial x_1} - \nu_1 \frac{\partial \Phi}{\partial x_2}) = \frac{1}{2}(\frac{\partial \psi}{\partial \nu} + i \frac{\partial \psi}{\partial \bar{z}}) \text{ and } (\nu_1 - i\nu_2) \frac{\partial \Phi}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial \psi}{\partial \nu} - i \frac{\partial \psi}{\partial \bar{z}}).\]

Hence

\[(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} = \frac{1}{2}(\frac{\partial \psi}{\partial \nu} + i \frac{\partial \psi}{\partial \bar{z}})(\varphi + i\psi) = \frac{1}{2}(\frac{\partial \varphi}{\partial \nu} - \frac{\partial \psi}{\partial \bar{z}}) + \frac{i}{2}(\frac{\partial \varphi}{\partial \bar{z}} + \frac{\partial \psi}{\partial \nu}) = - \frac{\partial \psi}{\partial \bar{z}} + i \frac{\partial \varphi}{\partial \bar{z}}.\]

Therefore

\[(\nu_1 - i\nu_2) \frac{\partial \Phi}{\partial \bar{z}} = \frac{\partial \psi}{\partial \nu} + i \frac{\partial \varphi}{\partial \nu} = - \frac{\partial \psi}{\partial \bar{z}} - i \frac{\partial \varphi}{\partial \bar{z}}.\]

Taking into account that \(\psi|_{\Gamma_0} = 0\) we obtain (5.46).

Then, using (5.46), on \(\Gamma_0\) we have

\[-(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} \left( e^{A_1 + \overline{\chi}_2} - \Psi \right) c(\bar{z}) \left( \frac{a_{2, +} e^{-2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} + \frac{a_{2, -} e^{2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} \right) \]

\[-(\nu_1 - i\nu_2) \frac{\partial \Phi}{\partial \bar{z}} \left( e^{B_1 + \overline{\Psi}} - \overline{\Psi} \right) b(z) \left( \frac{a_{2, +} e^{-2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} + \frac{a_{2, -} e^{2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} \right) \]

\[-(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} \left( e^{A_1 + \overline{\chi}_2} - \Psi \right) c(\bar{z}) \frac{a_{2, -} e^{-2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} + (e^{A_1 + \overline{\chi}_2} - \Psi e^{\overline{\chi}_2 - \overline{\xi}_2} - \overline{\Psi}) c(\bar{z}) \frac{d_{2, -} e^{2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} \]

\[-(\nu_1 - i\nu_2) \frac{\partial \Phi}{\partial \bar{z}} \left( e^{B_1 + \overline{\Psi}} - \overline{\Psi} \right) b(z) \left( \frac{a_{2, -} e^{-2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} + (e^{B_1 + \overline{\Psi}} - \overline{\Psi} e^{\overline{\chi}_2 - \overline{\xi}_2} - \overline{\Psi}) \frac{d_{2, +} e^{-2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} \right) \]

\[-(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} \left( e^{A_1 + \overline{\chi}_2} - \Psi \right) c(\bar{z}) \frac{a_{2, +} e^{-2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} + (e^{A_1 + \overline{\chi}_2} - \Psi e^{\overline{\chi}_2 - \overline{\xi}_2} - \overline{\Psi}) \frac{d_{2, +} e^{2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} \]

\[-(\nu_1 - i\nu_2) \frac{\partial \Phi}{\partial \bar{z}} \left( e^{B_1 + \overline{\Psi}} - \overline{\Psi} \right) b(z) \left( \frac{a_{2, +} e^{2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} + (e^{B_1 + \overline{\Psi}} - \overline{\Psi} e^{\overline{\chi}_2 - \overline{\xi}_2} - \overline{\Psi}) \frac{d_{2, -} e^{-2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} \right) \]

\[(5.47)\]

\[-(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} \left( e^{\overline{\chi}_2} - \Psi e^{-A_1} \right) c(\bar{z}) \frac{d_{2, -} e^{2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} \]

\[-(\nu_1 - i\nu_2) \frac{\partial \Phi}{\partial \bar{z}} \left( e^{\overline{\chi}_2} - \overline{\Psi} e^{-B_1} \right) b(z) \frac{d_{2, +} e^{-2i\varphi(\bar{z})}}{|\det \Im \Phi''(\bar{z})|^\frac{3}{2}} \]
and

\[-(\nu_1 + i\nu_2)(e^{A_1 + \overline{\mathcal{X}_2}} - \Psi) \frac{\partial \Phi}{\partial z} a(z) \left( \frac{c_{2+}e^{2\tau\nu(z)}}{|\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} + \frac{c_{2-}e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} \right) \]

\[-(\nu_1 - i\nu_2)(e^{B_1 + \overline{\mathcal{B}_2}} - \tilde{\Psi}) \frac{\partial \overline{\Phi}}{\partial \overline{z}} d(\overline{z}) \left( \frac{b_{2+}e^{2\tau\nu(z)}}{|\det \Im \Phi'(\overline{x})|^\frac{1}{2}} + \frac{b_{2-}e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\overline{x})|^\frac{1}{2}} \right) \]

\[-(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} a(z) \left( e^{A_1 + \overline{\mathcal{X}_2}} - \Psi \right) \left( \frac{c_{2+}e^{2\tau\nu(z)}}{|\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} + \left( e^{A_1 + \overline{\mathcal{B}_2}} - e^{-B_1 + A_1} \tilde{\Psi} \right) \frac{b_{2+}e^{2\tau\nu(z)}}{|\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} \right) \]

\[-(\nu_1 - i\nu_2) \frac{\partial \overline{\Phi}}{\partial \overline{z}} d(\overline{z}) \left( e^{B_1 + \overline{\mathcal{X}_2}} - \Psi e^{-A_2 - B_1} \right) \left( \frac{c_{2-}e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} + \left( e^{B_1 + \overline{\mathcal{B}_2}} - \tilde{\Psi} \right) \frac{b_{2-}e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} \right) \]

\[-(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} a(z) \left( e^{A_1 + \overline{\mathcal{X}_2}} - \Psi \right) \left( \frac{c_{2+}e^{2\tau\nu(z)}}{|\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} + \left( e^{A_1 + \overline{\mathcal{B}_2}} - e^{-B_1 + A_1} \tilde{\Psi} \right) \frac{b_{2+}e^{2\tau\nu(z)}}{|\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} \right) \]

\[-(\nu_1 - i\nu_2) \frac{\partial \overline{\Phi}}{\partial \overline{z}} d(\overline{z}) \left( e^{B_1 + \overline{\mathcal{X}_2}} - \Psi e^{-A_2 - B_1} \right) \left( \frac{c_{2-}e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} + \left( e^{B_1 + \overline{\mathcal{B}_2}} - \tilde{\Psi} \right) \frac{b_{2-}e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} \right) \]

(5.48)

Using (5.47), (5.48) and Proposition 7.2 in Section 7, we rewrite (5.45) as

\[ o\left( \frac{1}{\tau} \right) = \sum_{k=1}^{3} \tau^{2-k} \overline{F}_k + \]

\[ 2 \frac{1}{\tau |\det \Im \Phi'(\tilde{x})|^\frac{1}{2}} \left\{ \left( \overline{(Q_+ + (q_1 - q_2))a\tilde{b}\overline{\Phi}}(\overline{x}) e^{(A_1 + \overline{\mathcal{B}_2}) + 2\tau\Phi}(\overline{x}) \right) \right. \]

\[ + \left. \left( \overline{(Q_- + (q_1 - q_2))d\tilde{c}}(\overline{x}) e^{(B_1 + \overline{\mathcal{X}_2}) - 2\tau\Phi}(\overline{x}) \right) \right\} \]

\[ - \frac{1}{4\tau} \int_{\Gamma_0} (\nu_1 - i\nu_2)(e^{B_1 + \overline{\mathcal{X}_2}} - \tilde{\Psi}) d(\overline{z}) \left( \frac{e^{2\tau\nu(z)}}{|\det \Im \Phi'(\overline{x})|^\frac{1}{2}} \frac{\partial \nu_2 + e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\overline{x})|^\frac{1}{2}} d\sigma \right) \]

\[ - \frac{1}{4\tau} \int_{\Gamma_0} (\nu_1 + i\nu_2)(e^{A_1 + \overline{\mathcal{X}_2}} - \Psi) \left( \frac{e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\overline{x})|^\frac{1}{2}} \frac{\partial \nu_2 + e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\overline{x})|^\frac{1}{2}} d\sigma \right) \]

\[ - \frac{1}{4\tau} \int_{\Gamma_0} (\nu_1 + i\nu_2)(e^{A_1 + \overline{\mathcal{X}_2}} - \Psi) \left( \frac{e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\overline{x})|^\frac{1}{2}} \frac{\partial \nu_2 + e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\overline{x})|^\frac{1}{2}} d\sigma \right) \]

\[ - \frac{1}{4\tau} \int_{\Gamma_0} (\nu_1 - i\nu_2)(e^{B_1 + \overline{\mathcal{B}_2}} - \tilde{\Psi}) \left( \frac{e^{2\tau\nu(z)}}{|\det \Im \Phi'(\overline{x})|^\frac{1}{2}} \frac{\partial \nu_2 + e^{-2\tau\nu(z)}}{|\det \Im \Phi'(\overline{x})|^\frac{1}{2}} d\sigma \right) \]

(5.49)

\[ + o\left( \frac{1}{\tau} \right) \text{ as } |\tau| \to +\infty. \]
Using Proposition 5.3 we take $a = a^*_s, d = d_s$ and $b = b_s, c = c_s$ such that

$$\int_{\Gamma_0} (\nu_1 - i\nu_2)(e^{B_1 + \bar{B}_2} - \tilde{\Psi})b_s(z)\frac{1}{\bar{z} - z} d\sigma = 0 \quad \text{and} \quad \int_{\Gamma_0} (\nu_1 + i\nu_2)(e^{A_1 + \bar{A}_2} - \Psi)a_s(z)\frac{1}{\bar{z} - z} d\sigma = 0.$$  

Then we obtain from (5.49)

$$(5.50) \quad \frac{((Q_+ + (q_1 - q_2))a_s(z)e^{(A_1 + \bar{A}_2) + 2\tau \text{Im}\Phi(\bar{x})}}{\det \text{Im} \Phi''(\bar{x})} = 0.$$  

On the other hand using the Proposition 5.3 again we can take $a = a^*, d = d^*$ and $b = b^*, c = c^*$ such that

$$\int_{\Gamma_0} (\nu_1 - i\nu_2)(e^{B_1 + \bar{B}_2} - \tilde{\Psi})d^*(\bar{z})\frac{1}{\bar{z} - z} d\sigma = 0 \quad \text{and} \quad \int_{\Gamma_0} (\nu_1 + i\nu_2)(e^{A_1 + \bar{A}_2} - \Psi)c^*(\bar{z})\frac{1}{\bar{z} - z} d\sigma = 0.$$  

These equalities and (5.49) imply

$$(5.51) \quad \frac{((Q_+ + (q_1 - q_2))d^*c^*)(\bar{x})e^{(B_1 + \bar{B}_2) - 2\tau \text{Im}\Phi(\bar{x})}}{\det \text{Im} \Phi''(\bar{x})} = 0.$$  

Let $\bar{x}$ be an arbitrary point from $\Omega$. Consider the sequence of the functions $\Phi_\epsilon$ given by Proposition 2.1. From (5.50) and (5.51) we have

$$(Q_+ + (q_1 - q_2))a_s(z)e^{(A_1 + \bar{A}_2) + 2\tau \text{Im}\Phi(\bar{x})} = 0, \quad ((Q_+ + (q_1 - q_2))c^*(\bar{x})e^{(B_1 + \bar{B}_2) - 2\tau \text{Im}\Phi(\bar{x})} = 0.$$  

The proof of the theorem is completed. □

6. Appendix I

Consider the Cauchy problem for the Cauchy-Riemann equations

$$(6.1) \quad L(\phi, \psi) = (\frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi}{\partial x_1}) = 0 \quad \text{in} \quad \Omega, \quad (\phi, \psi)_{\Gamma_0} = (b_1(x), b_2(x)), \quad \phi(\bar{x}_j) = c_{0,j}, \quad \forall j \in \{1, \ldots, N\} \quad \text{and} \quad \forall l \in \{0, \ldots, 5\}.$$  

Here $\hat{x}_1, \ldots, \hat{x}_N$ be an arbitrary fixed points in $\Omega$. We consider the pair $b_1, b_2$ and complex numbers $\hat{C} = (c_{0,1}, c_{1,1}, c_{2,1}, c_{3,1}, c_{4,1}, c_{5,1}, \ldots, c_{0,N}, c_{1,N}, c_{2,N}, c_{3,N}, c_{4,N}, c_{5,N})$ as initial data for (6.1). The following proposition establishes the solvability of (6.1) for a dense set of Cauchy data.

**Proposition 6.1.** There exists a set $\mathcal{O} \subset (C^5(\Gamma_0))^2 \times \mathbb{C}^6$ such that for each $(b_1, b_2, \hat{C}) \in \mathcal{O}$, (6.1) has at least one solution $(\phi, \psi) \in (C^5(\Omega))^2$ and $\bar{\mathcal{O}} = (C^5(\Gamma_0))^2 \times \mathbb{C}^6$.

**Proof.** Denote $B = (b_1, b_2)$ an arbitrary element of the space $C^7(\Gamma_0) \times C^7(\Gamma_0)$. Consider the following extremal problem

$$J_\epsilon(\phi, \psi) = \|(\phi, \psi) - B\|^4_{B^4(\Gamma_0)} + \epsilon \sum_{k=0}^3 \left\| \frac{\partial^k \phi(\hat{x}_j)}{\partial x_1^k} \right\|^4_{B^4(\Gamma_0)} + \epsilon^2 \sum_{k=0}^5 \left( \frac{\partial^k \phi(\hat{x}_j)}{\partial x_1^k} \right)^2.$$  

+ $\frac{1}{\epsilon} \left\| \Delta^3 L(\phi, \psi) \right\|^4_{L^4(\Omega)} + \sum_{j=1}^N \sum_{k=0}^5 \left( \frac{\partial^k \phi(\hat{x}_j)}{\partial x_1^k} \right)^2 \rightarrow \inf,$
(6.3) \( (\phi, \psi) \in W^7_4(\Omega) \times W^7_4(\Omega) \).

Here \( B^j_k \) denotes the Besov space of the corresponding orders.

For each \( \epsilon > 0 \) there exists a unique solution to (6.2), (6.3) which we denote as \( (\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) \). This fact can be proved by standard arguments. We fix \( \epsilon > 0 \). Denote by \( U_{ad} \) the set of admissible elements of the problem (6.2), (6.3), namely

\[
U_{ad} = \{ (\phi, \psi) \in W^7_4(\Omega) \times W^7_4(\Omega) | J_\epsilon(\phi, \psi) < \infty \}.
\]

Denote \( \hat{J}_\epsilon = \inf_{(\phi, \psi) \in U_{ad}} J_\epsilon(\phi, \psi) \). Clearly the pair \((0, 0) \in U_{ad} \). Therefore there exists a minimizing sequence \( \{(\phi_k, \psi_k)\}_{k=1}^\infty \subset W^7_4(\Omega) \times W^7_4(\Omega) \) such that

\[
\hat{J}_\epsilon = \lim_{k \to +\infty} J_\epsilon(\phi_k, \psi_k).
\]

Observe that the minimizing sequence is bounded in \( W^7_4(\Omega) \times W^7_4(\Omega) \). Indeed, since the sequence \( \{\Delta^3 L(\phi_k, \psi_k), L(\phi_k, \psi_k)|_{\partial \Omega}, \ldots, \frac{\partial^{3^j}}{\partial \nu^{3^j}} L(\phi_k, \psi_k)|_{\partial \Omega} \} \) is bounded in \( L^4(\Omega) \times \Pi^3_{k=0} B^j_k \) the standard elliptic \( L^p \)-estimate implies that the sequence \( \{L(\phi_k, \psi_k)\} \) is bounded in the space \( W^{j_0}_6(\Omega) \times W^{j_0}_6(\Omega) \). By Sobolev imbedding theorem the sequence \( \{(\phi_k, \psi_k)\} \) is bounded in \( C^6(\Omega) \times C^6(\Omega) \). Then taking if necessary a subsequence, (which we denote again as \( \{(\phi_k, \psi_k)\} \) ) we obtain

\[
(\phi_k, \psi_k) \rightharpoonup (\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) \quad \text{weakly in } W^7_4(\Omega) \times W^7_4(\Omega),
\]

\[
\left( \frac{\partial^j \hat{\phi}_\epsilon}{\partial \nu^j}, \frac{\partial^j \hat{\psi}_\epsilon}{\partial \nu^j} \right) \rightharpoonup \left( \frac{\partial^j \hat{\phi}_\epsilon}{\partial \nu^j}, \frac{\partial^j \hat{\psi}_\epsilon}{\partial \nu^j} \right) \quad \text{weakly in } B^j_4(\partial \Omega) \times B^j_4(\partial \Omega) \quad \forall j \in \{0, 1, 2, 3\},
\]

\[
\frac{\partial^k}{\partial z^k}(\phi + i \psi)(\hat{x}_j) - c_{k,j} \rightharpoonup C_{k,j,\epsilon}, \quad k \in \{0, \ldots, 5\},
\]

\[
\Delta^3 L(\phi_k, \psi_k) \rightharpoonup r_\epsilon \quad \text{weakly in } L^4(\Omega) \times L^4(\Omega), \quad L(\phi_k, \psi_k) \rightharpoonup \tilde{r}_\epsilon \quad \text{weakly in } W^6_4(\Omega) \times W^6_4(\Omega).
\]

Obviously, \( r_\epsilon = \Delta^3 L(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon), \tilde{r}_\epsilon = L(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) \). Then, since the norms in the spaces \( L^4(\Omega) \) and \( B^j_4 \) are lower semicontinuous with respect to weak convergence we obtain that

\[
J_\epsilon(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) \leq \lim_{k \to +\infty} J_\epsilon(\phi_k, \psi_k) = \hat{J}_\epsilon.
\]

Thus the pair \( (\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) \) is a solution to the extremal problem (6.2), (6.3). Since the set of admissible elements is convex and the functional \( J_\epsilon \) is strictly convex, this solution is unique.

By Fermat’s theorem we have

\[
J'_\epsilon(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon)[\delta] = 0, \quad \forall \delta \in W^7_4(\Omega) \times W^7_4(\Omega).
\]

This equality can be written in the form

\[
I_{\Gamma_\alpha}^\ell(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) - B)[\delta] + \epsilon \sum_{k=0}^3 I_{\partial \Omega_\alpha}^\ell(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon)[\frac{\partial^k}{\partial \nu^k} \hat{\phi}_\epsilon, \hat{\psi}_\epsilon][\delta] + (p_\epsilon, \Delta^3 L \delta)_{L^2(\Omega)} + \sum_{j=1}^N \sum_{k=0}^5 \frac{\partial^k}{\partial z^k}(\hat{\phi}_\epsilon + i \hat{\psi}_\epsilon)\partial_j c_{k,j} + \frac{\partial^k}{\partial z^k} \delta_1 + i \delta_2 = 0,
\]

\[
+ \sum_{j=1}^N \sum_{k=0}^5 \frac{\partial^k}{\partial z^k}(\hat{\phi}_\epsilon + i \hat{\psi}_\epsilon)\partial_j c_{k,j} + \frac{\partial^k}{\partial z^k} \delta_1 + i \delta_2 = 0.
\]
where \( p_\epsilon = \frac{4}{\epsilon} ((\Delta^3 (\frac{\partial \phi}{\partial x_1}) - \frac{\partial \phi}{\partial x_2}))^3, (\Delta^3 (\frac{\partial \phi}{\partial x_2} + \frac{\partial \phi}{\partial x_1}))^3 \) and \( I'_{\epsilon, \infty}(\hat{w}) \) denotes the derivative of the functional \( w \to \|w\|_{B_3^4(\Gamma^*)}^4 \) at \( \hat{w} \).

Observe that the pair \( J_{\epsilon}(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) \leq J_{\epsilon}(0, 0) = \|B\|_{B_3^4(\Gamma_0)}^4 + \sum_{j=1}^N \sum_{k=0}^5 |c_{k,j}|^2 \). This implies that the sequence \( \{(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon)\} \) is bounded in \( B_3^4(\Gamma_0) \), the sequences \( \{\epsilon^3 \frac{\partial}{\partial x^ \epsilon}(\hat{\phi}_\epsilon + i\hat{\psi}_\epsilon)(\hat{x}_j) - c_{k,j}\} \) are bounded in \( \mathbb{C} \), the sequence \( \epsilon \sum_{k=0}^3 I'_{\epsilon, \infty}(\hat{\phi}_\epsilon + i\hat{\psi}_\epsilon) \frac{\partial}{\partial x^ \epsilon} \hat{\delta}(\hat{\tilde{\delta}}_1 + i\hat{\tilde{\delta}}_2)(\hat{x}_j) = 0 \) converges to zero for any \( \hat{\delta} \) from \( B_3^4(\partial \Omega) \times B_3^4(\Gamma_0) \). Then (6.4) implies that the sequence \( \{p_\epsilon\} \) is bounded in \( L^4(\Omega) \times L^4(\Omega) \).

Therefore there exist \( B \in B_3^4(\Gamma_0) \times B_3^4(\Gamma_0) \), \( C_{0,j}, C_{1,j}, \ldots, C_{5,j} \in \mathbb{C} \) and \( p = (p_1, p_2) \in L^4(\Omega) \times L^4(\Omega) \) such that

\[
(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) - B \to B \quad \text{weakly in } B_3^4(\Gamma_0) \times B_3^4(\Gamma_0), \quad p_\epsilon \to p \quad \text{weakly in } L^4(\Omega) \times L^4(\Omega),
\]

and

\[
\frac{\partial}{\partial x^ k}(\hat{\phi}_\epsilon + i\hat{\psi}_\epsilon)(\hat{x}_j) - c_{k,j} \to C_{k,j} \quad k \in \{0, 1, \ldots, 5\}, \quad j \in \{1, \ldots, N\}.
\]

Passing to the limit in (6.4) we obtain

\[
I'_{\epsilon, \infty}(B)[\hat{\delta}] + (p, \Delta^3 L\hat{\delta})_{L^2(\Omega)} + 2Re \sum_{j=1}^N \sum_{k=0}^5 C_{k,j} \frac{\partial}{\partial x^ k}(\hat{\delta}_1 + i\hat{\delta}_2)(\hat{x}_j) = 0 \quad \forall \hat{\delta}_1, \hat{\delta}_2 \in C_0^\infty(\Omega).
\]

Next we claim that

\[
\Delta^3 p = 0 \quad \text{in } \Omega \setminus \bigcup_{j=1}^N \{\hat{x}_j\}
\]

in the sense of distributions. Suppose that (6.8) is already proved. This implies

\[
(p, \Delta^3 L\hat{\delta})_{L^2(\Omega)} + 2Re \sum_{j=1}^N \sum_{k=0}^5 C_{k,j} \frac{\partial}{\partial x^ k}(\hat{\delta}_1 + i\hat{\delta}_2)(\hat{x}_j) = 0 \quad \forall \hat{\delta}_1, \hat{\delta}_2 \in C_0^\infty(\Omega).
\]

If \( p = (p_1, p_2) \), denoting \( P = p_1 - ip_2 \), we have

\[
\text{Re}(\Delta^3 P, \partial_z(\hat{\delta}_1 + i\hat{\delta}_2))_{L^2(\Omega)} + \text{Re} \sum_{j=1}^N \sum_{k=0}^5 C_{k,j} \frac{\partial}{\partial x^ k}(\hat{\delta}_1 + i\hat{\delta}_2)(\hat{x}_j) = 0 \quad \forall \hat{\delta}_1, \hat{\delta}_2 \in C_0^\infty(\Omega).
\]

Since by (6.8) \( \supp \Delta^3 P \subset \bigcup_{j=1}^N \{\hat{x}_j\} \) there exist some constants \( m_{\beta,j} \) and \( \hat{\ell}_j \) such that

\[
\Delta^3 P = \sum_{j=1}^N \sum_{|\beta|=1} \hat{\ell}_j m_{\beta,j} D^\beta \delta(x - \hat{x}_j).
\]

The above equality can be written in the form

\[
- \sum_{|\beta|=1} \hat{\ell}_j m_{\beta,j} \frac{\partial}{\partial z} D^\beta \delta(x - \hat{x}_j) = \sum_{k=0}^5 (-1)^k C_{k,j} \frac{\partial}{\partial x_k} \delta(x - \hat{x}_j).
\]

From this we obtain

\[
C_{0,j} = C_{1,j} = \ldots = C_{5,j} = 0 \quad j \in \{1, \ldots, N\}.
\]

Therefore

\[
\Delta^3 p = 0 \quad \text{in } \Omega.
\]
where

From (6.6) and (6.9) we obtain

\[ \{ \]

Therefore the sequence

\[ (6.15) \]

\[ \tilde{\phi}_{\epsilon_k}, \tilde{\psi}_{\epsilon_k} \]

\[ 0 = (p, \Delta^3 L \tilde{\delta})_{L^2(\Omega)} = (\tilde{x}p, \Delta^3 L \tilde{\delta})_{L^2(\Omega)} + (p, [\Delta^3 L, \tilde{x}] \delta)_{L^2(\Omega)} \forall \delta \in W^7_4(\Omega) \times W^7_4(\Omega). \]

Denote \( L \tilde{\delta} = \tilde{\delta} \). Consider the functional mapping \( \tilde{\delta} \in W^2_7(\text{supp} \tilde{x}) \) to \( (p, [\Delta^3 L, \tilde{x}] \tilde{\delta})_{L^2(\Omega)} \), where

\[ L \tilde{\delta} = \tilde{\delta} \quad \text{in} \ \Omega, \quad \text{Im} \tilde{\delta}|_{\partial \Omega} = 0, \quad \int_{\text{supp} \tilde{x}} \text{Re} \tilde{\delta} dx = 0, \]
where $S$ denotes the boundary of $\text{supp} \tilde{\chi}$. For each $\tilde{\delta} \in W^2_4(\text{supp} \tilde{\chi}) \times W^2_4(\text{supp} \tilde{\chi})$, there exists a unique solution $\tilde{\delta} \in W^3_4(\text{supp} \tilde{\chi}) \times W^3_4(\text{supp} \tilde{\chi})$. Hence the functional is well-defined and continuous on $W^2_4(\text{supp} \tilde{\chi})$. Therefore there exists $q, r, q_0 \in L^2\frac{3}{4}(\text{supp} \tilde{\chi})$ such that 

$$\int_{\text{supp} \tilde{\chi}} (\sum_{j,k=1}^2 \partial_{x_j,x_k}^2 \tilde{\delta} + (q, \tilde{\delta}) + q_0 \tilde{\delta}) dx = (p, [\Delta^3 L, \tilde{\chi}] \tilde{\delta})_{L^2(\text{supp} \tilde{\chi})}. $$

Consider the boundary value problem

$$\Delta^3 \tilde{P} = \tilde{f} \quad \text{in } \text{supp} \tilde{\chi}, \quad \tilde{P}|_S = \frac{\partial \tilde{P}}{\partial \nu}|_S = \frac{\partial^2 \tilde{P}}{\partial \nu^2}|_S = 0.$$

Here $\tilde{f} = 2 \text{div} (\nabla q) - q_0 - \sum_{j,k=1}^2 \partial_{x_j,x_k}^2 r_{jk}$. A solution to this problem exists and is unique, since $\tilde{f} \in (W^2_4(\text{supp} \tilde{\chi}))'$. Then $P \in W^1_4(\text{supp} \tilde{\chi}) \times W^1_4(\text{supp} \tilde{\chi})$. On the other hand, thanks to (6.16), $P = \tilde{\chi} p \in W^1_4(\text{supp} \tilde{\chi}) \times W^1_4(\text{supp} \tilde{\chi})$.

Next we take another smooth cut off function $\tilde{\chi}_1$ such that $\tilde{\chi}_1 \in \mathcal{A}$ and $\text{Int} (\text{supp} \chi_1)$ is a simply connected domain. A neighborhood of $\tilde{x}$ belongs to $\mathcal{A}_1 = \{x | \tilde{\chi}_1 = 1\}$, the interior of $\mathcal{A}_1$ is connected, and $\text{Int} \mathcal{A}_1 \cap \overline{\Gamma}$ contains an open subset $\mathcal{O}$ in $\partial \Omega$. Similarly to (6.16) we have

$$(\tilde{\chi}_1 p, \Delta^3 L \tilde{\delta})_{L^2(\Omega)} - (p, [\Delta^3 L, \tilde{\chi}_1] \tilde{\delta})_{L^2(\Omega)} = 0 \quad \forall \tilde{\delta} \in W^3_4(\Omega) \times W^3_4(\Omega).$$

This equality implies that $\tilde{\chi}_1 p \in W^3_4(\Omega)$, using a similar argument to the one above.

Next we take another smooth cut off function $\tilde{\chi}_2$ such that $\text{supp} \tilde{\chi}_2 \subset \mathcal{A}_2$ and $\text{Int} (\text{supp} \chi_2)$ is a simply connected domain. A neighborhood of $\tilde{x}$ belongs to $\mathcal{A}_3 = \{x | \tilde{\chi}_2 = 1\}$, the interior of $\mathcal{A}_1$ is connected, and $\text{Int} \mathcal{A}_3 \cap \overline{\Gamma}$ contains an open subset $\mathcal{O}$ in $\partial \Omega$. Similarly to (6.16) we have

$$(\tilde{\chi}_2 p, \Delta^3 L \tilde{\delta})_{L^2(\Omega)} - (p, [\Delta^3 L, \tilde{\chi}_2] \tilde{\delta})_{L^2(\Omega)} = 0 \quad \forall \tilde{\delta} \in W^3_4(\Omega) \times W^3_4(\Omega).$$

This equality implies that $\tilde{\chi}_2 p \in W^3_4(\Omega)$, using a similar argument to the one above. Let $\omega$ be a domain such that $\omega \cap \Omega = \emptyset$, $\partial \omega \cap \partial \Omega \subset \mathcal{O}$ contains an open set in $\partial \Omega$.

We extend $p$ on $\omega$ by zero. Then

$$(\Delta^3 (\tilde{\chi}_2 p), L \tilde{\delta})_{L^2(\Omega,\omega)} + (p, [\Delta^3 L, \tilde{\chi}_2] \tilde{\delta})_{L^2(\Omega,\omega)} = 0.$$ 

Hence, since $[\Delta^3 L, \tilde{\chi}_2]|_{\mathcal{A}_1} = 0$ we have

$$L^* \Delta^3 (\tilde{\chi}_2 p) = 0 \quad \text{in } \text{Int} \mathcal{A}_2 \cup \omega, \quad p|_\omega = 0.$$

By Holmgren’s theorem $\Delta^3 (\tilde{\chi}_2 p)|_{\text{Int} \mathcal{A}_1} = 0$, that is, $(\Delta^3 p)(\tilde{x}) = 0$. Thus (6.8) is proved. 

Consider the Cauchy problem for the Cauchy-Riemann equations

$$L(\phi, \psi) = \left( \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi}{\partial x_1} \right) = 0 \quad \text{in } \Omega, \quad (\phi, \psi)|_{\Gamma_0} = (b(x), 0),$$

$$\frac{\partial}{\partial x^l}(\phi + i \psi)(\tilde{x}_j) = c_{0,j}, \quad \forall j \in \{1, \ldots, N\} \quad \text{and } \forall l \in \{0, \ldots, 5\}.$$ 

Here $\tilde{x}_1, \ldots, \tilde{x}_N$ be an arbitrary fixed points in $\Omega$. We consider the function $b$ and complex numbers $\tilde{C} = (c_{0,1}, c_{1,1}, c_{2,1}, c_{3,1}, c_{4,1}, c_{5,1} \ldots c_{0,N}, c_{1,N}, c_{2,N}, c_{3,N}, c_{4,N}, c_{5,N})$ as initial data for (6.1). The following proposition establishes the solvability of (6.1) for a dense set of Cauchy data.
Corollary 6.1. There exists a set \( \mathcal{O} \subset C^5(\Gamma_0) \times C^6_{\gamma} \) such that for each \((b, \mathcal{C}) \in \mathcal{O}, \) problem (6.17) has at least one solution \((\phi, \psi) \in (C^5(\Gamma))^2 \) and \( \overline{\mathcal{O}} = C^5(\Gamma) \times C^6_{\gamma}. \)

The proof of Corollary 6.1 is similar to the proof of Proposition 6.1. The only difference is that instead of extremal problem considered there we have to use the following extremal problem

\[
J_c(\phi, \psi) = \| \phi - b \|_{B_{A^+}(\Gamma_0)}^4 + \epsilon \sum_{k=0}^{3} \| \frac{\partial^k(\phi, \psi)}{\partial \nu^k} \|_{L^2(\partial \Omega)}^4 + \frac{1}{\epsilon} \| \Delta^3 L(\phi, \psi) \|_{L^4(\Omega)}^4 + \sum_{j=1}^{N} \sum_{k=0}^{5} | \frac{\partial^k}{\partial z^j}(\phi + i\psi)(\tilde{x}_j) - c_{k,j} |^2 \to \inf,
\]

\[(\phi, \psi) \in W^4_0(\Omega) \times W^4_0(\Omega), \quad \psi|_{\Gamma_0} = 0.
\]

We have

Proposition 6.2. Let \( \alpha \in (0, 1), \) \( g_1, \ldots, g_p \) be linearly independent functions in \( L^2(\Gamma_0), \) \( y_1, \ldots, y_k \) be some arbitrary points from \( \Gamma_0, \) \( y_{k+1}, \ldots, y_l \) be some arbitrary points from \( \Omega \) and \( \tilde{x} \) be an arbitrary point from \( \Omega \setminus \{y_{k+1}, \ldots, y_l\}. \) Then there exist a holomorphic function \( a \in C^{5+\alpha}(\Omega) \) and an antiholomorphic function \( d \in C^{5+\alpha}(\Omega) \) such that \((ae^A + de^B)|_{\Gamma_0} = 0,
\]

\[
\int_{\Gamma_0} ag_\mu d\sigma = 0 \quad p \in \{1, \ldots, m\}; \quad \frac{\partial^{k+j}a}{\partial x_1^k \partial x_2^j}(y_\ell) = 0 \quad k + j \leq 5, \quad \forall \ell \in \{1, \ldots, k\},
\]

and

\[a(\tilde{x}) \neq 0 \quad \text{and} \quad d(\tilde{x}) \neq 0.
\]

Proof. Consider the operator

\[
R(\gamma) = (\int_{\Gamma_0} ag_1 d\sigma, \ldots, \int_{\Gamma_0} ag_p d\sigma, a(y_1), \ldots, \frac{\partial^5 a}{\partial z^5}(y_1), \ldots, a(y_k), \ldots, \frac{\partial^5 a}{\partial z^5}(y_k), \ldots, \frac{\partial^5 d}{\partial z^5}(y_1), \ldots, \frac{\partial^5 d}{\partial z^5}(y_k), a(\tilde{x}), d(\tilde{x})).
\]

Here \( \gamma \in C_0^\infty(\tilde{\Gamma}) \) and the functions \( a \) and \( d \) are solutions to the following problem

\[
\frac{\partial a}{\partial z} = 0 \quad \text{in} \ \Omega, \quad \frac{\partial d}{\partial z} = 0 \quad \text{in} \ \Omega, \quad (ae^A + de^B)|_{\partial \Omega} = \gamma.
\]

Consider the image of the operator \( R. \) Clearly it is closed. Let us show that the point \((0, \ldots, 0, 1, 1)\) belongs to the image of the operator \( R. \) Let a holomorphic function \( a \) satisfy

\[
\int_{\Gamma_0} ag_\mu d\sigma = \cdots = \int_{\Gamma_0} ag_p d\sigma = 0 \quad \text{and}
\]

\[
\frac{\partial^3 a}{\partial x_1^\beta \partial x_2^\gamma}(y_j) = 0 \quad \forall |\beta| \in \{0, \ldots, 5\}, \quad j \in \{1, \ldots, k\}, \quad |a(\tilde{x})| > 2.
\]

Consider the function \(-e^{A-B}a(z)\) and the pair \((b_1, b_2) = (\text{Re}\{e^{A-B}a\}, \text{Im}\{e^{A-B}a\})\). Using Proposition 6.1 we solve problem (6.1) with \( l = 0 \) approximately. Let \((\phi_\epsilon, \psi_\epsilon)\) be a sequence of functions such that

\[
\frac{\partial}{\partial z}(\phi_\epsilon + i\psi_\epsilon) = 0 \quad \text{in} \ \Omega, \quad (\phi_\epsilon, \psi_\epsilon)|_{\Gamma_0} \to (b_1, b_2) \quad \text{in} \ C^{5+\alpha}(\Gamma_0), \quad (\phi_\epsilon + i\psi_\epsilon)(\tilde{x}) \to 1.
\]
Denote $d_e = \phi_e - i\psi_e, \beta_e = ae^A + d_e e^R$. Then the sequence $\{\beta_e\}$ converges to zero in the space $C^{5+\alpha}(\Gamma_0)$.

By Proposition 2.5 there exists a solution to problem (2.46) with the initial data $\beta_e$, which we denote as $\{\tilde{\alpha}_e, \tilde{d}_e\}$ such that the sequence $\{\tilde{\alpha}_e, \tilde{d}_e\}$ converges to zero in $(C^5(\Omega))^2$. Denote by $\gamma_e = (a + \tilde{\alpha}_e, d_e + \tilde{d}_e)|_{\Gamma_0}$. Clearly $R(\gamma_e)$ converges to $(0, \ldots, 0, 1, 1)$. The proof of the proposition is completed.

7. Appendix II. Asymptotic Formulas

Proposition 7.1. Under the conditions of Theorem 1.1 for any point $x$ on the boundary of $\Omega$ we have

$$
-1 \int_{\Omega} e_1 \tilde{g}_1 e^{-\tau(\Phi(z) - \overline{\Phi(z)})} d\xi_1 d\xi_2 = \frac{1}{2\tau^2} \frac{\partial \tilde{g}_1(x)}{\partial \Phi(z)} \frac{e^{-2i\tau\psi(z)}}{|\det \text{Im} \Phi''(x)|^{\frac{1}{2}}} + o\left(\frac{1}{\tau^2}\right) + \frac{1}{4\tau^2} \frac{\partial \tilde{g}_1(x)}{\partial \Phi(z)} \frac{e^{-2i\tau\psi(z)}}{|\det \text{Im} \Phi''(x)|^{\frac{1}{2}}} as |\tau| \to +\infty.
$$

(7.1)

$$
-1 \int_{\Omega} e_1 \tilde{g}_2 e^{-\tau(\Phi(z) - \overline{\Phi(z)})} d\xi_1 d\xi_2 = \frac{1}{2\tau^2} \frac{\partial \tilde{g}_2(x)}{\partial \Phi(z)} \frac{e^{2i\tau\psi(z)}}{|\det \text{Im} \Phi''(x)|^{\frac{1}{2}}} + o\left(\frac{1}{\tau^2}\right) + \frac{1}{4\tau^2} \frac{\partial \tilde{g}_2(x)}{\partial \Phi(z)} \frac{e^{2i\tau\psi(z)}}{|\det \text{Im} \Phi''(x)|^{\frac{1}{2}}} as |\tau| \to +\infty.
$$

(7.2)

$$
-1 \int_{\Omega} e_1 \tilde{g}_3 e^{-\tau(\Phi(z) - \overline{\Phi(z)})} d\xi_1 d\xi_2 = -\frac{1}{2\tau^2} \frac{\partial \tilde{g}_3(x)}{\partial \Phi(z)} \frac{e^{-2i\tau\psi(z)}}{|\det \text{Im} \Phi''(x)|^{\frac{1}{2}}} + o\left(\frac{1}{\tau^2}\right) + \frac{1}{4\tau^2} \frac{\partial \tilde{g}_3(x)}{\partial \Phi(z)} \frac{e^{-2i\tau\psi(z)}}{|\det \text{Im} \Phi''(x)|^{\frac{1}{2}}} as |\tau| \to +\infty.
$$

(7.3)

$$
-1 \int_{\Omega} e_1 \tilde{g}_4 e^{\tau(\Phi(z) - \overline{\Phi(z)})} d\xi_1 d\xi_2 = -\frac{1}{2\tau^2} \frac{\partial \tilde{g}_4(x)}{\partial \Phi(z)} \frac{e^{2i\tau\psi(z)}}{|\det \text{Im} \Phi''(x)|^{\frac{1}{2}}} + o\left(\frac{1}{\tau^2}\right) + \frac{1}{4\tau^2} \frac{\partial \tilde{g}_4(x)}{\partial \Phi(z)} \frac{e^{2i\tau\psi(z)}}{|\det \text{Im} \Phi''(x)|^{\frac{1}{2}}} as |\tau| \to +\infty.
$$

(7.4)
Proof. Let $\delta > 0$ be a sufficiently small number and $\tilde{\epsilon} \in C^\infty_0(B(\tilde{x}, \delta))$, $\tilde{\epsilon}|_{B(\tilde{x}, \delta/2)} \equiv 1$. We compute the asymptotic formulae of the following integral as $|\tau|$ goes to infinity.

$$
- \frac{1}{\pi} \int \frac{e_1 \tilde{g}_1 e^{-\tau(\Phi(\zeta) - \Phi(\zeta))}}{\zeta - z} d\xi_1 d\xi_2 = \ldots
$$

(7.5)

Using stationary phase we obtain

$$
\int_{B(\tilde{x}, \delta)} \tilde{\epsilon} \frac{(\zeta - \tilde{z})}{(\zeta - z)^2} e^{-\tau(\Phi(\zeta) - \Phi(\zeta))} d\xi_1 d\xi_2 = o\left(\frac{1}{\tau^2}\right) \quad \text{as} \quad |\tau| \to +\infty.
$$

Another asymptotic calculation is
\[-1 \frac{1}{\pi} \int_{\Omega} \frac{e_1 \hat{g}_1 e^{\tau(\Phi(\zeta) - \Phi(\zeta))}}{\zeta - z} \, d\xi_1 d\xi_2 = \frac{1}{\pi \tau} \int_{B(\tilde{x}, \delta)} \tilde{e} \frac{\partial \hat{g}_1(\tilde{x})}{\partial \Phi(x)} (\zeta - z)^2 e^{-\tau(\Phi(\zeta) - \Phi(\zeta))} \, dx + o\left(\frac{1}{\tau^2}\right) +
\]

\[
\frac{1}{2\pi \tau} \int_{B(\tilde{x}, \delta)} \tilde{e} \frac{\partial \hat{g}_1(\tilde{x})}{\partial \Phi(x)} \frac{\partial^2 \Phi(x)}{\partial^2 \Phi(x)} + \frac{\partial^2 \hat{g}_1(\tilde{x})}{\partial^2 \Phi(x)} \frac{\partial^2 \Phi(x)}{\partial^2 \Phi(x)} e^{-\tau(\Phi(\zeta) - \Phi(\zeta))} \, dx =
\]

\[
\frac{1}{2\tau^2 (\tilde{z} - z)^2} \left| \det \text{Im}\Phi'(\tilde{x}) \right|^\frac{1}{2} + o\left(\frac{1}{\tau^2}\right) +
\]

\[
\frac{1}{4\tau^2} \frac{\partial \hat{g}_1(\tilde{x})}{\partial^2 \Phi(x)} \frac{\partial^2 \Phi(x)}{\partial^2 \Phi(x)} + \frac{\partial^2 \hat{g}_1(\tilde{x})}{\partial^2 \Phi(x)} - \frac{\partial \hat{g}_1(\tilde{x})}{\partial^2 \Phi(x)} \frac{\partial^2 \Phi(x)}{\partial^2 \Phi(x)} e^{-2i\tau \psi(\tilde{x})} \left| \det \text{Im}\Phi''(\tilde{x}) \right|^\frac{1}{2} \quad \text{as } |\tau| \to +\infty.
\]
Next we compute the asymptotics of the following integral as $|\tau|$ goes to infinity.

\[
\begin{align*}
&- \frac{1}{\pi} \int_{\Omega} \frac{e^{i\hat{\gamma}_{2}e^{\tau(\Phi(\zeta))}}}{\zeta - \bar{\zeta}} d\xi_{1} d\xi_{2} = \\
&- \frac{1}{\pi} \int_{B(\bar{x},\delta)} \frac{\hat{\gamma}_{2} e^{\tau(\Phi(\zeta))}}{\zeta - \bar{\zeta}} d\xi_{1} d\xi_{2} + o\left(\frac{1}{\tau^{2}}\right) = \\
&- \frac{1}{\pi} \int_{B(\bar{x},\delta)} \hat{\gamma}_{2} \left\{ \frac{\partial_{\zeta} \hat{g}_{2}(\bar{x})(\zeta - \bar{\zeta})(\zeta - \bar{\zeta}) + \frac{1}{2} \partial_{\zeta}^{2} \hat{g}_{2}(\bar{x})(\zeta - \bar{\zeta})^{2}}{\zeta - \bar{\zeta}} \right\} e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})} d\xi_{1} d\xi_{2} + o\left(\frac{1}{\tau^{2}}\right) = \\
&- \frac{1}{\pi} \int_{B(\bar{x},\delta)} \hat{\gamma}_{2} \left\{ \frac{\partial_{\zeta} \partial_{\bar{\zeta}} \hat{g}_{2}(\bar{x})(\zeta - \bar{\zeta})\partial_{\zeta} \Phi + \frac{1}{2} \partial_{\zeta}^{2} \hat{g}_{2}(\bar{x})\partial_{\zeta} \Phi(\zeta - \bar{\zeta})}{\zeta - \bar{\zeta}} \right\} e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})} d\xi_{1} d\xi_{2} + o\left(\frac{1}{\tau^{2}}\right) = \\
&- \frac{1}{\pi} \int_{B(\bar{x},\delta)} \hat{\gamma}_{2} \left\{ \frac{\partial_{\zeta} \partial_{\bar{\zeta}} \hat{g}_{2}(\bar{x}) (\zeta - \bar{\zeta})\partial_{\zeta} \Phi + \frac{1}{2} \partial_{\zeta}^{2} \hat{g}_{2}(\bar{x})\partial_{\zeta} \Phi(\zeta - \bar{\zeta})}{\zeta - \bar{\zeta}} \right\} e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})} d\xi_{1} d\xi_{2} + o\left(\frac{1}{\tau^{2}}\right) = \\
&- \frac{1}{\pi} \int_{B(\bar{x},\delta)} \hat{\gamma}_{2} \left\{ \frac{-1}{2} \frac{\partial_{\zeta} \partial_{\bar{\zeta}}^{2} \Phi(\bar{x})}{\partial_{\zeta} \Phi(\bar{x})} + \frac{1}{2} \frac{\partial_{\zeta}^{2} \hat{g}_{2}(\bar{x})}{\partial_{\zeta} \Phi(\bar{x})} \partial_{\zeta} \Phi(\zeta - \bar{\zeta}) - \frac{\partial_{\zeta} \partial_{\bar{\zeta}} \hat{g}_{2}(\bar{x})}{2 \partial_{\zeta} \Phi(\bar{x})} (\zeta - \bar{\zeta})}{(\zeta - \bar{\zeta})^{2}} \right\} e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})} d\xi_{1} d\xi_{2} + o\left(\frac{1}{\tau^{2}}\right). \\
\end{align*}
\]

(7.7)

Observe that by the stationary phase argument

\[
\int_{B(\bar{x},\delta)} \hat{\gamma}_{2} \frac{e^{\tau(\Phi(\zeta))}}{(\zeta - \bar{\zeta})^{2}} d\xi_{1} d\xi_{2} = o\left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to +\infty.
\]
Finally we have

\[- \frac{1}{\pi} \int_{\Omega} e^{\tau \Phi(\zeta)} \frac{\partial_{\bar{\zeta}} \hat{g}_2(x)}{\partial_{\bar{\Phi}(\hat{x})}} \frac{d\xi_1 d\xi_2}{\zeta - \bar{\zeta}} = \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_2(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} e^{\tau \Phi(\zeta)} d\xi_1 d\xi_2 \]

\[- \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_2(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \frac{1}{\zeta - \bar{\zeta}} \left[ \frac{e^{\tau \Phi(\zeta)} \hat{\Phi}(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} + \frac{2}{\partial_{\bar{\Phi}(\hat{x})}} \right] d\xi_1 d\xi_2 = \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_2(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} e^{\tau \Phi(\zeta)} d\xi_1 d\xi_2 + o\left( \frac{1}{\tau^2} \right) = \]

\[+ \frac{1}{2 \pi \tau^2} \left( \frac{e^{2i\tau \psi(\hat{x})}}{|\det \text{Im} \Phi''(\hat{x})|^2} \right) \]

\[- \frac{1}{4 \pi^2} \left( \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \right)^2 \frac{d\xi_1 d\xi_2}{\zeta - \bar{\zeta}} = \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \frac{d\xi_1 d\xi_2}{\zeta - \bar{\zeta}} + o\left( \frac{1}{\tau^2} \right) \]

We compute the asymptotic of the following integral as \(|\tau|\) goes to infinity.

\[- \frac{1}{\pi} \int_{\Omega} e^{\tau \Phi(\zeta)} \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \frac{d\xi_1 d\xi_2}{\zeta - \bar{\zeta}} = \]

\[- \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \frac{1}{\zeta - \bar{\zeta}} \left[ \frac{e^{\tau \Phi(\zeta)} \hat{\Phi}(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} + \frac{1}{\partial_{\bar{\Phi}(\hat{x})}} \right] d\xi_1 d\xi_2 = \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} e^{\tau \Phi(\zeta)} d\xi_1 d\xi_2 + o\left( \frac{1}{\tau^2} \right) = \]

\[- \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \frac{1}{\zeta - \bar{\zeta}} \left[ \frac{\partial_{\bar{\zeta}} \hat{\Phi}(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} + \frac{2}{\partial_{\bar{\Phi}(\hat{x})}} \right] d\xi_1 d\xi_2 = \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} e^{\tau \Phi(\zeta)} d\xi_1 d\xi_2 + o\left( \frac{1}{\tau^2} \right) = \]

\[- \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \frac{1}{\zeta - \bar{\zeta}} \left[ \partial_{\bar{\zeta}} \hat{\Phi}(\hat{x}) + \frac{1}{2} \frac{\partial^2 \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \partial_{\bar{\Phi}(\hat{x})} \frac{\partial_{\bar{\zeta}} \hat{\Phi}(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \right] d\xi_1 d\xi_2 = \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} e^{\tau \Phi(\zeta)} d\xi_1 d\xi_2 + o\left( \frac{1}{\tau^2} \right) = \]

\[- \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \frac{1}{\zeta - \bar{\zeta}} \left[ \partial_{\bar{\zeta}} \hat{\Phi}(\hat{x}) - \frac{1}{2} \frac{\partial^2 \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \partial_{\bar{\Phi}(\hat{x})} \frac{\partial_{\bar{\zeta}} \hat{\Phi}(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} \right] d\xi_1 d\xi_2 = \frac{1}{\pi \tau} \int_{B(\bar{x}, \delta)} \frac{\partial_{\bar{\zeta}} \hat{g}_4(\hat{x})}{\partial_{\bar{\Phi}(\hat{x})}} e^{\tau \Phi(\zeta)} d\xi_1 d\xi_2 + o\left( \frac{1}{\tau^2} \right) = \]
Finally we have

\[\int_{B(x, \delta)} e^{i\tau(\Phi(\zeta) - \Phi(\zeta))} d\xi_1 d\xi_2 = o\left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to +\infty.\]

(7.8)

Observe that

\[-\frac{1}{\pi \tau} \int_{B(x, \delta)} e^{i\tau(\Phi(\zeta) - \Phi(\zeta))} d\xi_1 d\xi_2 = o\left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to +\infty.

(7.9)

\[\frac{\partial \Phi(\zeta)}{\partial \Phi(\zeta)} \frac{\partial \Phi(\zeta)}{\partial \Phi(\zeta)} e^{2i\tau\psi(\tilde{z})} + o\left(\frac{1}{\tau^2}\right) \quad \text{as} \quad |\tau| \to +\infty.

We compute the asymptotic of the following integral as $|\tau|$ goes to infinity:

\[-\frac{1}{\pi \tau} \int_\Omega e^{i\tau(\Phi(\zeta) - \Phi(\zeta))} d\xi_1 d\xi_2 =

\[-\frac{1}{\pi \tau} \int_{B(x, \delta)} e^{i\tau(\Phi(\zeta) - \Phi(\zeta))} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right) =

\[-\frac{1}{\pi \tau} \int_{B(x, \delta)} e^{i\tau(\Phi(\zeta) - \Phi(\zeta))} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right) =

\[-\frac{1}{\pi \tau} \int_{B(x, \delta)} e^{i\tau(\Phi(\zeta) - \Phi(\zeta))} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right) =

\[-\frac{1}{\pi \tau} \int_{B(x, \delta)} e^{i\tau(\Phi(\zeta) - \Phi(\zeta))} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right) =

\[-\frac{1}{\pi \tau} \int_{B(x, \delta)} e^{i\tau(\Phi(\zeta) - \Phi(\zeta))} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right) =

\[-\frac{1}{\pi \tau} \int_{B(x, \delta)} e^{i\tau(\Phi(\zeta) - \Phi(\zeta))} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right) =
Using stationary phase we have

\[
\int_{B(x, \delta)} e^{\tau (\Phi(\zeta) - \overline{\Phi}(\zeta))} d\zeta_1 d\zeta_2 = o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \to +\infty.
\]

Finally we have

\[
\begin{align*}
&- \frac{1}{\pi} \int_{B(x, \delta)} e^{\tau (\Phi(\zeta) - \overline{\Phi}(\zeta))} d\zeta_1 d\zeta_2 = \\
&- \frac{1}{\pi} \int_{B(x, \delta)} e^{\tau (\Phi(\zeta) - \overline{\Phi}(\zeta))} \left\{ \frac{1}{2} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} - \frac{1}{2} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} \right\} d\zeta_1 d\zeta_2 + o\left(\frac{1}{\tau^2}\right) = \\
&- \frac{1}{\pi} \int_{B(x, \delta)} e^{\tau (\Phi(\zeta) - \overline{\Phi}(\zeta))} \frac{1}{2} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} + \frac{1}{\pi} \int_{B(x, \delta)} e^{\tau (\Phi(\zeta) - \overline{\Phi}(\zeta))} \frac{1}{2} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} e^{-2\tau \psi(\zeta)} d\zeta_1 d\zeta_2 + o\left(\frac{1}{\tau^2}\right) = \\
&- \frac{1}{4\tau^2} \left\{ \frac{1}{2} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} + \frac{1}{2} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} \frac{\partial^2 \tilde{g}_1(\zeta)}{\partial^2 \Phi(x)} \right\} e^{-2\tau \psi(\zeta)} d\zeta_1 d\zeta_2 + o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \to +\infty.
\end{align*}
\]
Proposition 7.2. For any $x$ from the boundary of $\Omega$, the following asymptotic formulae hold true as $|\tau|$ goes to $+\infty$.

\begin{align}
\mathbf{G}_1(x, \tau) &= -\frac{1}{\tau} \frac{e^{-2i\tau(\bar{z})}}{|\det \text{Im} \Phi'(\bar{z})|^\frac{1}{2}} \left( \frac{1}{s} \frac{\partial(g_1 e^{-A_1})}{\partial \bar{z}}(\bar{z}) + \frac{\partial \Phi}{\partial \bar{z}} \frac{m_1}{(\bar{z} - z)^2} + \frac{q_i \partial \Phi}{\partial \bar{z}} \right) + o(\frac{1}{\tau}), \\
\mathbf{G}_2(x, \tau) &= -\frac{1}{\tau} \frac{e^{2i\tau(\bar{z})}}{|\det \text{Im} \Phi'(\bar{z})|^\frac{1}{2}} \left( \frac{1}{s} \frac{\partial(g_2 e^{-B_1})}{\partial \bar{z}}(\bar{z}) + \frac{\partial \Phi}{\partial \bar{z}} \frac{m_1}{(\bar{z} - z)^2} + \frac{q_i \partial \Phi}{\partial \bar{z}} \right) + o(\frac{1}{\tau}), \\
\mathbf{G}_3(x, \tau) &= \frac{1}{\tau} \frac{e^{-2i\tau(\bar{z})}}{|\det \text{Im} \Phi'(\bar{z})|^\frac{1}{2}} \left( \frac{1}{s} \frac{\partial(g_2 e^{-B_2})}{\partial \bar{z}}(\bar{z}) + \frac{\partial \Phi}{\partial \bar{z}} \frac{m_1}{(\bar{z} - z)^2} + \frac{q_i \partial \Phi}{\partial \bar{z}} \right) + o(\frac{1}{\tau}), \\
\mathbf{G}_4(x, \tau) &= -\frac{1}{\tau} \frac{e^{2i\tau(\bar{z})}}{|\det \text{Im} \Phi'(\bar{z})|^\frac{1}{2}} \left( \frac{1}{s} \frac{\partial(g_1 e^{-B_2})}{\partial \bar{z}}(\bar{z}) + \frac{\partial \Phi}{\partial \bar{z}} \frac{m_1}{(\bar{z} - z)^2} + \frac{q_i \partial \Phi}{\partial \bar{z}} \right) + o(\frac{1}{\tau}).
\end{align}

Here $\bar{z} = \bar{x}_1 + i\bar{x}_2$. Moreover the following asymptotic formula holds true

\begin{equation}
\left\| \frac{\partial \mathbf{G}_2(\cdot, \tau)}{\partial \bar{z}} \right\|_{C(\bar{\Omega})} + \left\| \frac{\partial \mathbf{G}_1(\cdot, \tau)}{\partial \bar{z}} \right\|_{C(\bar{\Omega})} + \left\| \frac{\partial \mathbf{G}_3(\cdot, \tau)}{\partial \bar{z}} \right\|_{C(\bar{\Omega})} + \left\| \frac{\partial \mathbf{G}_4(\cdot, \tau)}{\partial \bar{z}} \right\|_{C(\bar{\Omega})} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \to +\infty.
\end{equation}

Proof. Observe that the functions $\mathbf{G}_1(x, \tau), \ldots, \mathbf{G}_4(x, \tau)$ are given by

\begin{align}
\mathbf{G}_1(x, \tau) &= -\frac{1}{2\pi} \int_{\Omega} \frac{\partial^2}{\partial \bar{z}}(\zeta, \bar{\zeta}) + \tau \frac{\partial \Phi}{\partial \bar{z}}(\zeta) - \frac{\partial A_1}{\partial \bar{z}}(z, \bar{z}) + \tau \frac{\partial B}{\partial \bar{z}}(z) (e_{1}g_1 e^{-A_1})(\xi_1, \xi_2) e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2, \\
\mathbf{G}_2(x, \tau) &= -\frac{1}{2\pi} \int_{\Omega} \frac{\partial^2}{\partial \bar{z}}(\zeta, \bar{\zeta}) + \tau \frac{\partial \Phi}{\partial \bar{z}}(\zeta) - \frac{\partial B_1}{\partial \bar{z}}(z, \bar{z}) + \tau \frac{\partial B}{\partial \bar{z}}(z) (e_{1}g_2 e^{-B_1})(\xi_1, \xi_2) e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2, \\
\mathbf{G}_3(x, \tau) &= \frac{1}{2\pi} \int_{\Omega} \frac{\partial^2}{\partial \bar{z}}(\zeta, \bar{\zeta}) + \tau \frac{\partial \Phi}{\partial \bar{z}}(\zeta) - \frac{\partial B_2}{\partial \bar{z}}(z, \bar{z}) + \tau \frac{\partial B}{\partial \bar{z}}(z) (e_{1}g_3 e^{-B_2})(\xi_1, \xi_2) e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2, \\
\mathbf{G}_4(x, \tau) &= \frac{1}{2\pi} \int_{\Omega} \frac{\partial^2}{\partial \bar{z}}(\zeta, \bar{\zeta}) + \tau \frac{\partial \Phi}{\partial \bar{z}}(\zeta) - \frac{\partial B_2}{\partial \bar{z}}(z, \bar{z}) + \tau \frac{\partial B}{\partial \bar{z}}(z) (e_{1}g_3 e^{-B_2})(\xi_1, \xi_2) e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2.
\end{align}

These explicit formulae and the stationary phase argument imply (7.15). Let $z = x_1 + ix_2$ where $x = (x_1, x_2) \in \partial \Omega$. The following asymptotic is valid.

\begin{align}
\frac{\tau}{2\pi} \int_{\Omega} \frac{\partial \Phi}{\partial \bar{z}}(\zeta) (e_{1}g_1 e^{-A_1}) e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 &= -\frac{1}{2\pi} \int_{\Omega} \frac{\partial}{\partial \bar{z}}(e_{1}g_1 e^{-A_1}) \frac{\partial}{\partial \bar{z}} e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 = \frac{1}{2\pi} \int_{\Omega} \frac{\partial^{2}}{\partial \bar{z}} \left( \frac{1}{\zeta - z} (e_{1}g_1 e^{-A_1}) \right) e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 = \frac{1}{2\pi} \int_{\Omega} \frac{\partial}{\partial \bar{z}} \left( \frac{1}{\zeta - z} (e_{1}g_1 e^{-A_1}) \right) e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 = \frac{1}{2\pi} \int_{\Omega} \frac{\partial}{\partial \bar{z}} \left( \frac{1}{\zeta - z} (e_{1}g_1 e^{-A_1}) \right) e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 = \frac{1}{2\pi} \int_{\Omega} \frac{\partial}{\partial \bar{z}} \left( \frac{1}{\zeta - z} (e_{1}g_1 e^{-A_1}) \right) e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 = \frac{1}{2\pi} \int_{\Omega} \frac{\partial}{\partial \bar{z}} \left( \frac{1}{\zeta - z} (e_{1}g_1 e^{-A_1}) \right) e^{\tau(\Phi(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2.
\end{align}
Proof of Proposition 5.2. Using (5.14) and (5.16) we have

\[
\mathcal{L}_0 \equiv (2(A_1 - A_2) \frac{\partial U_1}{\partial z}, b_r e^{B_2 - \tau \Phi} + c_r e^{A_2 - \tau \Phi})_{L^2(\Omega)}
\]

\[+ (2(B_1 - B_2) \frac{\partial U_1}{\partial z}, b_r e^{B_2 - \tau \Phi} + c_r e^{A_2 - \tau \Phi})_{L^2(\Omega)}
\]

\[= -2((A_1 - A_2) e^{\tau \Phi} \frac{\partial}{\partial z} \mathcal{R}_{\tau \Phi} \mathcal{A} \{e_1 g_1\}, b_r e^{B_2 - \tau \Phi} + c_r e^{A_2 - \tau \Phi})_{L^2(\Omega)}
\]

\[+ ((B_1 - B_2)(-e_1 g_1 + A_1 \mathcal{R}_{\tau \Phi} \mathcal{A} \{e_1 g_1\}) e^{\tau \Phi}, b_r e^{B_2 - \tau \Phi} + c_r e^{A_2 - \tau \Phi})_{L^2(\Omega)}
\]

\[= -2((A_1 - A_2) e^{\tau \Phi} \left( \frac{\partial (e_1 g_1)}{\partial z} \right) - e^{A_1 - \tau \Phi} \mathcal{G}_1 + o_{L^2(\Omega)}(\frac{1}{\tau})) , b_r e^{B_2 - \tau \Phi} + c_r e^{A_2 - \tau \Phi})_{L^2(\Omega)}
\]

(7.16) \[+ ((B_1 - B_2)(-e_1 g_1 + A_1 \mathcal{R}_{\tau \Phi} \mathcal{A} \{e_1 g_1\}) e^{\tau \Phi}, b_r e^{B_2 - \tau \Phi} + c_r e^{A_2 - \tau \Phi})_{L^2(\Omega)}
\]

By Proposition 3.4 we have

\[2((A_1 - A_2) e^{\tau \Phi} \frac{\partial (e_1 g_1)}{\partial z}, c_r e^{A_2 - \tau \Phi})_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty.
\]

By (7.11) and Proposition 3.4 we obtain

\[-2((A_1 - A_2) e^{\tau \Phi} e^{A_1} e^{-\tau (\Phi - \Phi)} \mathcal{G}_1, b_r e^{B_2 - \tau \Phi})_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty.
\]

Integrating by parts and using (7.15) and (3.2), we have

\[2((A_1 - A_2) e^{\tau \Phi} e^{A_1} e^{-\tau (\Phi - \Phi)} \mathcal{G}_1, c_r e^{A_2 - \tau \Phi})_{L^2(\Omega)} = 2((A_1 - A_2) e^{A_1} \mathcal{G}_1, c_r e^{A_2})_{L^2(\Omega)}
\]

\[= \int_\Omega 2(A_1 - A_2) e^{A_1 + A_2} c_r(z) \mathcal{G}_1(x, \tau) dx = -4 \int_\Omega \frac{\partial}{\partial z} e^{A_1 + A_2} c_r(z) \mathcal{G}_1(x, \tau) dx
\]

\[= -2 \int_{\partial \Omega} (\nu_1 + i \nu_2) e^{A_1 + A_2} c_r(z) \mathcal{G}_1(x, \tau) d\sigma + o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty.
\]
By (7.11), (7.17)-(7.19) and Propositions 3.3 and 3.5, we conclude

\[
\mathcal{L}_0 = (B_1 - B_2)(-e_1 g_1 + A_1 R_{-\tau, A_1 \{e_1 g_1\}}, b_\tau e^{B_2})_{L^2(\Omega)} + 2 ((A_1 - A_2)e^{\nabla_{\Phi} e^{-\tau(\Phi - \Phi)}} g_1, c_\tau e^{A_2 - \tau})_{L^2(\Omega)} - 2((A_1 - A_2)(-e_1 g_1 + \frac{\partial(e_1 g_1)}{\partial z} - e^{A_1 e^{-\tau(\Phi - \Phi)}} g_1), b_\tau e^{B_2})_{L^2(\Omega)} = (B_1 - B_2)(-e_1 g_1 + \frac{\partial(e_1 g_1)}{\partial z}, b_\tau e^{B_2})_{L^2(\Omega)} - 2((A_1 - A_2)(-e_1 g_1 + \frac{\partial(e_1 g_1)}{\partial z}, b_\tau e^{B_2})_{L^2(\Omega)} - 2\int_{\partial \Omega} (\nu_1 + i\nu_2)e^{A_1 + \tau} c_\tau(\overline{z})g_1(x, \tau)dx + o(\frac{1}{\tau}) \quad \text{as} \quad |\tau| \to +\infty.
\]

(7.20)

Using (5.14) and (5.18) we obtain after simple computations

\[
\mathcal{L}_1 = 2(A_1 - A_2)\frac{\partial U_2}{\partial z}, b_\tau e^{B_2 - \tau} + c_\tau e^{A_2 - \tau})_{L^2(\Omega)} + 2((B_1 - B_2)\frac{\partial U_2}{\partial z}, b_\tau e^{B_2 - \tau} + c_\tau e^{A_2 - \tau})_{L^2(\Omega)} = ((A_1 - A_2)(-e_1 g_2 + B_1 \tilde{R}_{-\tau, B_1 \{e_1 g_2\}}) e^{\tau} b_\tau e^{B_2 - \tau} + c_\tau e^{A_2 - \tau})_{L^2(\Omega)} - 2((B_1 - B_2)\frac{\partial}{\partial z} \tilde{R}_{-\tau, B_1 \{e_1 g_2\}} e^{\tau} b_\tau e^{B_2 - \tau} + c_\tau e^{A_2 - \tau})_{L^2(\Omega)} = ((A_1 - A_2)(-e_1 g_2 + B_1 \tilde{R}_{-\tau, B_1 \{e_1 g_2\}}) e^{\tau} b_\tau e^{B_2 - \tau} + c_\tau e^{A_2 - \tau})_{L^2(\Omega)} - 2((B_1 - B_2)\frac{\partial(e_1 g_2)}{\partial z} - e^{B_1 e^{\tau(\Phi - \Phi)}} g_2 + o_{L^2(\Omega)}(\frac{1}{\tau})) e^{\tau} b_\tau e^{B_2 - \tau} + c_\tau e^{A_2 - \tau})_{L^2(\Omega)}.
\]

(7.21)

Then, by (7.15) we have the asymptotic formula

\[
-2((B_1 - B_2)(\frac{\partial(e_1 g_2)}{\partial z} - e^{B_1 e^{\tau(\Phi - \Phi)}} g_2 + o_{L^2(\Omega)}(\frac{1}{\tau})) e^{\tau} b_\tau e^{B_2 - \tau})_{L^2(\Omega)} = 2((B_1 - B_2)e^{B_1} g_2 + b_\tau e^{B_2})_{L^2(\Omega)} + o(\frac{1}{\tau}) = 2 \int_{\Omega} (B_1 - B_2)e^{B_1 + B_2} b_\tau(\overline{z}) g_2(x, \tau)dx = -4 \int_{\Omega} \frac{\partial}{\partial z} e^{B_1 + B_2} b_\tau(\overline{z}) g_2(x, \tau)dx + o(\frac{1}{\tau}) = -2 \int_{\partial \Omega} (\nu_1 - i\nu_2)e^{B_1 + B_2} b_\tau(\overline{z}) g_2(x, \tau)dx + o(\frac{1}{\tau}).
\]

By (7.12) and Proposition 3.4 we obtain using (7.21):

\[
-2((B_1 - B_2)(\frac{\partial(e_1 g_2)}{\partial z} - e^{B_1 e^{\tau(\Phi - \Phi)}} g_2 + c_\tau e^{A_2 - \tau})_{L^2(\Omega)} = -2((B_1 - B_2)(\frac{\partial(e_1 g_2)}{\partial z} c_\tau e^{A_2 - \tau}) + o(\frac{1}{\tau}) + \chi \frac{1}{\tau}.
\]

(7.23)

Here \( \chi \) is a constant which is independent of \( \tau \).
By (7.22) and (7.23) we have

\[ \mathcal{L}_1 = (A_1 - A_2)(-e_1g_2 + B_1\overline{\mathcal{R}}_{\tau,B_1}\{e_1g_2\}), c_r e^{A_2})_{L^2(\Omega)} \]

\[ - 2((B_1 - B_2)(\frac{\partial(e_1g_2)}{\partial z} - e^{B_1e^{A_1+\tau\Phi}}\Phi, c_r e^{A_2})_{L^2(\Omega)} \]

\[ - ((A_1 - A_2)(-e_1g_2 + \frac{B_1e^{A_2}}{2\tau\partial_z\Phi}), c_r e^{A_2})_{L^2(\Omega)} \]

\[ - 2((B_1 - B_2)(\frac{\partial B_1 e^{A_2}}{\partial z} + \frac{\partial(e_1g_2)}{\partial z} - \frac{\partial(e_1g_2)}{2\tau\partial_z\Phi}), c_r e^{A_2})_{L^2(\Omega)} \]

(7.24)

\[ - 2 \int_{\partial\Omega} (\nu_1 - i\nu_2)e^{B_1 + \overline{B}_2b_r(z)}\Phi_2(x,\tau)d\sigma + o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \rightarrow +\infty. \]

Recall \( V_1 = -e^{-\tau\Phi}\overline{\mathcal{R}}_{-\tau,B_2}\{e_1g_1\} \) and \( V_2 = -e^{-\tau\Phi}\mathcal{R}_{-\tau,-B_2}\{e_1g_3\}. \)

By Proposition 3.2 we conclude

(7.25)

\[ 2\frac{\partial V_1}{\partial z} = (-e_1g_4 + \overline{A}_2\overline{\mathcal{R}}_{-\tau,A_2}\{e_1g_4\})e^{-\tau\Phi} \]

and

(7.26)

\[ 2\frac{\partial V_2}{\partial z} = (-e_1g_3 - \overline{B}_3\mathcal{R}_{-\tau,-B_2}\{e_1g_3\})e^{-\tau\Phi}. \]

Similarly to (5.15) and (5.17) we calculate \( \frac{\partial V_1}{\partial \overline{\tau}} \) and \( \frac{\partial V_2}{\partial \overline{\tau}} \):

(7.27)

\[ \frac{\partial V_1}{\partial \overline{\tau}} = -e^{-\tau\Phi}\overline{\mathcal{R}}_{-\tau,A_2}\left\{ \frac{\partial(e_1g_4)}{\partial \overline{\tau}} \right\} + e^{-\tau\Phi + A_2}\Phi_3(\cdot,\tau) \]

and

(7.28)

\[ \frac{\partial V_2}{\partial \overline{\tau}} = -e^{-\tau\Phi}\mathcal{R}_{-\tau,-B_2}\left\{ \frac{\partial(e_1g_3)}{\partial \overline{\tau}} \right\} + e^{-\tau\Phi + B_3}\Phi_4(\cdot,\tau). \]

Using (7.25) and (7.26) we obtain

\[ \mathcal{L}_2 \equiv \left(2(A_1 - A_2)\frac{\partial}{\partial z}(a_r e^{A_1+\tau\Phi} + d_r e^{B_1+\tau\Phi}), V_1 + V_2\right)_{L^2(\Omega)} \]

\[ = -((A_1 - A_2)d_r B_1e^{B_1e^{A_1+\tau\Phi}}, V_1 + V_2)_{L^2(\Omega)} \]

\[ + \left((A_1 - A_2)(\nu_1 - i\nu_2)a_r e^{A_1+\tau\Phi}, V_1 + V_2\right)_{L^2(\partial\Omega)} \]

\[ - \left(2\frac{\partial}{\partial z}(A_1 - A_2)a_r e^{A_1+\tau\Phi}, V_1 + V_2\right)_{L^2(\Omega)} \]

\[ - \left((A_1 - A_2)a_r e^{A_1+\tau\Phi}, 2\frac{\partial V_1}{\partial z} + \frac{\partial V_2}{\partial \overline{\tau}}\right)_{L^2(\Omega)}. \]

We observe that by (7.27), (7.15), Proposition 3.3 and Proposition 3.4

\[ -((A_1 - A_2)a_r e^{A_1+\tau\Phi}, 2\frac{\partial V_1}{\partial \overline{\tau}})_{L^2(\Omega)} = -4 \int_{\Omega} a_r(z)\frac{\partial}{\partial z} e^{A_1+\tau\Phi}\Phi_3(\overline{x},\tau)dx + o\left(\frac{1}{\tau}\right) = \]

(7.29)

\[ -2 \int_{\partial\Omega} a_r(z)\Phi_3(x,\tau)(\nu_1 + i\nu_2)e^{A_1+\tau\Phi}d\sigma + o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \rightarrow +\infty. \]
Hence, using (7.26) we have
\[
\mathcal{L}_2 = (A_1 - A_2)d_x B_1 e^{B_1}, \mathcal{R}_{\tau, -A_2} \{e_1 g_4\})_{L^2(\Omega)} + (A_1 - A_2)(\nu_1 - i\nu_2) a_x e^{A_1 + \tau \Phi}, V_1 + V_2)_{L^2(\partial \Omega)} + 2 \frac{\partial}{\partial z} (A_1 - A_2) a_x e^{A_1, \mathcal{R}_{\tau, -A_2} \{e_1 g_3\})_{L^2(\Omega)} + (A_1 - A_2) a_x e^{A_1, -e_1 g_3 - B_2 e_1 g_3 \mathcal{R}_{\tau, -A_2} \{e_1 g_3\})_{L^2(\Omega)} + o\left(\frac{1}{\tau}\right) - 2 \int_{\partial \Omega} a_x(z) \bar{\Phi}_3(x, \tau)(\nu_1 + i\nu_2) e^{A_1 + \tau x} d\sigma.
\]
By (4.34) and stationary phase we get
\[
((A_1 - A_2)(\nu_1 - i\nu_2) a_x e^{A_1 + \tau \Phi}, V_1 + V_2)_{L^2(\partial \Omega)} = o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \to +\infty.
\]
Therefore, by Proposition 4.1
\[
\mathcal{L}_2 = - (A_1 - A_2)d_x B_1 e^{B_1}, \mathcal{R}_{\tau, -A_2} \{e_1 g_4\})_{L^2(\Omega)} + \frac{\partial}{\partial z} (A_1 - A_2) a_x e^{A_1, \mathcal{R}_{\tau, -A_2} \{e_1 g_3\})_{L^2(\Omega)} + (A_1 - A_2) a_x e^{A_1, -e_1 g_3 + B_2 e_1 g_3 \mathcal{R}_{\tau, -A_2} \{e_1 g_3\})_{L^2(\Omega)} - 2 \int_{\partial \Omega} a_x(z) \bar{\Phi}_3(x, \tau)(\nu_1 + i\nu_2) e^{A_1 + \tau x} d\sigma + o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \to +\infty.
\]
Integrating by parts we compute
\[
\mathcal{L}_3 = (2(B_1 - B_2) \frac{\partial}{\partial \tau} (a_x e^{A_1 + \tau \Phi} + d_x e^{B_1 + \tau \Phi}), V_1 + V_2)_{L^2(\Omega)} = -2(B_1 - B_2) \frac{\partial}{\partial \tau} (a_x e^{A_1 + \tau \Phi}, V_1 + V_2)_{L^2(\Omega)} - (B_1 - B_2) a_x e^{A_1 + \tau \Phi}, V_1 + V_2)_{L^2(\Omega)} + (\nu_1 + i\nu_2)(B_1 - B_2) d_x e^{B_1 + \tau \Phi}, V_1 + V_2)_{L^2(\partial \Omega)} + (B_1 - B_2) d_x e^{B_1 + \tau \Phi}, 2(\frac{\partial V_1}{\partial z} + \frac{\partial V_2}{\partial z})_{L^2(\Omega)}.
\]
We observe that by (7.15), (7.28), Proposition 3.3 and Proposition 3.4:
\[
-2((B_1 - B_2) d_x e^{B_1 + \tau \Phi}, \frac{\partial V_2}{\partial z})_{L^2(\Omega)} = -4 \int_{\Omega} \frac{\partial}{\partial z} e^{B_1 + \tau \Phi} d_x(x, \tau) d\tau = -2 \int_{\partial \Omega} (\nu_1 - i\nu_2) e^{B_1 + \tau \Phi} d_x(x, \tau) d\sigma + o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \to +\infty.
\]
Hence
\[
\mathcal{L}_3 = 2 \frac{\partial}{\partial z} (B_1 - B_2) d_e e^{B_1}, \mathcal{R}_{\tau, -\mathcal{F}_2} \{e_1 g_4\} L^2(\Omega)
- (B_1 - B_2) A_1 a_\tau e^{A_1}, \mathcal{R}_{\tau, -\mathcal{F}_2} \{e_1 g_3\} L^2(\Omega)
+ (B_1 - B_2) d_e e^{B_1}, -e_1 g_4 + A_2 \mathcal{R}_{\tau, -\mathcal{F}_2} \{e_1 g_4\} L^2(\Omega)
+ (\nu_1 + i \nu_2)(B_1 - B_2) d_e e^{B_1 + \tau F}, V_1 + V_2) L^2(\partial \Omega)
\]

(7.32) + 2 \int_{\partial \Omega} (\nu_1 - i \nu_2) e^{B_1 + \tau F} d_e(x, \tau) \Phi_1(x, \tau) d\sigma + o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \to +\infty.

By (4.34) and the stationary phase argument

\[(\nu_1 + i \nu_2)(B_1 - B_2) d_e e^{B_1 + \tau F}, V_1 + V_2) L^2(\partial \Omega) = o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \to +\infty.
\]

Therefore, applying Proposition 3.3 we conclude finally that

\[
\mathcal{L}_3 = -2 \frac{\partial}{\partial z} (B_1 - B_2) d_e e^{B_1}, \frac{e_1 g_4}{2 \tau \partial_x \Phi} L^2(\Omega)
- (B_1 - B_2) A_1 a_\tau e^{A_1}, \frac{e_1 g_3}{2 \tau \partial_x \Phi} L^2(\Omega)
+ (B_1 - B_2) d_e e^{B_1}, -e_1 g_4 - A_2 e_1 g_4 \frac{1}{2 \tau \partial_x \Phi} L^2(\Omega)
\]

(7.33) + 2 \int_{\partial \Omega} (\nu_1 - i \nu_2) e^{B_1 + \tau F} d_e(x, \tau) \Phi_1(x, \tau) d\sigma + o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \to +\infty.

The sum \(\sum_{k=0}^{3} \mathcal{L}_k\) equal to the left hand side of (5.19). By (7.20), (7.24), (7.30) and (7.33), there exist numbers \(\kappa, \kappa_0\) such that the asymptotic formula (5.19) holds true. □

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