LINEARIZATION STABILITY RESULTS AND ACTIVE MEASUREMENTS FOR THE EINSTEIN-SCALAR FIELD EQUATIONS

YAROSLAV KURYLEV, MATTI LASSAS, GUNTHER UHLMANN

Abstract: *We consider linearization stability results for the coupled Einstein equations and the scalar field equations for the metric* g *and scalar fields* $\phi=(\phi^\ell)_{\ell=1}^L$ *on a 4-dimensional globally hyperbolic Lorentzian manifold* (M,g). *More precisely, we study the Einstein equations coupled with the scalar field equations and study the system* $Ein(g)=T$, $T=T(g,\phi)+\mathcal{F}^1$, *and* $\square_g\phi^\ell-m^2\phi^\ell=\mathcal{F}^2$, *where the sources* $\mathcal{F}=(\mathcal{F}^1,\mathcal{F}^2)$ *correspond to perturbations of the physical fields which we control. The sources* $\mathcal{F}$ *need to be such that the fields* $(g,\phi,\mathcal{F})$ *are solutions of this system and satisfy the conservation law* $div_g(T)=0$. *If* $(g_\varepsilon,\phi_\varepsilon)$ *solves the above equations, the derivatives* $\dot g=\partial_\varepsilon g_\varepsilon|_{\varepsilon=0}$, $\dot\phi=\phi_\varepsilon|_{\varepsilon=0}$, *and* $f=(f^1,f^2)=\partial_\varepsilon\mathcal{F}_\varepsilon|_{\varepsilon=0}$ *solve the linearized Einstein equations and the linearized conservation law*

$$\frac{1}{2}\widehat g^{pk}\widehat\nabla_p f^1_{kj}+\sum_{\ell=1}^L f^2_\ell\,\partial_j\widehat\phi_\ell=0,\quad j=1,2,3,4,$$

where $\widehat g=g_\varepsilon|_{\varepsilon=0}$ *and* $\widehat\phi=\phi_\varepsilon|_{\varepsilon=0}$. *In this case we say that* $(\widehat g,\widehat\phi)$ *and* f *have the linearization stability property. In linearization stability one ask the converse: If* $\dot g$, $\dot\phi$, *and* f *solve the linearized Einstein equations and the linearized conservation law, do there exist a family* $\mathcal{F}_\varepsilon=(\mathcal{F}^1_\varepsilon,\mathcal{F}^2_\varepsilon)$ *of sources and functions* $(g_\varepsilon,\phi_\varepsilon)$ *depending smoothly on* $\varepsilon\in[0,\varepsilon_0)$, $\varepsilon_0>0$, *such that* $(g_\varepsilon,\phi_\varepsilon)$ *solves the Einstein-scalar field equations and the conservation law. Under the condition that the background fields* $\widehat g$ *and* $\widehat\phi$ *vary enough and* $L\ge 5$, *we prove a microlocal version of this: When* $Y\subset M$ *is a 2-dimensional space-like surface and* (y,η) *is an element of the conormal bundle* N^*Y *of* Y, *one can find a linearized source* f *that is a conormal distibution with respect to the surface* Y *with a prescribed principal symbol at* (y,η) *such that* $(\widehat g,\widehat\phi)$ *and* f *have the linearization stability property. This result is proven by constructing a model with adaptive source functions. In this model the source term* $\mathcal{F}^1$ *corresponds to e.g. fluid fields consisting of particles which 4-velocity vectors are controlled and* $\mathcal{F}^2$ *contains a term corresponding to a secondary source function that adapts the changes of* g,ϕ, *and* $\mathcal{F}^1$ *so that the physical conservation law is satisfied. The obtained results can be applied to show that one can send gravitational*

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waves that propagates along the geodesic determined by (y, η) and the polarization of this wave can be controlled.
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Contents

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1. Introduction and main results

We consider the linearization stability result that is essential for the inverse problems for the non-linear Einstein equations coupled with matter field equations. In this paper, we consider for the matter fields the simplest possible model, the scalar field equations and study the perturbations of a globally hyperbolic Lorentzian manifold $(M, \widehat{g})$ of dimension $(1+3)$, where the metric signature of $\widehat{g}$ is $(-,+,+,+)$.

Our problem is related to the following inverse problem: Can an observer in a space-time determine the structure of the surrounding space-time by doing measurements near its world line. To study this, we need to produce a large number of measurements, or equivalently, a large number of sources.

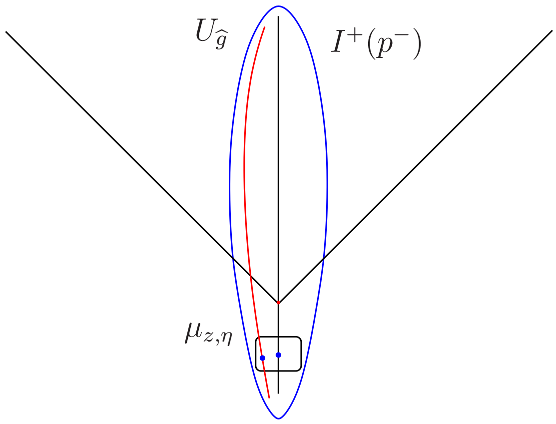

FIGURE 1. This is a schematic figure in $\mathbb{R}^{1+1}$. The black vertical line is the freely falling observer $\widehat{\mu}([-1,1])$. The rounded black square is $\pi(\mathcal{U}_{z_0,\eta_0})$ that is is a neighborhood of z_0, and the red curve passing through $z \in \pi(\mathcal{U}_{z_0,\eta_0})$ is the time-like geodesic $\mu_{z,\eta}([-1,1])$. The boundary of the domain $U_{\widehat{g}}$ where we observe waves is shown on blue. The black future cone is the set $I^+_{\widehat{g}}(p^-)$.

1.0.1. *Notations.* Let (M,g) be a C^∞-smooth $(1+3)$-dimensional time-orientable Lorentzian manifold. For $x, y \in M$ we say that x is in the chronological past of y and denote $x \ll y$ if $x \neq y$ and there is a time-like path from x to y. If $x \neq y$ and there is a causal path from x to y, we say that x is in the causal past of y and denote $x < y$. If $x < y$ or $x = y$ we denote $x \leq y$. The chronological future $I^+(p)$ of $p \in M$ consist of all points $x \in M$ such that $p \ll x$, and the causal future $J^+(p)$ of p consist of all points $x \in M$ such that $p \leq x$. One defines similarly the chronological past $I^-(p)$ of p and the causal past $J^-(p)$ of p. For a set A we denote $J^\pm(A) = \cup_{p\in A} J^\pm(p)$. We also denote $J(p,q) := J^+(p) \cap J^-(q)$ and $I(p,q) := I^+(p) \cap I^-(q)$. If we need to emphasize the metric g which is used to define the causality, we denote $J^\pm(p)$ by $J^\pm_g(p)$ etc.

Let $\gamma_{x,\xi}(t) = \gamma^g_{x,\xi}(t) = \exp_x(t\xi)$ denote a geodesics in (M,g). The projection from the tangent bundle TM to the base point of a vector is denoted by $\pi : TM \to M$. Let $L_x M$ denote the light-like directions of $T_x M$, and $L^+_x M$ and $L^-_x M$ denote the future and past pointing light-like vectors, respectively. We also denote ${}^{+}_{g}(x) = \exp_x(L^+_x M) \cup \{x\}$ the union of the image of the future light-cone in the exponential map of (M,g) and the point x.

By [10], an open time-orientable Lorenzian manifold (M,g) is globally hyperbolic if and only if there are no closed causal paths in M and for all $q^-, q^+ \in M$ such that $q^- < q^+$ the set $J(q^-,q^+) \subset M$ is compact. We assume throughout the paper that (M,g) is globally hyperbolic.

When g is a Lorentzian metric, having eigenvalues $\lambda_j(x)$ and eigenvectors $v_j(x)$ in some local coordinates, we will use also the corresponding Riemannian metric, denoted g^+ which has the eigenvalues

$|\lambda_j(x)|$ and the eigenvectors $v_j(x)$ in the same local coordinates. Let $B_{g^+}(x,r)=\{y\in M;\ d_{g^+}(x,y)<r\}$. Finally, when X is a set, let $P(X)=2^X=\{Z;\ Z\subset X\}$ denote the power set of X.

1.0.2. *Perturbations of a global hyperbolic metric.* Let $(M,\widehat g)$ be a C^∞-smooth globally hyperbolic Lorentzian manifold. We will call $\widehat g$ the background metric on M and consider its small perturbations. A Lorentzian metric g_1 dominates the metric g_2, if all vectors ξ that are light-like or time-like with respect to the metric g_2 are time-like with respect to the metric g_1, and in this case we denote $g_2<g_1$. As $(M,\widehat g)$ is globally hyperbolic, it follows from [34] that there is a Lorentzian metric $\widetilde g$ such that $(M,\widetilde g)$ is globally hyperbolic and $\widehat g<\widetilde g$. One can assume that the metric $\widetilde g$ is smooth. We use the positive definite Riemannian metric $\widehat g^+$ to define norms in the spaces $C^k_b(M)$ of functions with bounded k derivatives and the Sobolev spaces $H^s(M)$.

By [11], the globally hyperbolic manifold $(M,\widetilde g)$ has an isometry Φ to the smooth product manifold $(\mathbb{R}\times N,\widetilde h)$, where N is a 3-dimensional manifold and the metric $\widetilde h$ can be written as $\widetilde h=-\beta(t,y)dt^2+\kappa(t,y)$ where $\beta:\mathbb{R}\times N\to(0,\infty)$ is a smooth function and $\kappa(t,\cdot\,)$ is a Riemannian metric on N depending smoothly on $t\in\mathbb{R}$, and the submanifolds $\{t'\}\times N$ are C^∞-smooth Cauchy surfaces for all $t'\in\mathbb{R}$. We define the smooth time function $\mathbf{t}:M\to\mathbb{R}$ by setting $\mathbf{t}(x)=t$ if $\Phi(x)\in\{t\}\times N$. Let us next identify these isometric manifolds, that is, we denote $M=\mathbb{R}\times N$.

For $t\in\mathbb{R}$, let $M(t)=(-\infty,t)\times N$ and, for a fixed $t_0>0$ and $t_1>t_0$, let $M_j=M(t_j)$, $j=1,2$. Let $r_0>0$ be sufficiently small and $\mathcal{V}(r_0)$ be the set of metrics g on $M_1=(-\infty,t_1)\times N$, which $C^8_b(M_1)$-distance to $\widehat g$ is less that r_0 and coincide with $\widehat g$ in $M(0)=(-\infty,0)\times N$.

1.0.3. *Observation domain U.* For $g\in\mathcal{V}(r_0)$, let $\mu_g:[-1,1]\to M_1$ be a freely falling observer, that is, a time-like geodesic on (M,g). Let $-1<s_{-2}<s_{-1}<1$ be such that $p^-=\mu_g(s_{-1})\in\{0\}\times N$. Below, we denote $s_-=s_{-1}$ and $\widehat\mu=\mu_{\widehat g}$.

When $z_0=\widehat\mu(s_{-2})\in M(0)$ and $\eta_0=\partial_s\widehat\mu(s_{-2})$, let $\mathcal{U}_{z_0,\eta_0}(h)$ be the open h-neighborhood of (z_0,η_0) in the Sasaki metric of $(TM,\widehat g^+)$. We use below a small parameter $\widehat h>0$. For $(z,\eta)\in\mathcal{U}_{z_0,\eta_0}(2\widehat h)$ we define on (M,g) a freely falling observer $\mu_{g,z,\eta}:[-1,1]\to M$, such that $\mu_{g,z,\eta}(s_{-2})=z$, and $\partial_s\mu_{g,z,\eta}(s_{-2})=\eta$. We assume that $\widehat h$ is so small that $\pi(\mathcal{U}_{z_0,\eta_0}(2\widehat h))\subset M(0)$ and for all $g\in\mathcal{V}(r_0)$ and $(z,\eta)\in\mathcal{U}_{z_0,\eta_0}(2\widehat h)$ the geodesic $\mu_{g,z,\eta}([-1,1])\subset M$ is well defined and time-like. We denote, see Fig. 1, $\mathcal{U}_{z_0,\eta_0}=\mathcal{U}_{z_0,\eta_0}(\widehat h)$ and

$$U_g=\bigcup_{(z,\eta)\in\mathcal{U}_{z_0,\eta_0}}\mu_{g,z,\eta}([-1,1]),\quad U_g^+=U_g\cap I^+(\mu_{g,z,\eta}(s^+)),\tag{1}$$

and we also denote $\widehat U=U_{\widehat g}$ and $\widehat U^+=U^+_{\widehat g}$.

1.1. **Formulation of the linearization stability problem.**

1.1.1. *Einstein equations.* Below, we use the Einstein summation convention. The roman indexes i, j, k etc. run usually over the values $0, 1, 2, 3$ as the greek letters are reserved for other indexes in the sums. The Einstein tensor of a Lorentzian metric $g = g_{jk}(x)$ is

$$\mathrm{Ein}_{jk}(g) = \mathrm{Ric}_{jk}(g) - \frac{1}{2}(g^{pq}\,\mathrm{Ric}_{pq}(g))g_{jk}.$$

Here, $\mathrm{Ric}_{pq}(g)$ is the Ricci curvature of the metric g. We define the divergence of a 2-covariant tensor T_{jk} to be $(\mathrm{div}_g T)_k = \nabla_n(g^{nj}T_{jk})$.

Let us consider the Einstein equations in presence of matter,

$$\mathrm{Ein}_{jk}(g) = T_{jk}, \tag{2}$$

$$\mathrm{div}_g T = 0, \tag{3}$$

for a Lorentzian metric g and a stress-energy tensor T related to the distribution of mass and energy. We recall that by Bianchi's identity $\mathrm{div}_g(\mathrm{Ein}(g)) = 0$ and thus the equation (3), called the conservation law for the stress-energy tensor, follows automatically from (2).

1.1.2. *Reduced Einstein tensor.* Let $m \geq 5$, $t_1 > t_0 > 0$ and $g' \in \mathcal{V}(r_0)$ be a C^m-smooth metric that satisfy the Einstein equations $\mathrm{Ein}(g') = T'$ on $M(t_1)$. When r_0 above is small enough, there is a diffeomorphism $f : M(t_1) \to f(M(t_1)) \subset M$ that is a $(g', \widehat{g})$-wave map $f : (M(t_1), g') \to (M, \widehat{g})$ and satisfies $M(t_0) \subset f(M(t_1))$. Here, $f : (M(t_1), g') \to (M, \widehat{g})$ is a wave map, see [12, Sec. VI.7.2 and App. III, Thm. 4.2], if

$$\Box_{g',\widehat{g}} f = 0 \quad \text{in } M(t_1), \tag{4}$$

$$f = Id, \quad \text{in } (-\infty, 0) \times N, \tag{5}$$

where $\Box_{g',\widehat{g}} f = g' \cdot \widehat{\nabla}^2 f$ is the wave map operator, and $\widehat{\nabla}$ is the covariant derivative for the maps $(M_1, g') \to (M, \widehat{g})$, see [12, formula (VI.7.32)]. The existence and the properties of the map f is discussed below in Subsection 2.1.3. The wave map has the property that $\mathrm{Ein}(f_* g') = \mathrm{Ein}_{\widehat{g}}(f_* g')$, where $\mathrm{Ein}_{\widehat{g}}(g)$ is the $\widehat{g}$-reduced Einstein tensor,

$$(\mathrm{Ein}_{\widehat{g}} g)_{pq} = -\frac{1}{2} g^{jk}\widehat{\nabla}_j\widehat{\nabla}_k g_{pq} + \frac{1}{4}(g^{nm}g^{jk}\widehat{\nabla}_j\widehat{\nabla}_k g_{nm})g_{pq} + P_{pq}(g, \widehat{\nabla} g),$$

where $\widehat{\nabla}_j$ is the covariant differentation with respect to the metric $\widehat{g}$ and P_{pq} is a polynomial function of g_{nm}, g^{nm}, and $\widehat{\nabla}_j g_{nm}$ with coefficients depending on the metric $\widehat{g}_{nm}$ and its derivatives. Considering the wave map f as a transformation of coordinates, we see that $g = f_* g'$ and $T = f_* T'$ satisfy the $\widehat{g}$-reduced Einstein equation

$$\mathrm{Ein}_{\widehat{g}}(g) = T \quad \text{on } M(t_0). \tag{6}$$

In the literature, the above is often stated by saying that the reduced Einstein equations (6) is the Einstein equations written with the wave-gauge corresponding to the metric $\widehat{g}$. The equation (6) is a quasi-linear hyperbolic system of equations for g_{jk}. We emphasize that a solution of the reduced Einstein equations can be a solution of the original Einstein equations only if the stress energy tensor satisfies the conservation law $\nabla^g_j T^{jk} = 0$. It is usual also to assume that the energy density is non-negative. For instance, the weak energy condition requires that $T_{jk}X^jX^k \geq 0$ for all time-like vectors X. Next, we couple the Einstein equations with matter fields and formulate an initial value problem for the $\widehat{g}$-reduced Einstein equations with abstract sources.

1.1.3. *The direct problem.* We consider the coupled system of the Einstein equation and L scalar field equations with some sources $\mathcal{F}^1$ and $\mathcal{F}^2$. In these equations we use the wave gauge, that is, the equations are written for the metric tensor and the scalar fied that are pushed forward with the wave map so that the Einstein tensor of g is equal to the reduced Einstein tensor $\mathrm{Ein}_{\widehat{g}}(g)$.

Let $\widehat{g}$ and $\widehat{\phi} = (\widehat{\phi}_\ell)_{\ell=1}^L$ be C^∞-background fields on M. Consider

$$
\begin{aligned}
(7)\qquad & \mathrm{Ein}_{\widehat{g}}(g) = T, \quad T_{jk} = \mathbf{T}_{jk}(g,\phi) + \mathcal{F}^1_{jk}, \quad \text{in } M_0,\\
& \mathbf{T}_{jk}(g,\phi) = \sum_{\ell=1}^{L} (\partial_j\phi_\ell\, \partial_k\phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p\phi_\ell\, \partial_q\phi_\ell - V(\phi_\ell) g_{jk}),\\
& \square_g \phi_\ell - V'(\phi_\ell) = \mathcal{F}^2_\ell, \quad \ell = 1,2,3,\dots,L,\\
& g = \widehat{g} \text{ and } \phi_\ell = \widehat{\phi}_\ell \text{ in } (-\infty,0)\times N
\end{aligned}
$$

where $\mathcal{F}^1$ and $\mathcal{F}^2$ are supported in $U_g^+ \cap J_g^+(p^-)$, $V(s) = \frac{1}{2}m^2 s^2$. Above, $\square_g \phi = (-\det(g))^{-\frac{1}{2}} \partial_p((-\det(g))^{\frac{1}{2}} g^{pq} \partial_q \phi)$. We assume that the background fields $\widehat{g}$ and $\widehat{\phi}$ satisfy the equations (7) with $\mathcal{F}^1 = 0$ and $\mathcal{F}^2 = 0$. Note that above $J_g^+(p^-) \cap M_0 \subset J_{\widetilde{g}}^+(p^-)$ when $g \in \mathcal{V}(r_0)$.

To obtain a physically meaningful model, we need to assume that the physical conservation law in relativity,

$$
(8)\quad \nabla_p(g^{pk} T_{kj}) = 0, \quad \text{for } j = 1,2,3,4, \text{ where } T_{kj} = \mathbf{T}_{kj}(g,\phi) + \mathcal{F}^1_{kj}
$$

is satisfied. Here $\nabla = \nabla^g$ is the connection corresponding to g. In Subsection 2.1.4 we show that for the solution (g,ϕ) of the system (7) and the conservation law (8) the reduced Einstein tensor $\mathrm{Ein}_{\widehat{g}}(g)$ will then be equal to the Einstein tensor $\mathrm{Ein}(g)$.

We mainly need local existence results[1] for the system (7). The global existence problem for the related systems has recently attracted much interest in the mathematical community and many important results been obtained, see e.g. [18, 22, 63, 65, 69, 70].

[1]In this paper we do not use optimal smoothness for the solutions in classical C^k spaces or Sobolev space $W^{k,p}$ but just suitable smoothness for which the non-linear wave equations can be easily analyzed using L^2-based Sobolev spaces.

We encounter above the difficulty that the source $\mathcal{F}=(\mathcal{F}^1,\mathcal{F}^2)$ in (7) has to satisfy the condition (8) that depends on the solution g of (7). This makes the formulation of active measurements in relativity difficult. Later, we consider a model where the source term $\mathcal{F}^1$ corresponds to e.g. fluid fields consisting of particles whose4-velocity vectors are controlled and $\mathcal{F}^2$ contains a term corresponding to a secondary source function that adapts the changes of g,ϕ, and $\mathcal{F}^1$ so that the physical conservation law is satisfied.

1.1.4. *Linearized equations.* We need also to consider the linearized version of the equations (7) that have the form (in local coordinates)

$$\begin{aligned} &\square_{\widehat{g}}\dot g_{jk}+A_{jk}(\dot g,\dot\phi,\partial\dot g,\partial\dot\phi)=f^1,\quad \text{in } M_0,\\ &\square_{\widehat{g}}\dot\phi_\ell+B_\ell(\dot g,\dot\phi,\partial\dot g,\partial\dot\phi)=f^2,\quad \ell=1,2,3,\dots,L, \end{aligned}\tag{9}$$

where A_{jk} and B_ℓ are first order linear differential operators whose coefficients depend on $\widehat{g}$ and $\widehat{\phi}$. For more explicit formulation, see Subsection 2.2.1. When g_ε and ϕ_ε are solutions of (7) with source $\mathcal{F}_\varepsilon$ depending smoothly on $\varepsilon\in\mathbb{R}$ such that $(g_\varepsilon,\phi_\varepsilon,\mathcal{F}_\varepsilon)|_{\varepsilon=0}=(\widehat{g},\widehat{\phi},0)$, then $(\dot g,\dot\phi,f)=(\partial_\varepsilon g_\varepsilon,\partial_\varepsilon\phi_\varepsilon,\partial_\varepsilon\mathcal{F}_\varepsilon)|_{\varepsilon=0}$ solve (9).

Let us consider the concept of the *linearization stability (LS)* for the source problems, cf. [33]:

Definition 1.1. *Let $s_0>4$ and consider a C^{s_0+4}-smooth source $f=(f^1,f^2)$ that is supported in $U_{\widehat{g}}$ and satisfies the linearized conservation law*

$$\frac{1}{2}\widehat{g}^{pk}\widehat{\nabla}_p f^1_{kj}+\sum_{\ell=1}^{L} f^2_\ell\,\partial_j\widehat{\phi}_\ell=0,\quad j=1,2,3,4.\tag{10}$$

Let $(\dot g,\dot\phi)$ be the solution of the linearized Einstein equations (9) with source f. We say that f has the LS-property in $C^{s_0}(M_0)$ if there are $\varepsilon_0>0$ and a family $\mathcal{F}_\varepsilon=(\mathcal{F}^1_\varepsilon,\mathcal{F}^2_\varepsilon)$ of sources, supported in U_{g_ε} for all $\varepsilon\in[0,\varepsilon_0)$, and functions $(g_\varepsilon,\phi_\varepsilon)$ that depend smoothly on $\varepsilon\in[0,\varepsilon_0)$ in $C^{s_0}(M_0)$ such that

(11) *$(g_\varepsilon,\phi_\varepsilon)$ solves the equations (7) and the conservation law (8),*

$$(g_\varepsilon,\phi_\varepsilon,\mathcal{F}_\varepsilon)|_{\varepsilon=0}=(\widehat{g},\widehat{\phi},0),\ \textit{and}\ (\dot g,\dot\phi,f)=(\partial_\varepsilon g_\varepsilon,\partial_\varepsilon\phi_\varepsilon,\partial_\varepsilon\mathcal{F}_\varepsilon)|_{\varepsilon=0}.$$

In this case, we say that $f=(f^1,f^2)$ has the LS-property with the family $\mathcal{F}_\varepsilon$, $\varepsilon\in[0,\varepsilon_0)$.

Note that above (10) is obtained by linearization of the conservation law (8).

Next, we consider sources that are conormal distributions. When $Y\subset U_{\widehat{g}}$ is a 2-dimensional space-like submanifold, consider local coordinates defined in $V\subset M_0$ such that $Y\cap V\subset\{x\in\mathbb{R}^4;\ x^jb^1_j=0,\ x^jb^2_j=0\}$, where $b^1_j,b^2_j\in\mathbb{R}$. Next we slightly abuse the notation by identifying $x\in V$ with its coordinates $X(x)\in\mathbb{R}^4$. We denote

$f \in \mathcal{I}^n(Y)$, $n \in \mathbb{R}$, if in the above local coordinates, f can be written as

$$f(x^1, x^2, x^3, x^4) = \mathrm{Re} \int_{\mathbb{R}^2} e^{i(\theta_1 b^1_m + \theta_2 b^2_m) x^m} \sigma_f(x, \theta_1, \theta_2)\, d\theta_1 d\theta_2, \tag{12}$$

where $\sigma_f(x,\theta) \in S^n_{0,1}(V;\mathbb{R}^2)$, $\theta = (\theta_1,\theta_2)$ is a classical symbol. A function $c(x,\theta)$ that is n-positive homogeneous in θ, i.e., $c(x, s\theta) = s^n c(x,\theta)$ for $s > 0$, is the principal symbol of f if there is $\phi \in C^\infty_0(\mathbb{R}^2)$ being 1 near zero such that $\sigma_f(x,\theta) - (1-\phi(\theta))c(x,\theta) \in S^{n_1}_{0,1}(V;\mathbb{R}^2)$, $n_1 < n$. When $\eta = \theta_1 b_1 + \theta_2 b_2 \in N^*_x Y$, we say that $\widetilde{c}(x,\eta) = c(x,\theta)$ is the value of the principal symbol of f at $(x,\eta) \in N^*Y$.

We need a condition that we call *microlocal linearization stability*:

Definition of the μ-LS (Microlocal linearization stability) property: *We say that the set $U^+_{\widehat{g}}$, and $(M, \widehat{g})$ have the microlocal linearization stability property if the following holds:*

Let $Y \subset U^+_{\widehat{g}}$ be a 2-dimensional space-like submanifold, $V \subset U^+_{\widehat{g}}$ an open local coordinate neighborhood of $y \in Y$ with coordinates $X : V \to \mathbb{R}^4$, $X^j(x) = x^j$ such that $X(Y \cap V) \subset \{x \in \mathbb{R}^4;\ x^j b^1_j = 0,\ x^j b^2_j = 0\}$. Let, in addition, $(y,\eta) \in N^*Y$ be a light-like covector. $\mathcal{W} \subset N^*Y$ be a conic neighborhood of (y,η), $(c_{jk})^4_{j,k=1}$ be a symmetric matrix that satisfies

$$\widehat{g}^{lk}(y)\eta_l c_{kj} = 0, \quad \textit{for all } j = 1,2,3,4, \tag{13}$$

and $(d_\ell)^L_{\ell=1} \in \mathbb{R}^L$. Then there is $n_1 \in \mathbb{Z}_+$ such that for any $n \in \mathbb{Z}_-$, $n \le -n_1$ there are $f^1_{jk} \in \mathcal{I}^n(Y)$, $(j,k) \in \{1,2,3,4\}^2$, and $f^2_\ell \in \mathcal{I}^n(Y)$, $\ell = 1,2,\dots,L$, supported in V with symbols that are in $S^{-\infty}$ outside the neighborhood $\mathcal{W}$ of (y,η), and whose principal symbols at (y,η) are equal to $\widetilde{f}^1_{jk}(y,\eta) = c_{jk}$ and $\widetilde{f}^2_\ell(y,\eta) = d_\ell$, respectively. Moreover, the source $f = (f^1, f^2)$ satisfies the linearized conservation law (10) and f has the LS property (11) in $C^{s_1}(M_0)$, $s_1 \ge 13$, with a family $\mathcal{F}_\varepsilon$, $\varepsilon \in [0,\varepsilon_0)$ such that $\mathcal{F}_\varepsilon$ are supported in V.

1.1.5. *Main results.* We consider the following condition that is valid when the background fields vary sufficiently:

Condition A: Assume that at any $x \in \overline{U_{\widehat{g}}}$ there is a permutation $\sigma : \{1,2,\dots,L\} \to \{1,2,\dots,L\}$, denoted σ_x, such that the 5×5 matrix $[B^\sigma_{jk}(\widehat{\phi}(x), \nabla\widehat{\phi}(x))]_{j,k\le 5}$ is invertible, where

$$[B^\sigma_{jk}(\phi(x), \nabla\phi(x))]_{k,j\le 5} = \begin{bmatrix} (\partial_j \phi_\ell(x))_{\ell \le 5,\ j \le 4} \\ (\phi_\ell(x))_{\ell \le 5} \end{bmatrix}.$$

Our main result is that when the Condition A is valid, also the condition μ-LS is valid:

Theorem 1.2. *(Microlocal linearization stability) Let $(M,\widehat{g})$ be a smooth, globally hyperbolic Lorentzian manifold with a global time function $\mathbf{t}: M\to\mathbb{R}$ and $M_0=\mathbf{t}^{-1}(-\infty,t_0)$. Moreover, let $U_{\widehat{g}}^+, U_{\widehat{g}}\subset M_0$ be the sets of the form (1) and let $(\widehat{g},\widehat{\phi})$ satisfy the equation (7) with vanishing sources $\mathcal{F}^1=0$ and $\mathcal{F}^2=0$. Assume that Condition A is valid in $U_{\widehat{g}}$. Then the set $U_{\widehat{g}}^+$ and $(M,\widehat{g})$ have the microlocal linearization stability property.*

2. Basic properties of the Einstein equations

2.1. **Reduced Einstein equation and wave map.** In this section we review well known results for the Einstein equation and the wave maps that we will need later.

2.1.1. *Geometric considerations.* Let us recall some definitions given in the Introduction. Let $(M,\widehat{g})$ be a C^∞-smooth globally hyperbolic Lorentzian manifold and $\widetilde{g}$ be a C^∞-smooth globally hyperbolic metric on M such that $\widehat{g}<\widetilde{g}$. Let us start by explaining how one can construct a C^∞-smooth metric $\widetilde{g}$ such that $\widehat{g}<\widetilde{g}$ and $(M,\widetilde{g})$ is globally hyperbolic: Let $v(x)$ be an eigenvector corresponding to the negative eigenvalue of $\widehat{g}(x)$. We can choose a smooth, strictly positive function $\eta: M\to\mathbb{R}_+$ such that

$$\widetilde{g}':=\widehat{g}-\eta v\otimes v<\widetilde{g}.$$

Then $(M,\widetilde{g}')$ is globally hyperbolic, $\widetilde{g}'$ is smooth and $\widehat{g}<\widetilde{g}'$. Thus we can replace $\widetilde{g}$ by the smooth metric $\widetilde{g}'$ having the same properties that are required for $\widetilde{g}$.

Recall that there is an isometry $\Phi:(M,\widetilde{g})\to(\mathbb{R}\times N,\widetilde{h})$, where N is a 3-dimensional manifold and the metric $\widetilde{h}$ can be written as $\widetilde{h}=-\beta(t,y)dt^2+\kappa(t,y)$ where $\beta:\mathbb{R}\times N\to(0,\infty)$ is a smooth function and $\kappa(t,\cdot\,)$ is a Riemannian metric on N depending smoothly on $t\in\mathbb{R}$. As in the main text we identify these isometric manifolds and denote $M=\mathbb{R}\times N$. Also, for $t\in\mathbb{R}$, recall that $M(t)=(-\infty,t)\times N$. We use parameters $t_1>t_0>0$ and denote $M_j=M(t_j)$, $j\in\{0,1\}$. We use the time-like geodesic $\widehat{\mu}=\mu_{\widehat{g}}$, $\mu_{\widehat{g}}:[-1,1]\to M_0$ on $(M_0,\widehat{g})$ and the set $\mathcal{K}_j:=J_{\widetilde{g}}^+(\widehat{\mu}(-1))\cap M_j$ with $\widehat{\mu}(-1)\in(-\infty,t_0)\times N$. Then $J_{\widetilde{g}}^+(\widehat{\mu}(-1))\cap M_j$ is compact. Also, there exists $\varepsilon_0>0$ such that if g is a Lorentz metric in M_1 such that $\|g-\widehat{g}\|_{C_b^0(M_1;\widehat{g}^+)}<\varepsilon_0$, then $g|_{\mathcal{K}_1}<\widetilde{g}|_{\mathcal{K}_1}$. In particular, this implies that we have $J_g^+(p)\cap M_1\subset\mathcal{K}_1$ for all $p\in\mathcal{K}_1$. Later, we use this property to deduce that when g satisfies the $\widehat{g}$-reduced Einstein equations in M_1, with a source that is supported in $\mathcal{K}_1$ and has small enough norm is a suitable space, then g coincides with $\widehat{g}$ in $M_1\setminus\mathcal{K}_1$ and satisfies $g<\widetilde{g}$.

Let us use local coordinates on M_1 and denote by $\nabla_k=\nabla_{X_k}$ the covariant derivative with respect to the metric g in the direction $X_k=$

$\frac{\partial}{\partial x^k}$ and by $\widehat\nabla_k=\widehat\nabla_{X_k}$ the covariant derivative with respect to the metric $\widehat g$ to the direction X_k.

2.1.2. *Reduced Ricci and Einstein tensors.* Following [32] we recall that

$$\mathrm{Ric}_{\mu\nu}(g) \;=\; \mathrm{Ric}^{(h)}_{\mu\nu}(g)+\frac12(g_{\mu q}\frac{\partial\Gamma^q}{\partial x^\nu}+g_{\nu q}\frac{\partial\Gamma^q}{\partial x^\mu}) \tag{14}$$

where $\Gamma^q=g^{mn}\Gamma^q_{mn}$,

$$\mathrm{Ric}^{(h)}_{\mu\nu}(g)=-\frac12 g^{pq}\frac{\partial^2 g_{\mu\nu}}{\partial x^p\partial x^q}+P_{\mu\nu}, \tag{15}$$

$$P_{\mu\nu}=g^{ab}g_{ps}\Gamma^p_{\mu b}\Gamma^s_{\nu a}+\frac12(\frac{\partial g_{\mu\nu}}{\partial x^a}\Gamma^a+g_{\nu l}\Gamma^l_{ab}g^{aq}g^{bd}\frac{\partial g_{qd}}{\partial x^\mu}+g_{\mu l}\Gamma^l_{ab}g^{aq}g^{bd}\frac{\partial g_{qd}}{\partial x^\nu}).$$

Note that $P_{\mu\nu}$ is a polynomial of g_{jk} and g^{jk} and first derivatives of g_{jk}. The harmonic Einstein tensor is

$$\mathrm{Ein}^{(h)}_{jk}(g)=\mathrm{Ric}^{(h)}_{jk}(g)-\frac12 g^{pq}\mathrm{Ric}^{(h)}_{pq}(g)\,g_{jk}. \tag{16}$$

The harmonic Einstein tensor is extensively used to study the Einstein equations in local coordinates where one can use the Minkowski space $\mathbb{R}^4$ as the background space. To do global constructions with a background space $(M,\widehat g)$ one uses the reduced Einstein tensor. The $\widehat g$-reduced Einstein tensor $\mathrm{Ein}_{\widehat g}(g)$ and the $\widehat g$-reduced Ricci tensor $\mathrm{Ric}_{\widehat g}(g)$ are given by

$$(\mathrm{Ein}_{\widehat g}(g))_{pq}=(\mathrm{Ric}_{\widehat g}(g))_{pq}-\frac12(g^{jk}(\mathrm{Ric}_{\widehat g}g)_{jk})g_{pq}, \tag{17}$$

$$(\mathrm{Ric}_{\widehat g}(g))_{pq}=\mathrm{Ric}_{pq}(g)-\frac12(g_{pn}\widehat\nabla_q\widehat F^n+g_{qn}\widehat\nabla_p\widehat F^n) \tag{18}$$

where $\widehat F^n$ are the harmonicity functions given by

$$\widehat F^n=\Gamma^n-\widehat\Gamma^n,\quad \text{where } \Gamma^n=g^{jk}\Gamma^n_{jk},\quad \widehat\Gamma^n=g^{jk}\widehat\Gamma^n_{jk}, \tag{19}$$

with Γ^n_{jk} and $\widehat\Gamma^n_{jk}$ being the Christoffel symbols for g and $\widehat g$, correspondingly. Note that $\widehat\Gamma^n$ depends also on g^{jk}. As $\Gamma^n_{jk}-\widehat\Gamma^n_{jk}$ is the difference of two connection coefficients, it is a tensor. Thus $\widehat F^n$ is tensor (actually, a vector field), implying that both $(\mathrm{Ric}_{\widehat g}(g))_{jk}$ and $(\mathrm{Ein}_{\widehat g}(g))_{jk}$ are 2-covariant tensors. A direct calculation shows that the $\widehat g$-reduced Einstein tensor is the sum of the harmonic Einstein tensor and a term that is a zeroth order in g,

$$(\mathrm{Ein}_{\widehat g}(g))_{\mu\nu}=\mathrm{Ein}^{(h)}_{\mu\nu}(g)+\frac12(g_{\mu q}\frac{\partial\widehat\Gamma^q}{\partial x^\nu}+g_{\nu q}\frac{\partial\widehat\Gamma^q}{\partial x^\mu}). \tag{20}$$

We also use the wave operator

(21)

$$\square_g\phi=\sum_{p,q=1}^4(-\det(g(x)))^{-\frac12}\frac{\partial}{\partial x^p}\left((-\det(g(x)))^{\frac12}g^{pq}(x)\frac{\partial}{\partial x^q}\phi(x)\right),$$

which can be written as

$$\square_g \phi = g^{jk}\partial_j\partial_k\phi - g^{pq}\Gamma^n_{pq}\partial_n\phi = g^{jk}\partial_j\partial_k\phi - \Gamma^n\partial_n\phi. \tag{22}$$

2.1.3. *Wave maps and reduced Einstein equations.* Let us consider the manifold $M_1 = (-\infty, t_1) \times N$ with a C^m-smooth metric g', $m \geq 5$, which is a perturbation of the metric $\widehat{g}$ and satisfies the Einstein equation

$$\mathrm{Ein}(g') = T' \quad \text{on } M_1, \tag{23}$$

or equivalently,

$$\mathrm{Ric}(g') = \rho', \quad \rho'_{jk} = T'_{jk} - \frac{1}{2}((g')^{nm}T'_{nm})g'_{jk} \quad \text{on } M_1.$$

Assume also that $g' = \widehat{g}$ in the domain A, where $A = M_1 \setminus \mathcal{K}_1$ and $\|g' - \widehat{g}\|_{C^2_b(M_1,\widehat{g}^+)} < \varepsilon_0$, so that (M_1, g') is globally hyperbolic. Note that then $T' = \widehat{T}$ in the set A. Then the metric g' coincides with $\widehat{g}$ in particular in the set $M^- = \mathbb{R}_- \times N$

We recall next the considerations of [12]. Let us consider the Cauchy problem for the wave map $f : (M_1, g') \to (M, \widehat{g})$, namely

$$\square_{g',\widehat{g}} f = 0 \quad \text{in } M_1, \tag{24}$$

$$f = Id, \quad \text{in } \mathbb{R}_- \times N, \tag{25}$$

where $M_1 = (-\infty, t_1) \times N \subset M$. In (24), $\square_{g',\widehat{g}} f = g' \cdot \widehat{\nabla}^2 f$ is the wave map operator, where $\widehat{\nabla}$ is the covariant derivative of a map $(M_1, g') \to (M, \widehat{g})$, see [12, Ch. VI, formula (7.32)]. In local coordinates $X : V \to \mathbb{R}^4$ of $V \subset M_1$, denoted $X(z) = (x^j(z))_{j=1}^4$ and $Y : W \to \mathbb{R}^4$ of $W \subset M$, denoted $Y(z) = (y^A(z))_{A=1}^4$, the wave map $f : M_1 \to M$ has the representation $Y(f(X^{-1}(x))) = (f^A(x))_{A=1}^4$ and the wave map operator in equation (24) is given by

$$(\square_{g',\widehat{g}} f)^A(x) = (g')^{jk}(x)\Big(\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}f^A(x) - \Gamma'^n_{jk}(x)\frac{\partial}{\partial x^n}f^A(x) + \widehat{\Gamma}^A_{BC}(f(x))\,\frac{\partial}{\partial x^j}f^B(x)\,\frac{\partial}{\partial x^k}f^C(x)\Big) \tag{26}$$

where $\widehat{\Gamma}^A_{BC}$ denotes the Christoffel symbols of metric $\widehat{g}$ and Γ'^j_{kl} are the Christoffel symbols of metric g'. When (24) is satisfied, we say that f is wave map with respect to the pair $(g', \widehat{g})$.

It follows from [12, App. III, Thm. 4.2 and sec. 4.2.2], that if $g' \in C^m(M_0)$, $m \geq 5$ is sufficiently close to $\widehat{g}$ in $C^m(M_0)$, then (4)-(5) has a unique solution $f \in C^0([0, t_1]; H^{m-1}(N)) \cap C^1([0, t_1]; H^{m-2}(N))$. This comes from the fact that the Christoffel symbols of g' are in $C^{m-1}(M_0)$. Moreover, when m is even, using [58, Thm. 7] for f and $\partial_t^p f$, we see that the solution $f \in \cap_{p=0}^{m-1} C^p([0, t_1]; H^{m-1-p}(N)) \subset C^{m-3}(M_0)$ and f depends in $C^{m-3}(M_0)$ continuously on $g' \in C^m(M_0)$. We note that these smoothness results for f are not optimal.

The wave map operator $\square_{g',\widehat{g}}$ is a coordinate invariant operator. The important property of the wave maps is that, if f is wave map with respect to the pair $(g',\widehat{g})$ and $g=f_*g'$ then, as follows from (26), the identity map $Id:x\mapsto x$ is a wave map with respect to the pair $(g,\widehat{g})$ and, the wave map equation for the identity map is equivalent to (cf. [12, p. 162])

$$\Gamma^n=\widehat{\Gamma}^n,\quad \text{where } \Gamma^n=g^{jk}\Gamma^n_{jk},\quad \widehat{\Gamma}^n=g^{jk}\widehat{\Gamma}^n_{jk} \tag{27}$$

where the Christoffel symbols $\widehat{\Gamma}^n_{jk}$ of the metric $\widehat{g}$ are smooth functions.

Since $g=g'$ outside a compact set $\mathcal{K}_1\subset(0,t_1)\times N$, we see that this Cauchy problem is equivalent to the same equation restricted to the set $(-\infty,t_1)\times B_0$, where $B_0\subset N$ is an open relatively compact set such that $\mathcal{K}_1\subset(0,t_1]\times B_0$ with the boundary condition $f=Id$ on $(0,t_1]\times\partial B_0$. Moreover, by the uniqueness of the wave map, we have $f|_{M_1\setminus\mathcal{K}_1}=id$ so that $f(\mathcal{K}_1)\cap M_0\subset\mathcal{K}_0$.

As the inverse function of the wave map f depends continuously, in $C^{m-3}_b([0,t_1]\times N,g^+)$, on the metric $g'\in C^m(M_0)$ we can also assume that ε_1 is so small that $M_0\subset f(M_1)$.

Denote next $g:=f_*g'$, $T:=f_*T'$, and $\rho:=f_*\rho'$ and define $\widehat{\rho}=\widehat{T}-\frac{1}{2}(\mathrm{Tr}\,\widehat{T})\widehat{g}$. Then g is C^{m-6}-smooth and the equation (23) implies

$$\mathrm{Ein}(g)=T\quad\text{on } M_0. \tag{28}$$

Since f is a wave map and $g=f_*g'$, we have that the identity map is a $(g,\widehat{g})$-wave map and thus g satisfies (27) and thus by the definition of the reduced Einstein tensor, (18)-(17), we have

$$\mathrm{Ein}_{pq}(g)=(\mathrm{Ein}_{\widehat{g}}(g))_{pq}\quad\text{on } M_0.$$

This and (28) yield the $\widehat{g}$-reduced Einstein equation

$$(\mathrm{Ein}_{\widehat{g}}(g))_{pq}=T_{pq}\quad\text{on } M_0. \tag{29}$$

This equation is useful for our considerations as it is a quasilinear, hyperbolic equation on M_0. Recall that g coincides with $\widehat{g}$ in $M_0\setminus\mathcal{K}_0$. The unique solvability of this Cauchy problem is studied in e.g. [12, Thm. 4.6 and 4.13], [45].

2.1.4. *Relation of the reduced Einstein equations and the original Einstein equation.* The metric g which solves the $\widehat{g}$-reduced Einstein equation $\mathrm{Ein}_{\widehat{g}}(g)=T$ is a solution of the original Einstein equations $\mathrm{Ein}(g)=T$ if the harmonicity functions $\widehat{F}^n$ vanish identically. Next we recall the result that the harmonicity functions vanish on M_0 when

$$\begin{aligned}&(\mathrm{Ein}_{\widehat{g}}(g))_{jk}=T_{jk},\quad\text{on } M_0,\\ &\nabla_pT^{pq}=0,\quad\text{on } M_0,\\ &g=\widehat{g},\quad\text{on } M_0\setminus\mathcal{K}_0.\end{aligned} \tag{30}$$

To see this, let us denote $\mathrm{Ein}_{jk}(g) = S_{jk}$, $S^{jk} = g^{jn}g^{km}S_{nm}$, and $T^{jk} = g^{jn}g^{km}T_{nm}$. Following standard arguments, see [12], we see from (17) that in local coordinates

$$S_{jk} - (\mathrm{Ein}_{\widehat{g}}(g))_{jk} = \frac{1}{2}(g_{jn}\widehat{\nabla}_k\widehat{F}^n + g_{kn}\widehat{\nabla}_j\widehat{F}^n - g_{jk}\widehat{\nabla}_n\widehat{F}^n).$$

Using equations (30), the Bianchi identity $\nabla_p S^{pq} = 0$, and the basic property of Lorentzian connection, $\nabla_k g^{nm} = 0$, we obtain

$$\begin{aligned} 0 &= 2\nabla_p(S^{pq} - T^{pq}) \\ &= \nabla_p(g^{qk}\widehat{\nabla}_k F^p + g^{pm}\widehat{\nabla}_m\widehat{F}^q - g^{pq}\widehat{\nabla}_n\widehat{F}^n) \\ &= g^{pm}\nabla_p\widehat{\nabla}_m\widehat{F}^q + (g^{qp}\nabla_n\widehat{\nabla}_p\widehat{F}^n - g^{qp}\nabla_p\widehat{\nabla}_n\widehat{F}^n) \\ &= g^{pm}\nabla_p\widehat{\nabla}_m\widehat{F}^q + W^q(\widehat{F}) \end{aligned}$$

where $\widehat{F} = (\widehat{F}^q)_{q=1}^4$ and the operator

$$W : (\widehat{F}^q)_{q=1}^4 \mapsto (g^{qk}(\nabla_p\widehat{\nabla}_k\widehat{F}^p - \nabla_k\widehat{\nabla}_p\widehat{F}^p))_{q=1}^4$$

is a linear first order differential operator whose coefficients are polynomial functions of $\widehat{g}_{jk}$, $\widehat{g}^{jk}$, g_{jk}, g^{jk} and their first derivatives.

Thus the harmonicity functions $\widehat{F}^q$ satisfy on M_0 the hyperbolic initial value problem

$$\begin{aligned} &g^{pm}\nabla_p\widehat{\nabla}_m\widehat{F}^q + W^q(\widehat{F}) = 0, \quad \text{on } M_0, \\ &\widehat{F}^q = 0, \quad \text{on } M_0 \setminus \mathcal{K}_0, \end{aligned}$$

and as this initial Cauchy problem is uniquely solvable by [12, Thm. 4.6 and 4.13] or [45], we see that $\widehat{F}^q = 0$ on M_0. Thus equations (30) yield that the Einstein equations $\mathrm{Ein}(g) = T$ hold on M_0.

We note that in the $(g, \widehat{g})$-wave map coordinates, where $\widehat{F}^q = 0$, the wave operator (22) has the form

$$\square_g\phi = g^{jk}\partial_j\partial_k\phi - g^{pq}\widehat{\Gamma}^n_{pq}\partial_n\phi. \tag{31}$$

Thus, the scalar field equation $\square_g\phi - m^2\phi = 0$ does not involve derivatives of g.

2.2. **Linearization of Einstein equation and conservation law.**

2.2.1. *Linearized Einstein-scalar field equations.* Next we consider the linearized equations that are obtained as the derivatives of the solutions of the non-linear, generalized Einstein-matter field equations (7).

Observe that if a family $\mathcal{F}_\varepsilon = (\mathcal{F}^1_\varepsilon, \mathcal{F}^2_\varepsilon)$ of sources and a family $(g_\varepsilon, \phi_\varepsilon)$ of functions are solutions of the non-linear reduced Einstein-scalar field equations (7) that depend smoothly on $\varepsilon \in [0, \varepsilon_0)$ in $C^{15}(M_0)$ and satisfy $\mathcal{F}_\varepsilon|_{\varepsilon=0} = 0$, $(g_\varepsilon, \phi_\varepsilon)|_{\varepsilon=0} = (\widehat{g}, \widehat{\phi})$, then $\partial_\varepsilon\mathcal{F}_\varepsilon|_{\varepsilon=0} = f = (f^1, f^2)$, and

$\partial_\varepsilon(g_\varepsilon,\phi_\varepsilon)|_{\varepsilon=0}=(\dot g,\dot\phi)$, satisfies the linearized version of the equation (7) that has the form (in the local coordiantes),

$$\begin{aligned}(32)\qquad &\square_{\widehat g}\dot g_{jk}+A_{jk}(\dot g,\dot\phi,\partial\dot g,\partial\dot\phi)=f^1_{jk},\quad \text{in } M_0,\\ &\square_{\widehat g}\dot\phi_\ell+B_\ell(\dot g,\dot\phi,\partial\dot g,\partial\dot\phi)=f^2_\ell,\quad \ell=1,2,3,\dots,L.\end{aligned}$$

Here A_{jk} and B_ℓ are first order linear differential operators which coefficients depend on $\widehat g$ and $\widehat\phi$. Let us write these equations in more explicit form. We see that the linearized reduced Einstein tensor is in local coordinates of the form

$$\begin{aligned}e_{pq}(\dot g)\ &:=\ \partial_\varepsilon(\mathrm{Ein}_{\widehat g}g_\varepsilon)_{pq}|_{\varepsilon=0}\\ &=\ -\frac12\widehat g^{jk}\widehat\nabla_j\widehat\nabla_k\dot g_{pq}+\frac14(\widehat g^{nm}\widehat g^{jk}\widehat\nabla_j\widehat\nabla_k\dot g_{nm})\widehat g_{pq}\\ &\qquad +A^{abn}_{pq}\widehat\nabla_n\dot g_{ab}+B^{ab}_{pq}\dot g_{ab},\end{aligned}$$

where $A^{abn}_{pq}(x)$ and $B^{ab}_{pq}(x)$ depend on $\widehat g_{jk}$ and its derivatives (these terms can be computed explicitly using (15)). The linearized scalar field stress-energy tensor is the linear first order differential operator

$$\begin{aligned}&t^{(1)}_{pq}(\dot g)+t^{(2)}_{pq}(\dot\phi):=\partial_\varepsilon(\mathbf{T}_{jk}(g_\varepsilon,\phi_\varepsilon))|_{\varepsilon=0}\\ =\ &\sum_{\ell=1}^L\Big(\partial_j\widehat\phi_\ell\,\partial_k\dot\phi_\ell+\partial_j\dot\phi_\ell\,\partial_k\widehat\phi_\ell\\ &-\frac12\dot g_{jk}\widehat g^{pq}\partial_p\widehat\phi_\ell\,\partial_q\widehat\phi_\ell-\frac12\widehat g_{jk}\dot g^{pq}\partial_p\widehat\phi_\ell\,\partial_q\widehat\phi_\ell\\ &-\frac12\widehat g_{jk}\widehat g^{pq}\partial_p\dot\phi_\ell\,\partial_q\widehat\phi_\ell-\frac12\widehat g_{jk}\widehat g^{pq}\partial_p\widehat\phi_\ell\,\partial_q\dot\phi_\ell\\ &-m^2\widehat\phi_\ell\dot\phi_\ell\widehat g_{jk}-\frac12 m^2\widehat\phi_\ell^2\dot g_{jk}\Big).\end{aligned}$$

Thus when $\mathcal F_\varepsilon=(\mathcal F^1_\varepsilon,\mathcal F^2_\varepsilon)$ is a family of sources and $(g_\varepsilon,\phi_\varepsilon)$ a family of functions that satisfy the non-linear reduced Einstein-scalar field equations (7), the ε-derivatives $\dot u=(\dot g,\dot\phi)$ and $\partial_\varepsilon\mathcal F_\varepsilon|_{\varepsilon=0}=f=(f^1,f^2)$ satisfy in local $(g,\widehat g)$-wave coordinates

$$\begin{aligned}(33)\qquad &e_{pq}(\dot g)-t^{(1)}_{pq}(\dot g)-t^{(2)}_{pq}(\dot\phi)=f^1_{pq},\\ &\square_{\widehat g}\dot\phi_\ell-\widehat g^{nm}\widehat g^{kj}(\partial_n\partial_j\widehat\phi_\ell+\widehat\Gamma^p_{nj}\partial_p\widehat\phi_\ell)\dot g_{mk}-m^2\phi_\ell=f^2_\ell.\end{aligned}$$

We call this linearized Einstein-scalar field equations.

2.2.2. *Linearization of the conservation law.* Assume that (g,ϕ) and $\mathcal F=(\mathcal F^1,\mathcal F^2)$ satisfy equation (7). Then the conservation law (8) gives for all $j = 1,2,3,4$ equations (see [12, Sect. 6.4.1])

$$\begin{aligned} 0 &= \frac{1}{2}\nabla^g_p(g^{pk}T_{jk}) \\ &= \frac{1}{2}\nabla^g_p(g^{pk}(\mathbf{T}_{jk}(g,\phi)+\mathcal{F}^1_{jk})) \\ &= \sum_{\ell=1}^{L}(g^{pk}\nabla^g_p\partial_k\phi_\ell)\,\partial_j\phi_\ell-(m_\ell^2\phi_\ell\partial_p\phi_\ell)\delta^p_j+\frac{1}{2}\nabla^g_p(g^{pk}\mathcal{F}^1_{jk}) \\ &= \sum_{\ell=1}^{L}(g^{pk}\nabla^g_p\partial_k\phi_\ell-m_\ell^2\phi_\ell)\,\partial_j\phi_\ell+\frac{1}{2}\nabla^g_p(g^{pk}\mathcal{F}^1_{jk}). \end{aligned}$$

This yields by (7)

$$\frac{1}{2}g^{pk}\nabla^g_p\mathcal{F}^1_{jk}+\sum_{\ell=1}^{L}\mathcal{F}^2_\ell\,\partial_j\phi_\ell=0,\quad j=1,2,3,4. \tag{34}$$

Next assume that $(g_\varepsilon,\phi_\varepsilon)$ and $\mathcal{F}_\varepsilon$ satisfy equation (7) and C^1-smooth functions of $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$ taking values in $H^1(N)$-tensor fields, and $(g_\varepsilon,\phi_\varepsilon)|_{\varepsilon=0}=(\widehat g,\widehat\phi)$ and $\mathcal{F}_\varepsilon|_{\varepsilon=0}=0$. Denote $(f^1,f^2)=\partial_\varepsilon\mathcal{F}_\varepsilon|_{\varepsilon=0}$. Then by taking ε-derivative of (34) at $\varepsilon=0$ we get

$$\frac{1}{2}\widehat g^{pk}\widehat\nabla_p f^1_{jk}+\sum_{\ell=1}^{L}f^2_\ell\,\partial_j\widehat\phi_\ell=0,\quad j=1,2,3,4. \tag{35}$$

We call this the linearized conservation law.

3. Analysis for the Einstein-scalar field equations

Let we consider the solutions (g,ϕ) of the equations (7) with source $\mathcal{F}$. To consider their local existence, let us denote $u:=(g,\phi)-(\widehat g,\widehat\phi)$.

It follows from by [5, Cor. A.5.4] that $\mathcal{K}_j=J^+_{\widetilde g}(p^-)\cap\overline{M}_j$ is compact. Since $\widehat g<\widetilde g$, we see that if r_0 above is small enough, for all $g\in\mathcal{V}(r_0)$, see subsection 1.0.2, we have $g|_{\mathcal{K}_1}<\widetilde g|_{\mathcal{K}_1}$. In particular, we have $J^+_g(p^-)\cap M_1\subset J^+_{\widetilde g}(p^-)$.

Let us assume that $\mathcal{F}$ is small enough in the norm $C^4_b(M_0)$ and that it is supported in a compact set $\mathcal{K}=J_{\widetilde g}(p^-)\cap([0,t_0]\times N)\subset\overline{M}_0$. Then we can write the equations (7) for u in the form

$$\begin{aligned} &P_{g(u)}(u)=\mathcal{F},\quad x\in M_0,\\ &u=0 \text{ in } (-\infty,0)\times N, \text{ where}\\ &P_{g(u)}(u):=g^{jk}(x;u)\partial_j\partial_k u(x)+H(x,u(x),\partial u(x)), \end{aligned} \tag{36}$$

$(g^{jk}(x;u))^4_{j,k=1}=(g_{jk}(x))^{-1}$, where $(g,\phi)=u+(\widehat g,\widehat\phi)$, and $(x,v,w)\mapsto H(x,v,w)$ is a smooth function which is a second order polynomial in w with coefficients being smooth functions of v and derivatives of $\widehat g$, [97]. Note that when the norm of $\mathcal{F}$ in $C^4_b(M_0)$ is small enough, we have $\mathrm{supp}\,(u) \subset \mathcal{K}$. We note that one could also consider non-compactly supported sources or initial data, see [19]. Also, the scalar field-Einstein system can be considered with much less regularity that is done below, see [14, 15].

Let $s_0 \geq 4$ be an even integer. Below we will consider the solutions $u = (g - \widehat{g}, \phi - \widehat{\phi})$ and the sources $\mathcal{F}$ as sections of the bundle $\mathcal{B}^L$ on M_0. We will consider these functions as elements of the section-valued Sobolev spaces $H^s(M_0; \mathcal{B}^L)$ etc. Below, we omit the bundle $\mathcal{B}^L$ in these notations and denote $H^s(M_0; \mathcal{B}^L) = H^s(M_0)$. We use the same convention for the spaces

$$E^s = \bigcap_{j=0}^{s} C^j([0, t_0]; H^{s-j}(N)), \quad s \in \mathbb{N}.$$

Note that $E^s \subset C^p([0, t_0] \times N)$ when $0 \leq p < s - 2$. Local existence results for (36) follow from the standard techniques for quasi-linear equations developed e.g. in [45] or [58], or [90, Section 9]. These yield that when $\mathcal{F}$ is supported in the compact set $\mathcal{K}$ and $\|\mathcal{F}\|_{E^{s_0}} < c_0$, where $c_0 > 0$ is small enough, there exists a unique function u satisfying equation (36) on M_0 with the source $\mathcal{F}$. Moreover,

$$\|u\|_{E^{s_0}} \leq C_1 \|\mathcal{F}\|_{E^{s_0}}. \tag{37}$$

For a detailed analysis, see Appendix B in [61].

3.1. **Linearized conservation law and harmonicity conditions.**

3.1.1. *Lagrangian distributions.* Let us recall definition of the conormal and Lagrangian distributions that we will use below. Let X be a manifold of dimension n and $\Lambda \subset T^*X \setminus \{0\}$ be a Lagrangian submanifold. Let $\phi(x, \theta)$, $\theta \in \mathbb{R}^N$ be a non-degenerate phase function that locally parametrizes Λ. We say that a distribution $u \in \mathcal{D}'(X)$ is a Lagrangian distribution associated with Λ and denote $u \in \mathcal{I}^m(X; \Lambda)$, if in local coordinates u can be represented as an oscillatory integral,

$$u(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x, \theta)\, d\theta, \tag{38}$$

where $a(x, \theta) \in S^{m+n/4-N/2}(X; \mathbb{R}^N)$, see [38, 49, 78].

In particular, when $S \subset X$ is a submanifold, its conormal bundle $N^*S = \{(x, \xi) \in T^*X \setminus \{0\};\ x \in S,\ \xi \perp T_xS\}$ is a Lagrangian submanifold. If u is a Lagrangian distribution associated with Λ_1 where $\Lambda_1 = N^*S$, we say that u is a conormal distribution.

Let us next consider the case when $X = \mathbb{R}^n$ and let $(x^1, x^2, \ldots, x^n) = (x', x'', x''')$ be the Euclidean coordinates with $x' = (x_1, \ldots, x_{d_1})$, $x'' = (x_{d_1+1}, \ldots, x_{d_1+d_2})$, $x''' = (x_{d_1+d_2+1}, \ldots, x_n)$. If $S_1 = \{x' = 0\} \subset \mathbb{R}^n$, $\Lambda_1 = N^*S_1$ then $u \in \mathcal{I}^m(X; \Lambda_1)$ can be represented by (38) with $N = d_1$ and $\phi(x, \theta) = x' \cdot \theta$.

Next we recall the definition of $\mathcal{I}^{p,l}(X;\Lambda_1,\Lambda_2)$, the space of the distributions u in $\mathcal{D}'(X)$ associated to two cleanly intersecting Lagrangian manifolds $\Lambda_1,\Lambda_2\subset T^*X\setminus\{0\}$, see [21, 38, 78]. These classes have been widely used in the study of inverse problems, see [16, 29]. Let us start with the case when $X=\mathbb{R}^n$.

Let $S_1,S_2\subset\mathbb{R}^n$ be the linear subspaces of codimensions d_1 and d_1+d_2, respectively, $S_2\subset S_1$, given by $S_1=\{x'=0\}$, $S_2=\{x'=x''=0\}$. Let us denote $\Lambda_1=N^*S_1$, $\Lambda_2=N^*S_2$. Then $u\in\mathcal{I}^{p,l}(\mathbb{R}^n;N^*S_1,N^*S_2)$ if and only if

$$u(x)=\int_{\mathbb{R}^{d_1+d_2}} e^{i(x'\cdot\theta'+x''\cdot\theta'')}a(x,\theta',\theta'')\,d\theta' d\theta'',$$

where the symbol $a(x,\theta',\theta'')$ belongs in the product type symbol class $S^{\mu_1,\mu_2}(\mathbb{R}^n;(\mathbb{R}^{d_1}\setminus 0)\times\mathbb{R}^{d_2})$ that is the space of function $a\in C^\infty(\mathbb{R}^n\times\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ that satisfy

$$|\partial_x^\gamma\partial_{\theta'}^\alpha\partial_{\theta''}^\beta a(x,\theta',\theta'')|\leq C_{\alpha\beta\gamma K}(1+|\theta'|+|\theta''|)^{\mu_1-|\alpha|}(1+|\theta''|)^{\mu_2-|\beta|}\tag{39}$$

for all $x\in K$, multi-indexes α,β,γ, and compact sets $K\subset\mathbb{R}^n$. Above, $\mu_1=p+l-d_1/2+n/4$ and $\mu_2=-l-d_2/2$.

When X is a manifold of dimension n and $\Lambda_1,\Lambda_2\subset T^*X\setminus\{0\}$ are two cleanly intersecting Lagrangian manifolds, we define the class $\mathcal{I}^{p,l}(X;\Lambda_1,\Lambda_2)\subset\mathcal{D}'(X)$ to consist of locally finite sums of distributions of the form $u=Au_0$, where $u_0\in\mathcal{I}^{p,l}(\mathbb{R}^n;N^*S_1,N^*S_2)$ and $S_1,S_2\subset\mathbb{R}^n$ are the linear subspace of codimensions d_1 and d_1+d_2, respectively, such that $S_2\subset S_1$, and A is a Fourier integral operator of order zero with a canonical relation Σ for which $\Sigma\circ(N^*S_1)'\subset\Lambda_1'$ and $\Sigma\circ(N^*S_2)'\subset\Lambda_2'$. Here, for $\Lambda\subset T^*X$ we denote $\Lambda'=\{(x,-\xi)\in T^*X;\ (x,\xi)\in\Lambda\}$, and for $\Sigma\subset T^*X\times T^*X$ we denote $\Sigma'=\{(x,\xi,y,-\eta);\ (x,\xi,y,\eta)\in\Sigma\}$.

In most cases, below $X=M$. We denote then $\mathcal{I}^p(M;\Lambda_1)=\mathcal{I}^p(\Lambda_1)$ and $\mathcal{I}^{p,l}(M;\Lambda_1,\Lambda_2)=\mathcal{I}^{p,l}(\Lambda_1,\Lambda_2)$. Also, $\mathcal{I}(\Lambda_1)=\cup_{p\in\mathbb{R}}\mathcal{I}^p(\Lambda_1)$.

By [38, 78], microlocally away from Λ_1 and Λ_0,

$$\mathcal{I}^{p,l}(\Lambda_0,\Lambda_1)\subset\mathcal{I}^{p+l}(\Lambda_0\setminus\Lambda_1)\quad\text{and}\quad\mathcal{I}^{p,l}(\Lambda_0,\Lambda_1)\subset\mathcal{I}^{p}(\Lambda_1\setminus\Lambda_0),\tag{40}$$

respectively. Thus the principal symbol of $u\in\mathcal{I}^{p,l}(\Lambda_0,\Lambda_1)$ is well defined on $\Lambda_0\setminus\Lambda_1$ and $\Lambda_1\setminus\Lambda_0$. We denote $\mathcal{I}(\Lambda_0,\Lambda_1)=\cup_{p,q\in\mathbb{R}}\mathcal{I}^{p,q}(\Lambda_0,\Lambda_1)$.

Below, when $\Lambda_j=N^*S_j$, $j=1,2$ are conormal bundles of smooth cleanly intersecting submanifolds $S_j\subset M$ of codimension m_j, where $\dim(M)=n$, we use the traditional notations,

$$\mathcal{I}^\mu(S_1)=\mathcal{I}^{\mu+m_1/2-n/4}(N^*S_1),\quad\mathcal{I}^{\mu_1,\mu_2}(S_1,S_2)=\mathcal{I}^{p,l}(N^*S_1,N^*S_2),\tag{41}$$

where $p=\mu_1+\mu_2+m_1/2-n/4$ and $l=-\mu_2-m_2/2$, and call such distributions conormal distributions associated to S_1 or product type conormal distributions associated to S_1 and S_2, respectively. By [38], $\mathcal{I}^\mu(X;S_1)\subset L^p_{loc}(X)$ for $\mu<-m_1(p-1)/p$, $1\leq p<\infty$.

For the wave operator $\square_g$ on the globally hyperbolic manifold (M,g) $\mathrm{Char}\,(\square_g)$ is the set of light-like vectors with respect to g, and $(y,\eta)\in$

$\Theta_{x,\xi}$ if and only if there is $t \in \mathbb{R}$ such that $(y,\eta) = (\gamma^g_{x,b}(t), \dot\gamma^g_{x,b}(t))$ where $\gamma^g_{x,b}$ is a light-like geodesic with respect to the metric g with the initial data $(x,b) \in TM$, and $a = \eta^\flat$, $b = \xi^\flat$.

Let $P = \square_g + B^0 + B^j\partial_j$, where B^0 and B_j are tensors. Then P is a classical pseudodifferential operator of real principal type and order $m = 2$ on M, and [78], see also [64], P has a parametrix $Q \in \mathcal{I}^{p,l}(\Delta'_{T^*M}, \Lambda_P)$, $p = \frac{1}{2} - m$, $l = -\frac{1}{2}$, where $\Delta_{T^*M} = N^*(\{(x,x);\ x \in M\})$ and $\Lambda_g \subset T^*M \times T^*M$ is the Lagrangian manifold associated to the canonical relation of the operator P, that is,

$$\Lambda_g = \{(x,\xi,y,-\eta);\ (x,\xi) \in \mathrm{Char}\,(P),\ (y,\eta) \in \Theta_{x,\xi}\}, \tag{42}$$

where $\Theta_{x,\xi} \subset T^*M$ is the bicharacteristic of P containing (x,ξ). When (M,g) is a globally hyperbolic manifold, the operator P has a causal inverse operator, see e.g. [5, Thm. 3.2.11]. We denote it by P^{-1} and by [78], we have $P^{-1} \in \mathcal{I}^{-3/2,-1/2}(\Delta'_{T^*M}, \Lambda_g)$. We will repeatedly use the fact (see [38, Prop. 2.1]) that if $F \in \mathcal{I}^p(\Lambda_0)$ and Λ_0 intersects $\mathrm{Char}(P)$ transversally so that all bicharacterestics of P intersect Λ_0 only finitely many times, then $(\square_g + B^0 + B^j\partial_j)^{-1}F \in \mathcal{I}^{p-3/2,-1/2}(\Lambda_0,\Lambda_1)$ where $\Lambda_1' = \Lambda_g \circ \Lambda_0'$ is called the flowout from Λ_0 on $\mathrm{Char}(P)$, that is,

$$\Lambda_1 = \{(x,-\xi);\ (x,\xi,y,-\eta) \in \Lambda_g,\ (y,\eta) \in \Lambda_0\}.$$

3.1.2. *The linearized Einstein equations and the linearized conservation law.* We will below consider sources $\mathcal{F} = \varepsilon\mathbf{f}(x)$ and solution u_ε satisfying (36), where $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)})$.

We consider the linearized Einstein equations and the linearized wave $u^{(1)} = \partial_\varepsilon u_\varepsilon|_{\varepsilon=0}$. It satisfies the linearized Einstein equations (9) that we write as

$$\square_{\widehat g} u^{(1)} + V(x,\partial_x)u^{(1)} = \mathbf{f}, \tag{43}$$

where $v \mapsto V(x,\partial_x)v$ is a linear first order partial differential operator with coefficients depending on the derivatives of $\widehat g$.

Assume that $Y \subset M_0$ is a 2-dimensional space-like submanifold and consider local coordinates defined in $V \subset M_0$. Moreover, assume that in these local coordinates $Y \cap V \subset \{x \in \mathbb{R}^4;\ x^jb_j = 0,\ x^jb'_j = 0\}$, where $b'_j \in \mathbb{R}$ and let $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)}) \in \mathcal{I}^n(Y)$, $n \le n_0 = -17$, be defined by

$$\mathbf{f}(x^1,x^2,x^3,x^4) = \mathrm{Re}\int_{\mathbb{R}^2} e^{i(\theta_1 b_m + \theta_2 b'_m)x^m}\sigma_{\mathbf{f}}(x,\theta_1,\theta_2)\,d\theta_1 d\theta_2. \tag{44}$$

Here, we assume that $\sigma_{\mathbf{f}}(x,\theta)$, $\theta = (\theta_1,\theta_2)$ is a $\mathcal{B}^L$-valued classical symbol and we denote the principal symbol of $\mathbf{f}$ by $c(x,\theta) = \sigma_p(\mathbf{f})(x,\theta)$, or component-wise, $((c^{(1)}_{jk}(x,\theta))^4_{j,k=1}, (c^{(2)}_\ell(x,\theta))^L_{\ell=1})$. When $x \in Y$ and $\xi = (\theta_1 b_m + \theta_2 b'_m)dx^m$ so that $(x,\xi) \in N^*Y$, we denote the value of the principal symbol $\mathbf{f}$ at (x,ξ) by $\widetilde c(x,\xi) = c(x,\theta)$, that is component-wise, $\widetilde c^{(1)}_{jk}(x,\xi) = c^{(1)}_{jk}(x,\theta)$ and $\widetilde c^{(2)}_\ell(x,\xi) = c^{(2)}_\ell(x,\theta)$. We say that this is the principal symbol of $\mathbf{f}$ at (x,ξ), associated to the phase function

$\phi(x,\theta_1,\theta_2) = (\theta_1 b_m + \theta_2 b'_m)x^m$. The above defined principal symbols can be defined invariantly, see [42].

We will below consider what happens when $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)}) \in \mathcal{I}^n(Y)$ satisfies the *linearized conservation law* (10). Roughly speaking, these four linear conditions imply that the principal symbol of the source $\mathbf{f}$ satisfies four linear conditions. Furthermore, the linearized conservation implies that also the linearized wave $u^{(1)}$ produced by $\mathbf{f}$ satisfies four linear conditions that we call the linearized harmonicity conditions, and finally, the principal symbol of the wave $u^{(1)}$ has to satisfy four linear conditions. Next we explain these conditions in detail.

When (10) is valid, we have

$$\widehat{g}^{lk}\xi_l \widetilde{c}^{(1)}_{kj}(x,\xi) = 0, \quad \text{for } j \leq 4 \text{ and } \xi \in N^*_x Y. \tag{45}$$

We say that this is the *linearized conservation law for the principal symbols.* Note that $\mathcal{I}^\mu(Y) \subset C^s(M_0)$ when $s \leq -\mu - 3$. We will later use such indexes μ so that we can use $s = 13$.

3.1.3. *A harmonicity condition for the linearized solutions.* Assume that (g,ϕ) satisfy equations (7) and the conservation law (8) is valid. The conservation law (8) and the $\widehat{g}$-reduced Einstein equations (7) imply, see e.g. [12, 90], that the harmonicity functions $\Gamma^j = g^{nm}\Gamma^j_{nm}$ satisfy

$$g^{nm}\Gamma^j_{nm} = g^{nm}\widehat{\Gamma}^j_{nm}. \tag{46}$$

Next we denote $u^{(1)} = (g^1,\phi^1) = (\dot{g},\dot{\phi})$, and discuss the implications of this for the metric component $\dot{g}$ of the solution of the linearized Einstein equations.

We do next calculations in local coordinates of M_0 and denote $\partial_k = \frac{\partial}{\partial x^k}$. Direct calculations show that $h^{jk} = g^{jk}\sqrt{-\det(g)}$ satisfies $\partial_k h^{kq} = -\Gamma^q_{kn}h^{nk}$. Then (46) implies that

$$\partial_k h^{kq} = -\widehat{\Gamma}^q_{kn}h^{nk}. \tag{47}$$

We call (47) the *harmonicity condition* for the metric g.

Assume now that g_ε and ϕ_ε satisfy (7) with source $\mathcal{F} = \varepsilon f$ where $\varepsilon > 0$ is a small parameter. We define $h^{jk}_\varepsilon = g^{jk}_\varepsilon\sqrt{-\det(g_\varepsilon)}$ and denote $\dot{g}_{jk} = \partial_\varepsilon (g_\varepsilon)_{jk}|_{\varepsilon=0}$, $\dot{g}^{jk} = \partial_\varepsilon (g_\varepsilon)^{jk}|_{\varepsilon=0}$, and $\dot{h}^{jk} = \partial_\varepsilon h^{jk}_\varepsilon|_{\varepsilon=0}$.

The equation (47) yields then[2]

$$\partial_k \dot{h}^{kq} = -\widehat{\Gamma}^q_{kn}\dot{h}^{nk}. \tag{48}$$

A direct computation shows that

$$\dot{h}^{ab} = (-\det(\widehat{g}))^{1/2}\kappa^{ab},$$

[2]The treatment on this de Donder-type gauge condition is known in the folklore of the field. For a similar gauge condition to (48) in harmonic coordinates, see [72, pages 6 and 250], or [67, formulas 107.5, 108.7, 108.8], or [46, p. 229-230].

where $\kappa^{ab}=\dot g^{ab}-\frac 12\widehat g^{ab}\widehat g_{qp}\dot g^{pq}$. Thus (48) gives

$$\partial_a((-\det(\widehat g))^{1/2}\kappa^{ab})=-\widehat\Gamma^b_{ac}(-\det(\widehat g))^{1/2}\kappa^{ac} \tag{49}$$

that implies $\partial_a\kappa^{ab}+\kappa^{nb}\widehat\Gamma^a_{an}+\kappa^{an}\widehat\Gamma^b_{an}=0$, or equivalently,

$$\widehat\nabla_a\kappa^{ab}=0. \tag{50}$$

We call (50) the *linearized harmonicity condition* for g. Writing this for $\dot g$, we obtain

$$-\widehat g^{an}\partial_a\dot g_{nj}+\frac12\widehat g^{pq}\partial_j\dot g_{pq}=m_j^{pq}\dot g_{pq} \tag{51}$$

where m_j depend on $\widehat g_{pq}$ and its derivatives. On similar conditions for the polarization tensor, see [85, form. (9.58) and example 9.5.a, p. 416].

3.1.4. *Properties of the principal symbols of the waves.* Let $K\subset M_0$ be a light-like submanifold of dimension 3 that in local coordinates $X:V\to\mathbb R^4$, $x^k=X^k(y)$ is given by $K\cap V\subset\{x\in\mathbb R^4;\ b_kx^k=0\}$, where $b_k\in\mathbb R$ are constants. Assume that the solution $u^{(1)}=(\dot g,\dot\phi)$ of the linear wave equation (43) with the right hand side vanishing in V is such that $u^{(1)}\in\mathcal I^\mu(K)$ with $\mu\in\mathbb R$. Below we use $\mu=n-\frac32$ where $n\in\mathbb Z_-$, $n\le n_0=-18$. Let us write $\dot g_{jk}$ as an oscillatory integral using a phase function $\varphi(x,\theta)=b_kx^k\theta$, and a symbol $a_{jk}(x,\theta)\in S^n_{clas}(\mathbb R^4,\mathbb R)$,

$$\dot g_{pq}(x^1,x^2,x^3,x^4)=\mathrm{Re}\int_{\mathbb R}e^{i(\theta b_mx^m)}a_{pq}(x,\theta)\,d\theta, \tag{52}$$

where $n=\mu+\frac12$. We denote the (positively homogeneous) principal symbol of $a_{pq}(x,\theta)$ by $\sigma_p(\dot g_{pq})(x,\theta)$. When $x\in K$ and $\xi=\theta b_kdx^k$ so that $(x,\xi)\in N^*K$, we denote the value of $\sigma_p(\dot g_{pq})$ at (x,θ) by $\widetilde a_{jk}(x,\xi)$, that is, $\widetilde a_{jk}(x,\xi)=\sigma_p(\dot g_{pq})(x,\theta)$.

Then, if $\dot g_{jk}$ satisfies the linearized harmonicity condition (46), its principal symbol $\widetilde a_{jk}(x,\xi)$ satisfies

$$-\widehat g^{mn}(x)\xi_mv_{nj}+\frac12\xi_j(\widehat g^{pq}(x)v_{pq})=0,\quad v_{pq}=\widetilde a_{pq}(x,\xi), \tag{53}$$

where $j=1,2,3,4$ and $\xi=\theta b_kdx^k\in N^*_xK$. If (53) holds, we say that the *harmonicity condition for the symbol* is satisfied for $\widetilde a(x,\xi)$ at $(x,\xi)\in N^*K$.

4. A model with adaptive source function

4.1. **Initial value problem with adaptive source functions.** Let us define some physical fields and introduce a model as a system of partial differential equations. Later we will motivate this system by discussion of the corresponding Lagrangians, but we postpone this discussion to a last section as it is not completely rigorous.

We assume that there are C^∞-background fields $\widehat g$, $\widehat\phi$, on M.

We consider a Lorentzian metric g on M_0 and $\phi = (\phi_\ell)_{\ell=1}^L$ where ϕ_ℓ are scalar fields on $M_0 = (-\infty, t_0) \times N$.

Let $P = P_{jk}(x)dx^j dx^k$ be a symmetric tensor on M_0, corresponding below to a direct perturbation to the stress energy tensor, and $Q = (Q_\ell(x))_{\ell=1}^K$ where $Q_\ell(x)$ are real-valued functions on M_0, where $K \geq L+1$. We denote by $\mathcal{V}(\phi_\ell; S_\ell)$ the potential functions of the fields ϕ_ℓ,

$$\mathcal{V}(\phi_\ell; S_\ell) = \frac{1}{2}m^2\Big(\phi_\ell + \frac{1}{m^2}S_\ell\Big)^2. \tag{54}$$

These potentials depend on the source variables S_ℓ. The way how S_ℓ, called below the adaptive source functions, depend on other fields is explained later. We assume that there are smooth background fields $\widehat{P}$ and $\widehat{Q}$. For a while we consider the case when $\widehat{P} = 0$ and $\widehat{Q} = 0$, and discuss later the generalization to non-vanishing background fields.

Using the ϕ and P fields, we define the stress-energy tensor

$$T_{jk} = \sum_{\ell=1}^L (\partial_j\phi_\ell\,\partial_k\phi_\ell - \frac{1}{2}g_{jk}g^{pq}\partial_p\phi_\ell\,\partial_q\phi_\ell - \mathcal{V}(\phi_\ell; S_\ell)g_{jk}) + P_{jk}. \tag{55}$$

We assume that P and Q are supported on $\mathcal{K} = J^+_{\widetilde{g}}(\widehat{p}^-) \cap M_0$. Let us represent the stress energy tensor (55) in the form

$$\begin{aligned}
T_{jk} &= P_{jk} + Zg_{jk} + \mathbf{T}_{jk}(g,\phi), \quad Z = -\sum_{\ell=1}^L (S_\ell\phi_\ell + \frac{1}{2m^2}S_\ell^2),\\
\mathbf{T}_{jk}(g,\phi) &= \sum_{\ell=1}^L (\partial_j\phi_\ell\,\partial_k\phi_\ell - \frac{1}{2}g_{jk}g^{pq}\partial_p\phi_\ell\,\partial_q\phi_\ell - \frac{1}{2}m^2\phi_\ell^2 g_{jk}),
\end{aligned}$$

where we call Z the stress energy density caused by the sources S_ℓ.

Now we are ready to formulate the direct problem for the adaptive Einstein-scalar field equations. Let g and ϕ satisfy

$$\begin{aligned}
&\mathrm{Ein}_{\widehat{g}}(g) = P_{jk} + Zg_{jk} + \mathbf{T}_{jk}(g,\phi), \quad Z = -\sum_{\ell=1}^L (S_\ell\phi_\ell + \frac{1}{2m^2}S_\ell^2),\\
&\square_g\phi_\ell - \mathcal{V}'(\phi_\ell; S_\ell) = 0 \quad \text{in } M_0, \quad \ell = 1,2,3,\dots,L,\\
&S_\ell = \mathcal{S}_\ell(g,\phi,\nabla\phi,Q,\nabla Q,P,\nabla^g P), \quad \text{in } M_0,\\
&g = \widehat{g}, \quad \phi_\ell = \widehat{\phi}_\ell, \quad \text{in } (-\infty,0)\times N.
\end{aligned} \tag{56}$$

Above, $\mathcal{V}'(\phi; s) = \partial_\phi\mathcal{V}(\phi; s)$ so that $\mathcal{V}'(\phi_\ell; S_\ell) = m^2\phi_\ell + S_\ell$. We assume that the background fields $\widehat{g}$, $\widehat{\phi}$, satisfy these equations with $\widehat{Q} = 0$ and $\widehat{P} = 0$.

We consider here $P = (P_{jk})_{j,k=1}^4$ and $Q = (Q_\ell)_{\ell=1}^K$ as fields that we can control and call those the controlled source fields. Local existence of the solution for small sources P and Q is considered in Appendix B.

To obtain a physically meaningful model, we need to consider how the adaptive source functions $\mathcal{S}_\ell$ should be chosen so that the physical conservation law in relativity

$$\nabla_k(g^{kp}T_{pq}) = 0 \tag{57}$$

is satisfied. Here $\nabla = \nabla^g$ is the connection corresponding to the metric g.

We note that the conservation law is a necessary condition for the equation (56) to have solutions for which $\mathrm{Ein}_{\widehat{g}}(g) = \mathrm{Ein}(g)$, i.e., that the solutions of (56) are solutions of the Einstein field equations.

The functions $\mathcal{S}_\ell(g, \phi, \nabla\phi, Q, \nabla Q, P, \nabla^g P)$ model the devices that we use to perform active measurements. Thus, even though the Condition S below may appear quite technical, this assumption can be viewed as the instructions on how to build a device that can be used to measure the structure of the spacetime far away. Outside the support of the measurement device (that contain the union of the supports of Q and P) we have just assumed that the standard coupled Einstein-scalar field equations hold, c.f. (58). We can consider them in the form

$$\mathcal{S}_\ell(g, \phi, \nabla\phi, Q, \nabla Q, P, \nabla^g P) = Q_\ell + \mathcal{S}_\ell^{2nd}(g, \phi, \nabla\phi, Q, \nabla Q, P, \nabla^g P)$$

for $\ell = 1, 2, \dots, L$ where Q_ℓ are the primary sources and $\mathcal{S}_\ell^{2nd}$, that depend also on Q_ℓ with $\ell = L+1, L+2, \dots, K$, corresponds to the response of the measurement device that forces the conservation law to be valid.

The solution (g, ϕ) of (56) is a solution of the equations (7) when we denote

$$\mathcal{F}^1_{jk} = P_{jk} + Zg_{jk},$$
$$\mathcal{F}^2_\ell = \mathcal{V}'(\phi_\ell; S_\ell) - \mathcal{V}'(\phi_\ell; 0) = S_\ell.$$

Our next goal is to construct suitable adaptive source functions $\mathcal{S}_\ell$ and consider what kind of sources $\mathcal{F}^1$ and $\mathcal{F}^2$ of the above form can be obtained by varying P and Q.

We will consider adaptive source functions $\mathcal{S}_\ell$ satisfying the following conditions:

Condition S:
The adaptive source functions $\mathcal{S}_\ell(g, \phi, \nabla\phi, Q, \nabla Q, P, \nabla^g P)$ have the following properties:

(i) Denoting $c = \nabla\phi$, $C = \nabla^g P$, and $H = \nabla Q$ we assume that $\mathcal{S}_\ell(g, \phi, c, Q, H, P, C)$ are smooth non-linear functions, of the pointwise values $g_{jk}(x), \phi(x), \nabla\phi(x)$, $Q(x), \nabla Q(x), P(x)$, and $\nabla^g P(x)$, defined near $(g, \phi, c, Q, H, P, C) = (\widehat{g}, \widehat{\phi}, \nabla\widehat{\phi}, 0, 0, 0, 0)$, that satisfy

$$\mathcal{S}_\ell(g, \phi, c, 0, 0, 0, 0) = 0. \tag{58}$$

(ii) We assume that $\mathcal{S}_\ell$ is independent of $P(x)$ and the dependency of $\mathcal{S}$ on $\nabla^g P$ and ∇Q is only due to the dependency in the term

$g^{pk}\nabla^g_p(P_{jk}+Zg_{jk}) = g^{pk}\nabla^g_p P_{jk} + \nabla^g_j Q_K$, associated to the divergence of the perturbation of T, that is, there exist functions $\widetilde{\mathcal{S}}_\ell$ so that

$$\mathcal{S}_\ell(g,\phi,c,Q,H,P,C) = \widetilde{\mathcal{S}}_\ell(g,\phi,c,Q,R), \quad R = (g^{pk}\nabla^g_p(P_{jk}+Q_K g_{jk}))_{j=1}^4.$$

Below, denote $\widehat{R} = \widehat{g}^{pk}\widehat{\nabla}_p\widehat{P}_{jk} + \widehat{\nabla}_j\widehat{Q}_K$. Note that we still are considering the case when $\widehat{Q}=0$ and $\widehat{P}=0$ so that $\widehat{R}=0$, too. This implies that for the background fields that adaptive source functions $\mathcal{S}_\ell$ vanish.

To simplify notations, we also denote below $\widetilde{\mathcal{S}}_\ell$ just by $\mathcal{S}_\ell$ and indicate the function which we use by the used variables in these functions.

Below we will denote $Q=(Q',Q_K)$, $Q'=(Q_\ell)_{\ell=1}^{K-1}$. There are examples when the background fields $(\widehat{g},\widehat{\phi})$ and the adaptive source functions $\mathcal{S}_\ell$ exists and satisfy the Condition S.

Our next aim is to prove the following:

Theorem 4.1. *Let $L\geq 5$ and assume that $\widehat{Q}=0$ and $\widehat{P}=0$ so that $\widehat{R}=0$. Moreover, assume that Condition A is valid. Then for all permutations $\sigma:\{1,2,\dots,L\}\to\{1,2,\dots,L\}$ there exists functions $\mathcal{S}_{\ell,\sigma}$ satisfying Condition S such that*

(i) For all $x\in U_{\widehat{g},\sigma}$ the differential of

$$\mathcal{S}_\sigma(\widehat{g},\widehat{\phi},\nabla\widehat{\phi},Q,R) = (\mathcal{S}_{\ell,\sigma}(\widehat{g},\widehat{\phi},\nabla\widehat{\phi},Q,R))_{\ell=1}^L$$

with respect to Q and R, that is, the map

$$D_{Q,R}\mathcal{S}_\sigma(\widehat{g}(x),\widehat{\phi}(x),\nabla\widehat{\phi}(x),Q,R)|_{Q=\widehat{Q}(x),R=\widehat{R}(x)} : \mathbb{R}^{K+4}\to\mathbb{R}^L \tag{59}$$

is surjective.

(ii) The adaptive source functions $\mathcal{S}_\sigma$ are such that for $(Q_\ell)_{\ell=1}^K$ and (P_{jk}) that are sufficiently close to $\widehat{Q}=0$ and $\widehat{P}=0$ in the $C^3_b(M_0)$-topology and supported in $U_{\widehat{g},\sigma}$ the equations (56) with source functions $\mathcal{S}_\sigma$ have a unique solution (g,ϕ) and the conservation law (57) is valid.

(iii) Under the same assumptions as in (ii), when (g,ϕ) is a solution of (56) with the controlled source functions P and Q, we have $Q_K=Z$. This means that the physical field Z can be directly controlled.

Proof. As one can enumerate the ℓ-indexes of the fields ϕ_ℓ as one wishes, it is enough to prove the claim with one σ. We consider below the case when $\sigma = Id$.

Consider a symmetric (0,2)-tensor P and a scalar functions Q_ℓ that are C^3-smooth and compactly supported in $U_{\widehat{g},\sigma}$. Let $[P_{jk}(x)]_{j,k=1}^4$ be the coefficients of P in local coordinates and $Q(x)=(Q_\ell(x))_{\ell=1}^L$.

To obtain adaptive required adaptive source functions, let us start to consider implications of the conservation law (57). To this end, consider C^2-smooth functions $S_\ell(x)$ on $U_{\widehat{g},\sigma}$.

Note that since $[\nabla_p,\nabla_n]=[\partial_p,\partial_n]=0$ (see [12, Sect. III.6.4.1]),

$$\begin{aligned}
&\nabla_p(g^{pj}\mathbf{T}_{jk}(g,\phi)=\sum_{\ell=1}^L \nabla_p g^{pj}(\partial_j\phi_\ell\,\partial_k\phi_\ell-\frac12 g_{jk}g^{nm}\partial_n\phi_\ell\,\partial_m\phi_\ell-\frac12 m^2\phi_\ell^2 g_{jk})\\
=\ &\sum_{\ell=1}^L(g^{pj}\nabla_p\partial_j\phi_\ell)\,\partial_k\phi_\ell+\sum_{\ell=1}^L(g^{pj}\partial_j\phi_\ell\,\nabla_p\partial_k\phi_\ell)\\
&-\frac12\sum_{\ell=1}^L\delta_k^p(g^{nm}(\nabla_p\partial_n\phi_\ell)\,\partial_m\phi_\ell+g^{nm}\partial_n\phi_\ell\,(\nabla_p\partial_m\phi_\ell))-\sum_{\ell=1}^L m^2\delta_k^p\phi_\ell\partial_p\phi_\ell\\
=\ &\sum_{\ell=1}^L(g^{pj}\nabla_p\partial_j\phi_\ell)\,\partial_k\phi_\ell+\sum_{\ell=1}^L(g^{pj}\partial_j\phi_\ell\,(\nabla_p\partial_k\phi_\ell))\\
&-\frac12\sum_{\ell=1}^L(g^{nm}\partial_m\phi_\ell\,(\nabla_n\partial_k\phi_\ell)+g^{nm}\partial_n\phi_\ell\,(\nabla_m\partial_k\phi_\ell))-\sum_{\ell=1}^L m^2\phi_\ell\partial_k\phi_\ell\\
=\ &\sum_{\ell=1}^L(g^{pj}\nabla_p\partial_j\phi_\ell-m^2\phi_\ell)\,\partial_k\phi_\ell.
\end{aligned}$$

Thus conservation law (57) gives for all $j=1,2,3,4$ equations

$$\begin{aligned}
0\ &=\ \nabla_p^g(g^{pk}T_{jk})\\
&=\ \nabla_p^g(g^{pk}(\mathbf{T}_{jk}(g,\phi)+P_{jk}+Zg_{jk}))\\
&=\ \sum_{\ell=1}^L(g^{pk}\nabla_p^g\partial_k\phi_\ell)\,\partial_j\phi_\ell-(m_\ell^2\phi_\ell\partial_p\phi_\ell)\delta_j^p-\nabla_p^g(g^{pk}g_{jk}(S_\ell\phi_\ell+\frac{1}{2m^2}S_\ell^2)+g^{pk}P_{jk})\\
&=\ \sum_{\ell=1}^L(g^{pk}\nabla_p^g\partial_k\phi_\ell-m_\ell^2\phi_\ell)\,\partial_j\phi_\ell-\nabla_p^g(g^{pk}g_{jk}(S_\ell\phi_\ell+\frac{1}{2m^2}S_\ell^2)+g^{pk}P_{jk})\\
&=\ \sum_{\ell=1}^L S_\ell\,\partial_j\phi_\ell-\nabla_p^g(g^{pk}g_{jk}(S_\ell\phi_\ell+\frac{1}{2m^2}S_\ell^2))+g^{pk}\nabla_p^g P_{jk}\\
&=\ \left(\sum_{\ell=1}^L S_\ell\,\partial_j\phi_\ell\right)-\partial_j\left(\sum_{\ell=1}^L S_\ell\phi_\ell+\frac{1}{2m^2}S_\ell^2\right)+g^{pk}\nabla_p^g P_{jk}.
\end{aligned}$$

Summarizing, the conservation law gives

$$\left(\sum_{\ell=1}^L S_\ell\,\partial_j\phi_\ell\right)-\partial_j\left(\sum_{\ell=1}^L S_\ell\phi_\ell+\frac{1}{2m^2}S_\ell^2\right)+g^{pk}\nabla_p^g P_{jk}=0, \tag{60}$$

for $j=1,2,3,4$.

Recall that the field Z has the definition

$$\sum_{\ell=1}^L S_\ell\phi_\ell+\frac{1}{2m^2}S_\ell^2=-Z. \tag{61}$$

Then, the conservation law (57) holds if we have

$$\sum_{\ell=1}^{L} S_\ell\, \partial_j \phi_\ell = -g^{pk}\nabla^g_p V_{jk}, \quad V_{jk} = (P_{jk} + g_{jk} Z), \tag{62}$$

for $j = 1, 2, 3, 4$.

Equations (61) and (62) give together five point-wise equations for the functions $S_1, \dots, S_L$.

Recall that we consider here the case when $\sigma = Id$. By Condition A, at any $x \in U_{\widehat{g},\sigma}$ that the 5×5 matrix $(B^\sigma_{jk}(\widehat{\phi}(x), \nabla\widehat{\phi}(x)))_{j,k\le 5}$ is invertible, where

$$(B^\sigma_{jk}(\phi(x), \nabla\phi(x)))_{j,k\le 5} = \left(\begin{array}{c} (\partial_j \phi_\ell(x))_{j\le 4,\ \ell\le 5} \\ (\phi_\ell(x))_{\ell\le 5} \end{array} \right).$$

We consider a $\mathbb{R}^K$ valued function $Q(x) = (Q'(x), Q_K(x))$, where

$$Q' = (Q_\ell)_{\ell=1}^{K-1}.$$

Also, below $R_j = g^{pk}\nabla^g_p V_{jk}$, $V_{jk} = P_{jk} + g_{jk} Z$ and we require that the identity

$$Q_K = Z \tag{63}$$

holds.

Motivated by equations (61), (62), and (63), our next aim is to consider a point $x \in U_{\widehat{g},\sigma}$, and construct scalar functions $\mathcal{S}_{\sigma,\ell}(\phi, \nabla\phi, Q', Q_K, R, g)$, $\ell = 1, 2, \dots, L$ that satisfy

$$\begin{aligned} &\sum_{\ell=1}^{5} \mathcal{S}_{\sigma,\ell}(\phi, \nabla\phi, Q', Q_K, R, g)\, \partial_j \phi_\ell = -R_j - \sum_{\ell=6}^{L} Q_{\sigma,\ell}\, \partial_j \phi_\ell, \\ &\sum_{\ell=1}^{5} \mathcal{S}_{\sigma,\ell}(\phi, \nabla\phi, Q', Q_K, R, g)\, \phi_\ell = -\Big(Q_K + \sum_{\ell=6}^{L} Q_{\sigma,\ell}\phi_\ell + \\ &\qquad + \sum_{\ell=1}^{L} \frac{1}{2m^2} \mathcal{S}_{\sigma,\ell}(\phi, \nabla\phi, Q', Q_K, R, g)^2 \Big). \end{aligned} \tag{64}$$

Let

$$\begin{aligned} &(Y_\sigma(\phi, \nabla\phi))(x) = \psi(x)(B^\sigma(\phi, \nabla\phi))^{-1}, \quad \text{for } x \in U_{\widehat{g},\sigma}, \\ &(Y_\sigma(\phi, \nabla\phi))(x) = 0, \quad \text{for } x \notin U_{\widehat{g},\sigma}, \end{aligned}$$

where $\psi \in C^\infty_0(U_{\widehat{g},\sigma})$ has value 1 in $\mathrm{supp}\,(Q) \cup \mathrm{supp}\,(P)$.

Then we define $\mathcal{S}_{\sigma,\ell} = \mathcal{S}_{\sigma,\ell}(g, \phi, \nabla\phi, Q', Q_K, R)$, $\ell = 1, 2, \dots, L$, to be the solution of the system

$$\begin{aligned} &(S_{\sigma,\ell})_{\ell\le 5} = Y_\sigma(\phi, \nabla\phi) \left(\begin{array}{c} (-R_j - \sum_{\ell=6}^{L} Q_{\sigma,\ell}\, \partial_j \phi_\ell)_{j\le 4} \\ -Q_K - \sum_{\ell=6}^{L} Q_{\sigma,\ell}\phi_\ell - \sum_{\ell=1}^{L} \frac{1}{2m^2} S^2_{\sigma,\ell} \end{array} \right) \\ &(S_{\sigma,\ell})_{\ell\ge 6} = (Q_\ell)_{\ell\ge 6}. \end{aligned} \tag{65}$$

When Q and R are sufficiently small, this equation can be solved pointwisely, at each point $x \in U_{\widehat{g},\sigma}$, using iteration by the Banach fixed point theorem.

Let

$$(K^\sigma_{jk}(\phi(x),\nabla\phi(x)))_{j,k\le 5} = \begin{pmatrix} (\,\partial_j\phi_\ell(x))_{j\le 4,\ 6\le\ell\le L} \\ (\phi_\ell(x))_{6\le\ell\le L} \end{pmatrix}.$$

Then we see that the differential of $\mathcal{S}_\sigma = (\mathcal{S}_{\sigma,\ell})_{\ell=1}^L$ with respect to (Q',Q_K,R) at $(Q,R)=(0,0)$, that is,

$$\begin{aligned} &(66)\\ &D_{Q',Q_K,R}\mathcal{S}_\sigma(\widehat{g},\widehat{\phi},\nabla\widehat{\phi},Q',Q_K,R)|_{Q=0,R=0} : \mathbb{R}^{K+4}\to\mathbb{R}^L,\\ &(Q',Q_K,R)\mapsto -\begin{pmatrix} Y_\sigma(\widehat{\phi},\nabla\widehat{\phi}) & Y_\sigma(\widehat{\phi},\nabla\widehat{\phi})K(\widehat{\phi},\nabla\widehat{\phi}) \\ 0 & I_\sigma \end{pmatrix}\left(\begin{pmatrix} R \\ Q_K \end{pmatrix} \atop Q'\right), \end{aligned}$$

is surjective, where $I_\sigma = [\delta_{k,j+5}]_{k\le K-1,\ j\le L-5,} \in \mathbb{R}^{(K-1)\times(L-5)}$. Hence (i) is valid.

By their construction, the functions $\mathcal{S}_\sigma = (\mathcal{S}_{\sigma,\ell})_{\ell=1}^L$ satisfy the equations (61) and (62) for all $x\in U_{\widehat{g},\sigma}$ and also equation (63) holds.

Hence (iii) is valid.

Above, the equation (62) is valid by the construction of functions $(\mathcal{S}_\ell)_{\ell=1}^L$. Thus the conservation law is valid. This proves (ii). □

Note that as the adaptive source functions $\mathcal{S}_\ell$ were constructed in Theorem 4.1 using inverse function theorem, the results of Theorem 4.1 are valid also if $\widehat{Q}$ and $\widehat{P}$ are sufficiently small non-vanishing fields and $\widehat{g}$ and $\widehat{\phi}$ satisfy the Einstein scalar field equations (56) with these background fields. Next we return to the case when $\widehat{P}=0$ and $\widehat{Q}=0$.

5. Proof of the microlocal linearization stabililty

Below we consider the case when $\widehat{P}=0$ and $\widehat{Q}=0$ and use the adaptive source functions $\mathcal{S}_\ell$ constructed in Theorem 4.1 and its proof.

Assume that $Y\subset M_0$ is a 2-dimensional space-like submanifold and consider local coordinates defined in $V\subset M_0$. Moreover, assume that in these local coordinates $Y\cap V\subset\{x\in\mathbb{R}^4;\ x^jb_j=0,\ x^jb'_j=0\}$, where $b'_j\in\mathbb{R}$ and let $\mathbf{p}\in\mathcal{I}^n(Y)$, $n\le n_0=-17$, be defined by

$$\mathbf{p}_{jk}(x^1,x^2,x^3,x^4) = \mathrm{Re}\int_{\mathbb{R}^2} e^{i(\theta_1 b_m+\theta_2 b'_m)x^m} v_{jk}(x,\theta_1,\theta_2)\,d\theta_1 d\theta_2. \tag{67}$$

Here, we assume that $v_{jk}(x,\theta)$, $\theta=(\theta_1,\theta_2)$ are classical symbols and we denote their principal symbols by $\sigma_p(\mathbf{p}_{jk})(x,\theta)$. When $x\in Y$ and $\xi=(\theta_1 b_m+\theta_2 b'_m)dx^m$ so that $(x,\xi)\in N^*Y$, we denote the value of the principal symbol $\sigma_p(\mathbf{p})$ at (x,θ_1,θ_2) by $\widetilde{v}^{(a)}_{jk}(x,\xi)$, that is, $\widetilde{v}^{(a)}_{jk}(x,\xi) = \sigma_p(\mathbf{p}_{jk})(x,\theta_1,\theta_2)$, and say that it is the principal symbol of $\mathbf{p}_{jk}$ at (x,ξ), associated to the phase function $\phi(x,\theta_1,\theta_2) = (\theta_1 b_m + \theta_2 b'_m)x^m$. The above defined principal symbols can be defined invariantly, see [42].

We assume that also $\mathbf{q}', \mathbf{z} \in \mathcal{I}^n(Y)$ have representations (67) with classical symbols. Below we consider symbols in local coordinates. Let us denote the principal symbols of $\mathbf{p}, \mathbf{q}', \mathbf{z} \in \mathcal{I}^n(Y)$ by $\widetilde{v}^{(a)}(x,\xi)$, $\widetilde{w}_1^{(a)}(x,\xi)$, $\widetilde{w}_2^{(a)}(x,\xi)$, respectively and let $\widetilde{v}^{(b)}(x,\xi)$ and $\widetilde{w}_2^{(b)}(x,\xi)$ denote the sub-principal symbols of $\mathbf{p}$ and $\mathbf{z}$, correspondingly, at $(x,\xi) \in N^*Y$.

We will below consider what happens when $(\mathbf{p}_{jk} + \mathbf{z}\widehat{g}_{jk}) \in \mathcal{I}^n(Y)$ satisfies

$$\widehat{g}^{lk}\nabla_l^{\widehat{g}}(\mathbf{p}_{jk} + \mathbf{z}\widehat{g}_{jk}) \in \mathcal{I}^n(Y), \quad j = 1,2,3,4. \tag{68}$$

Note that a priori this function is only in $\mathcal{I}^{n+1}(Y)$, so the assumption (68) means that $\widehat{g}^{lk}\nabla_l^{\widehat{g}}(\mathbf{p}_{jk} + \mathbf{z}\widehat{g}_{jk})$ is one degree smoother than it a priori should be.

When (68) is valid, we say that *the leading order of singularity of the wave satisfies the linearized conservation law.* This corresponds to the assumption that the principal symbol of the sum of divergence of the first two terms appearing in the stress energy tensor on the right hand side of (56) vanishes.

By [42], the identity (68) is equivalent to the vanishing of the principal symbol on N^*Y, that is,

$$\widehat{g}^{lk}\xi_l(\widetilde{v}_{kj}^{(a)}(x,\xi) + \widehat{g}_{kj}(x)\widetilde{w}_2^{(a)}(x,\xi)) = 0, \text{ for } j \le 4 \text{ and } \xi \in N_x^*Y. \tag{69}$$

We say that this is the *linearized conservation law for the principal symbol of R*.

Let us consider source fields that have the form $Q'_\varepsilon = ((Q_\varepsilon)_\ell)_{\ell=1}^{K-1} = \varepsilon\mathbf{q}'$, $(Q_\varepsilon)_K = \varepsilon\mathbf{z}$ and $P_\varepsilon = \varepsilon\mathbf{p}$. We denote $\mathbf{q} = (\mathbf{q}', \mathbf{z})$. We assume that $\mathbf{q}'$, $\mathbf{z}$, and $\mathbf{p}$ are supported in $\widehat{V} \subset\subset \widehat{U}$.

Let $u_\varepsilon = (g_\varepsilon, \phi_\varepsilon)$ be the solution of (56) with source P_ε and Q_ε. Then u_ε depends C^4-smoothly on ε and $(g_\varepsilon,\phi_\varepsilon)|_{\varepsilon=0} = (\widehat{g},\widehat{\phi})$. Denote $\partial_\varepsilon(g_\varepsilon,\phi_\varepsilon)|_{\varepsilon=0} = (\dot{g},\dot{\phi})$. When ε_0 is small enough, P_ε and Q_ε are supported in U_{g_ε} for all $\varepsilon \in (0,\varepsilon_0)$.

Let

$$R_\varepsilon = g_\varepsilon^{pk}\nabla_p^{g_\varepsilon}((P_\varepsilon)_{jk} + g_{jk}^\varepsilon(Q_\varepsilon)_K)$$

and

$$(S_\varepsilon)_\ell = \mathcal{S}_\ell(g_\varepsilon,\phi_\varepsilon,\nabla\phi_\varepsilon,Q'_\varepsilon,(Q_\varepsilon)_K,R_\varepsilon),$$

where $\mathcal{S}_\ell$ are the adaptive source functions constructed in Theorem 4.1 and its proof.

Then $S_\varepsilon|_{\varepsilon=0} = 0$ and $\partial_\varepsilon S_\varepsilon|_{\varepsilon=0} = \dot{S}$ satisfy

(70)

$$\dot{S}_\ell = D_{Q',Q_K,R}\mathcal{S}_\ell(\widehat{g},\widehat{\phi},\nabla\widehat{\phi},Q',Q_K,R)\Big|_{Q'=0,Q_K=0,R=0}\begin{pmatrix}\mathbf{q}'\\ \mathbf{z}\\ \mathbf{r}\end{pmatrix},$$

where $\mathbf{r}=\widehat g^{pk}\widehat\nabla_p(\mathbf{p}_{jk}+\widehat g_{jk}\mathbf{z})$.

The functions $\dot u=(\dot g,\dot\phi)$ satisfy the linearized Einstein-scalar field equation (33). The linearized Einstein-scalar field equation (33) is

$$\begin{aligned}&e_{pq}(\dot g)-t^{(1)}_{pq}(\dot g)-t^{(2)}_{pq}(\dot\phi)=\mathbf{f}^1_{pq}\\ &\square_{\widehat g}\dot\phi^\ell-\widehat g^{nm}\widehat g^{kj}(\partial_n\partial_j\widehat\phi_\ell+\widehat\Gamma^p_{nj}\partial_p\widehat\phi_\ell)\dot g_{mk}-m^2\dot\phi^\ell=\mathbf{f}^2_\ell,\end{aligned}$$

where

$$\begin{aligned}&\mathbf{f}^1_{pq}=\mathbf{p}_{pq}-(\sum_{\ell=1}^L\dot S_\ell\widehat\phi_\ell)\widehat g_{pq},\quad\text{where }-(\sum_{\ell=1}^L\dot S_\ell\widehat\phi_\ell)=\mathbf{z},\\ &\mathbf{f}^2_\ell=\dot S_\ell.\end{aligned}\tag{71}$$

By Theorem 4.1 (ii), $u_\varepsilon=(g_\varepsilon,\phi_\varepsilon)$ satisfy the conservation law (57). This implies that $\dot u=(\dot g,\dot\phi)$ satisfies the linearized Einstein-scalar field equation (33) and linearized of the conservation law (10) is valid, too. The linearized of the conservation law (10) gives, by considerations before (71),

$$\left(\sum_{\ell=1}^L\mathbf{f}^2_\ell\,\partial_j\widehat\phi_\ell\right)+\widehat g^{pk}\widehat\nabla_p\mathbf{f}^1_{kj}=0.\tag{72}$$

Below, we use the adaptive source functions $\mathcal{S}_\ell$ constructed in Theorem 4.1 and its proof.

We see that

$$\mathbf{f}=F(x;\mathbf{p},\mathbf{q})=(F^{(1)}(x;\mathbf{p},\mathbf{q}),F^{(2)}(x;\mathbf{p},\mathbf{q}))\tag{73}$$

has by formulas (70) and (71) and Condition S the form

$$F^{(1)}_{jk}(x;\mathbf{p},\mathbf{q})=\mathbf{p}_{jk}+\mathbf{z}(x)\widehat g_{jk}(x)\tag{74}$$

and

$$F^{(2)}(x;\mathbf{p},\mathbf{q})\ =\ M_{(2)}\mathbf{q}'+L_{(2)}\mathbf{z}+N^j_{(2)}\widehat g^{lk}\widehat\nabla_l(\mathbf{p}_{jk}+\mathbf{z}\widehat g_{jk}),\tag{75}$$

where

$$\begin{aligned}M_{(2)}&=M_{(2)}(\widehat\phi(x),\widehat\nabla\widehat\phi(x),\widehat g(x)),\\ L_{(2)}&=L_{(2)}(\widehat\phi(x),\widehat\nabla\widehat\phi(x),\widehat g(x)),\\ N^j_{(2)}&=N^j_{(2)}(\widehat\phi(x),\widehat\nabla\widehat\phi(x),\widehat g(x))\end{aligned}$$

are, in local coordinates, matrices whose elements are smooth functions of $\widehat\phi(x),\widehat\nabla\widehat\phi(x)$, and $\widehat g(x)$. By Thm. 4.1 (i), the union of the image spaces of the matrices $M_{(2)}(x)$ and $L_{(2)}(x)$, and $N^j_{(2)}(x)$, $j=1,2,3,4$, span the space $\mathbb{R}^L$ for all $x\in U_{\widehat g,\sigma}$.

Consider $n\in\mathbb{Z}$, $t_0,s_0>0$, $Y=Y(x_0,\zeta_0;t_0,s_0)$, $K=K(x_0,\zeta_0;t_0,s_0)$ and $(x,\xi)\in N^*Y$ (to recall the definitions of these notations, see formula (77) and definitions below it). Let $\mathcal{Z}=\mathcal{Z}(x,\xi)$ be set of the values of the principal symbol $\widetilde f(x,\xi)=(\widetilde f_1(x,\xi),\widetilde f_2(x,\xi))$, at (x,ξ), of the source $\mathbf{f} = (f_1(x), f_2(x)) \in \mathcal{I}^n(Y)$ that satisfy the linearized conservation law for principal symbols (13).

We use the following auxiliary result:

Lemma 5.1. *Assume the the Condition A is satisfied and $\widehat{Q} = 0$ and $\widehat{P} = 0$. Let $k_0 \geq 8$, $s_1 \geq k_0 + 5$, and $Y \subset U_{\widehat{g}}$ be a 2-dimensional space-like submanifold and $y \in Y$, $\xi \in N_y^*Y$, and let $\mathcal{W}$ be a conic neighborhood of (y, ξ) in T^*M. Also, let $y \in U_{\widehat{g},\sigma}$ with some permutation σ. Let us consider an open, relatively compact local coordinate neighborhood $V \subset U_{\widehat{g},\sigma} \cap U_{\widehat{g}}^+$ of y such that in the coordinates $X : V \to \mathbb{R}^4$, $X^j(x) = x^j$, we have $X(Y \cap V) \subset \{x \in \mathbb{R}^4;\ x^j b_j^1 = 0,\ x^j b_j^2 = 0\}$. Let $n_1 \in \mathbb{Z}_+$ be sufficiently large and $n \leq -n_1$. Let us consider $\mathbf{p}, \mathbf{q}', \mathbf{z} \in \mathcal{I}^{n+1}(Y)$, supported in V, that have classical symbols with principal symbols $\widetilde{v}^{(a)}(x, \xi)$, $\widetilde{w}_1^{(a)}(x, \xi)$, $\widetilde{w}_2^{(a)}(x, \xi)$, correspondingly, at $(x, \xi) \in N^*Y$. Moreover, assume that the principal symbols of $\mathbf{p}$ and $\mathbf{z}$ satisfy the linearized conservation law for the principal symbols, that is, (69), at all $N^*Y \cap N^*K$ and assume that they vanish outside the conic neighborhood $\mathcal{W}$ of (y, ξ) in T^*M. Let $\mathbf{f} = (f^1, f^2) \in \mathcal{I}^{n+1}(Y)$ be given by (70) and (71).*

Then the principal symbol $\widetilde{f}(y, \xi) = (\widetilde{f}_1(y, \xi), \widetilde{f}_2(y, \xi))$ of the source $\mathbf{f}$ at (y, ξ) is the set $\mathcal{Z} = \mathcal{Z}(y, \xi)$. Moreover, by varying $\mathbf{p}, \mathbf{q}', \mathbf{z}$ so that the linearized conservation law (69) for principal symbols is satisfied, the principal symbol $\widetilde{f}(y, \xi)$ at (y, ξ) achieves all values in the $(L + 6)$ dimensional space $\mathcal{Z}$.

Proof. Let us use local coordinates $X : V \to \mathbb{R}^4$ where $V \subset M_0$ is a neighborhood of x. In these coordinates, let $\widetilde{v}^{(b)}(x, \xi)$ and $\widetilde{w}_2^{(b)}(x, \xi)$ denote the sub-principal symbols of $\mathbf{p}$ and $\mathbf{z}$, respectively, at $(x, \xi) \in N^*Y$. Moreover, let $\widetilde{v}_j^{(c)}(x, \xi) = \frac{\partial}{\partial x^j} \widetilde{v}^{(a)}(x, \xi)$ and $\widetilde{d}_j^{(c)}(x, \xi) = \frac{\partial}{\partial x^j} \widetilde{d}_2^{(a)}(x, \xi)$, $j = 1, 2, 3, 4$ be the x-derivatives of the principal symbols and let us denote

$$\widetilde{v}^{(c)}(x, \xi) = (\widetilde{v}_j^{(c)}(x, \xi))_{j=1}^4, \quad \widetilde{d}^{(c)}(x, \xi) = (\widetilde{d}_j^{(c)}(x, \xi))_{j=1}^4.$$

Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) = F(x; \mathbf{p}, \mathbf{q})$ be defined by (74) and (75). When the principal symbols of $\mathbf{p}, \mathbf{q}', \mathbf{z} \in \mathcal{I}^n(Y)$ are such that the linearized conservation law (69) for principal symbols is satisfied, we see that $\mathbf{f} \in \mathcal{I}^n(Y)$ has the principal symbol $\widetilde{f}(x, \xi) = (\widetilde{f}_1(x, \xi), \widetilde{f}_2(x, \xi))$ at (x, ξ), given by

$$\begin{aligned} \widetilde{f}_1(x, \xi) &= s_1(x, \xi), \\ \widetilde{f}_2(x, \xi) &= s_2(x, \xi), \end{aligned}$$

where we use the notations

$$
\begin{aligned}
(76)\qquad s_1(x,\xi) &= (\widetilde v^{(a)}+\widehat g\widetilde w_2^{(a)})(x,\xi),\\
s_2(x,\xi) &= \Big(M_{(2)}(x)\widetilde w_1^{(a)}+J_{(2)}(\widetilde v^{(c)}+\widehat g\widetilde d^{(c)})+\\
&\quad +L_{(2)}\widetilde w_2^{(a)}+N^j_{(2)}\,\widehat g^{lk}\xi_l(\widetilde v_1^{(b)}+\widehat g\widetilde w_2^{(b)})_{jk}\Big)(x,\xi).
\end{aligned}
$$

Here, roughly speaking, the $J_{(2)}$ term appears when the ∇-derivatives in R hit to the symbols of the conormal distributions having the form (67). We emphasize that here the symbols $s_1(x,\xi)$ and $s_2(x,\xi)$ are well defined objects (in fixed local coordinates) also when the linearized conservation law (69) for principal symbols is not valid. When (69) is valid, $\mathbf{f}\in\mathcal{I}^n(Y)$ and $s_1(x,\xi)$ and $s_1(x,\xi)$ coincide with the principal symbols of $\mathbf{f}_1$ and $\mathbf{f}_2$.

Observe that the map $(c^{(b)}_{jk})\mapsto(\widehat g^{lk}\xi_l c^{(b)}_{jk})_{j=1}^4$, defined as $\mathrm{Symm}(\mathbb{R}^{4\times4})\to\mathbb{R}^4$, is surjective. Denote

$$
\begin{aligned}
\widetilde m^{(a)} &= (\widetilde v^{(a)}+\widehat g\widetilde w_2^{(a)})(x,\xi),\\
\widetilde m^{(b)} &= (\widetilde v^{(b)}+\widehat g\widetilde w_2^{(b)})(x,\xi),\\
\widetilde m^{(c)} &= (\widetilde v^{(c)}+\widehat g\widetilde d^{(c)})(x,\xi).
\end{aligned}
$$

As noted above, by (66), the union of the image spaces of the matrices $M_{(2)}(x)$ and $L_{(2)}(x)$, and $N^j_{(2)}(x)$, $j=1,2,3,4$, span the space $\mathbb{R}^L$ for all $x\in\widehat U$. Hence the map

$$
\mathbf{A}:(\widetilde m^{(a)},\widetilde m^{(b)},\widetilde m^{(c)},\widetilde w_1,\widetilde w_2^{(a)})|_{(x,\xi)}\mapsto(s_1(x,\xi),s_2(x,\xi)),
$$

given by (76), considered as a map $\mathbf{A}:\mathcal{Y}=(\mathrm{Symm}(\mathbb{R}^{4\times4}))^{1+1+4}\times\mathbb{R}^K\times\mathbb{R}\to\mathrm{Symm}(\mathbb{R}^{4\times4})\times\mathbb{R}^L$, is surjective. Let $\mathcal{X}$ be the set of elements $(\widetilde m^{(a)}(x,\xi),\widetilde m^{(b)}(x,\xi),\widetilde m^{(c)}(x,\xi),\widetilde w_1^{(a)}(x,\xi),\widetilde w_2^{(a)}(x,\xi))\in\mathcal{Y}$ where $\widetilde m^{(a)}(x,\xi)=(\widetilde v^{(a)}+\widehat g\widetilde w_2^{(a)})(x,\xi)$ is such that the pair $(\widetilde v^{(a)}(x,\xi),\widetilde w_2^{(a)}(x,\xi))$ satisfies the linearized conservation law for principal symbols, see (69). Then $\mathcal{X}$ has codimension 4 in Y, we see that the image $\mathbf{A}(\mathcal{X})$ has in $\mathrm{Symm}(\mathbb{R}^{4\times4})\times\mathbb{R}^L$ co-dimension less or equal to 4.

By (72)) and considerations above it, we have that $\mathbf{f}$ satisfies the linearized conservation law (10). This implies that its principal symbol $\mathbf{A}((\widetilde m^{(a)}(x,\xi),\widetilde m^{(b)}(x,\xi),\widetilde m^{(c)}(x,\xi),\widetilde w_1^{(a)}(x,\xi),\widetilde w_2^{(a)}(x,\xi)))$ has to satisfy the linearized conservation law for principal symbols (13) and hence $\mathbf{A}(\mathcal{X})\subset\mathcal{Z}$. As $\mathcal{Z}$ has codimension 4, this and the above prove that $\mathbf{A}(\mathcal{X})=\mathcal{Z}$. □

Now we are ready to prove the microlocal stability result for the Einstein-scalar field equation (7). Note that the claim of the following theorem does not involve the adaptive source functions constructed in Theorem 4.1 as these functions are needed only as an auxiliary tool in the proof.

Next we prove Theorem 1.2.

Proof. Let $\sigma \in \Sigma(K)$ be such that $y \in U_{\widehat{g},\sigma}$. Let $\mathbf{p}$ and $\mathbf{q}$ be the functions constructed in Lemma 5.1. We can assume that these functions are supported in $W_0 = V_0 \cap V \cap U_{\widehat{g},\sigma}$. Let $P_\varepsilon = \varepsilon \mathbf{p}$ and $Q_\varepsilon = \varepsilon \mathbf{q}$ be sources depending on $\varepsilon \in \mathbb{R}$ and $u_\varepsilon = (g_\varepsilon, \phi_\varepsilon)$ be the solution of (56) with the sources P_ε and Q_ε. Also, let

$$\mathcal{F}^1_\varepsilon = P_\varepsilon + Z_\varepsilon g_\varepsilon, \quad Z_\varepsilon = -(\sum_{\ell=1}^{L} S^\varepsilon_\ell \phi^\varepsilon_\ell + \frac{1}{2m^2}(S^\varepsilon_\ell)^2),$$
$$(\mathcal{F}^2_\varepsilon)_\ell = S^\varepsilon_\ell,$$

where

$$S^\varepsilon_\ell = \mathcal{S}_\ell(g_\varepsilon, \phi_\varepsilon, \nabla \phi_\varepsilon, Q_\varepsilon, \nabla Q_\varepsilon, P_\varepsilon, \nabla^{g_\varepsilon} P_\varepsilon),$$

where $\mathcal{S}_\ell$ are the adaptive source functions constructed in Theorem 4.1 and its proof.

By (58) also S^ε_ℓ and the family $\mathcal{F}_\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$ of non-linear sources are supported in V_0 and we have shown that $u_\varepsilon = (g_\varepsilon, \phi_\varepsilon)$ and $\mathcal{F}_\varepsilon$ satisfy the reduced Einstein-scalar field equation (7) and the conservation law (57). This proves Theorem 1.2. $\square$

6. Application: Gravitational wave packets

Next we consider a distorted plane wave whose singular support is concentrated near a geodesic. These waves, sketched in Fig. 1(Right), propagate near the geodesic $\gamma_{x_0,\zeta_0}([t_0, \infty))$ and are singular on a surface $K(x_0, \zeta_0; t_0, s_0)$, defined below in (77), that is a subset of the light cone $^{+}_{g}(x')$, $x' = \gamma_{x_0,\xi_0}(t_0)$. The parameter s_0 gives a "width" of the wave packet and when $s_0 \to 0$, its singular support tends to the set $\gamma_{x_0,\zeta_0}([2t_0, \infty))$. Next we will define these wave packets.

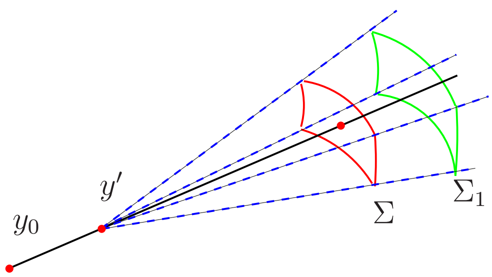

FIGURE 2. This is a schematic figure in the space $\mathbb{R}^3$. It describes the location of a distorted plane wave (or a piece of a spherical wave) $\dot{u}$ at different time moments. This wave propagates near the geodesic $\gamma_{x_0,\zeta_0}((0,\infty)) \subset \mathbb{R}^{1+3}$, $x_0 = (y_0, t_0)$ and is singular on a subset of a light cone emanated from $x' = (y', t')$. The piece of the distorted plane wave is sent from the surface $\Sigma \subset \mathbb{R}^3$, it starts to propagate, and at a later time its singular support is the surface Σ_1.

We define the 3-submanifold $K(x_0,\zeta_0;t_0,s_0)\subset M_0$ associated to $(x_0,\zeta_0)\in L^+(M_0,\widehat g)$, $x_0\in U_{\widehat g}$ and parameters $t_0,s_0\in\mathbb{R}_+$ as

$$K(x_0,\zeta_0;t_0,s_0)=\{\gamma_{x',\eta}(t)\in M_0;\ \eta\in\mathcal{W},\ t\in(0,\infty)\},\tag{77}$$

where $(x',\zeta')=(\gamma_{x_0,\zeta_0}(t_0),\dot\gamma_{x_0,\zeta_0}(t_0))$ and $\mathcal{W}\subset L^+_{x'}(M_0,\widehat g)$ is a neighborhood of ζ' consisting of vectors $\eta\in L^+_{x'}(M_0)$ satisfying $\|\eta-\zeta'\|_{\widehat g^+}<s_0$. Note that $K(x_0,\zeta_0;t_0,s_0)\subset {}^{+}_{\widehat g}(x')$ is a subset of the light cone starting with $x'=\gamma_{x_0,\xi_0}(t_0)$ and that it is singular at the point x'. Let $S=\{x\in M_0;\mathbf{t}(x)=\mathbf{t}(\gamma_{x_0,\zeta_0}(2t_0))\}$ be a Cauchy surface which intersects $\gamma_{x_0,\zeta_0}(\mathbb{R})$ transversally at the point $\gamma_{x_0,\zeta_0}(2t_0)$. When $t_0>0$ is small enough, $Y(x_0,\zeta_0;t_0,s_0)=S\cap K(x_0,\zeta_0;t_0,s_0)$ is a smooth 2-dimensional space-like surface that is a subset of $U_{\widehat g}$.

Let $\Lambda(x_0,\zeta_0;t_0,s_0)$ be the Lagranginan manifold that is the flowout from $N^*Y(x_0,\zeta_0;t_0,s_0)\cap N^*K(x_0,\zeta_0;t_0,s_0)$ on $\mathrm{Char}(\square_{\widehat g})$ in the future direction. When $K^{reg}\subset K=K(x_0,\zeta_0;t_0,s_0)$ is the set of points x that have a neighborhood W such that $K\cap W$ a smooth 3-dimensional submanifold, we have $N^*K^{reg}\subset\Lambda(x_0,\zeta_0;t_0,s_0)$. Below, we represent locally the elements $w\in\mathcal{B}_x$ in the fiber of the bundle $\mathcal{B}$ as a $(10+L)$-dimensional vector, $w=(w_m)_{m=1}^{10+L}$.

Lemma 6.1. *Let n_1 be a sufficiently large integer, $n\le -n_1$, $t_0,s_0>0$, $Y=Y(x_0,\zeta_0;t_0,s_0)$, $K=K(x_0,\zeta_0;t_0,s_0)$, $\Lambda_1=\Lambda(x_0,\zeta_0;t_0,s_0)$, and $(x,\xi)\in N^*Y\cap\Lambda_1$. Assume that $\mathbf{f}=(\mathbf{f}_1,\mathbf{f}_2)\in\mathcal{I}^n(Y)$, is a $\mathcal{B}^L$-valued conormal distribution that is supported in a neighborhood $V\subset M_0$ of $\gamma_{x_0,\zeta_0}\cap Y=\{\gamma_{x_0,\zeta_0}(2t_0)\}$ and has a $\mathbb{R}^{10+L}$-valued classical symbol. Denote the principal symbol of $\mathbf{f}$ by $\widetilde f(x,\xi)=(\widetilde f_k(x,\xi))_{k=1}^{10+L}$, and assume that the symbol of $\mathbf{f}$ vanishes near the light-like directions in $N^*Y\setminus N^*K$.*

Let $(\dot g,\dot\phi)$ be a solution of the linear wave equation (43) with the source $\mathbf{f}$. Then $u^{(1)}$, considered as a vector valued Lagrangian distribution on the set $M_0\setminus Y$, satisfies $u^{(1)}\in\mathcal{I}^{n-3/2}(M_0\setminus Y;\Lambda_1)$, and its principal symbol $\widetilde a(y,\eta)=(\widetilde a_j(y,\eta))_{j=1}^{10+L}$ at $(y,\eta)\in\Lambda_1$ is given by

$$\widetilde a_j(y,\eta)=\sum_{k=1}^{10+L}R_j^k(y,\eta,x,\xi)\widetilde f_k(x,\xi),\tag{78}$$

where the pairs (x,ξ) and (y,η) are on the same bicharacteristics of $\square_{\widehat g}$, and $x<y$, that is, $((x,\xi),(y,\eta))\in\Lambda'_{\widehat g}$, and in addition, $(x,\xi)\in N^*Y\cap N^*K$. Moreover, the matrix $(R_j^k(y,\eta,x,\xi))_{j,k=1}^{10+L}$ is invertible.

We call the solution $u^{(1)}$ a distorted plane wave that is associated to the submanifold $K(x_0,\zeta_0;t_0,s_0)$.

Proof. As noted above (42), the parametrix of the scalar wave equation satisfies $(\square_{\widehat g}+V(x,D))^{-1}\in\mathcal{I}^{-3/2,-1/2}(\Delta'_{T^*M_0},\Lambda_{\widehat g})$, where $V(x,D)$ is a 1st order differential operator, $\Delta_{T^*M_0}$ is the conormal bundle of the diagonal of $M_0 \times M_0$ and $\Lambda_{\widehat{g}}$ is the flow-out of the canonical relation of $\square_{\widehat{g}}$. A geometric representation for its kernel is given in [64]. An analogous result holds for the matrix valued wave operator, $\square_{\widehat{g}} I + V(x,D)$, when $V(x,D)$ is a 1st order differential operator, that is, $(\square_{\widehat{g}} I + V(x,D))^{-1} \in \mathcal{I}^{-3/2,-1/2}(\Delta'_{T^*M_0}, \Lambda_{\widehat{g}})$, see [78] and [24]. By [38, Prop. 2.1], this yields $u^{(1)} \in \mathcal{I}^{n-3/2}(\Lambda_1)$ and the formula (78) where $R = (R_j^k(y,\eta,x,\xi))_{j,k=1}^{10+L}$ is obtained by solving a system of ordinary differential equation along a bicharacteristic curve. Making similar considerations for the adjoint of the $(\square_{\widehat{g}} I + V(x,D))^{-1}$, i.e., considering the propagation of singularities using reversed causality, we see that the matrix R is invertible. $\square$

Let $\mathcal{B}_y^L$ be the fiber of the bundle $\mathcal{B}^L$ at y and $\mathfrak{S}_{y,\eta}$ be the space of the elements in $\mathcal{B}_y^L$ satisfying the harmonicity condition for the symbols (53) at (y,η). Let $(x,\xi) \in N^*Y$ and $\mathfrak{C}_{x,\xi}$ the set of elements in $\mathcal{B}_x^L$ that satisfy the linearized conservation law for symbols, i.e., equation (45).

Let $n \leq n_0$, $t_0, s_0 > 0$, $Y = Y(x_0,\zeta_0;t_0,s_0)$, $K = K(x_0,\zeta_0;t_0,s_0)$, $\Lambda_1 = \Lambda(x_0,\zeta_0;t_0,s_0)$, and $v \in \mathfrak{C}_{x,\xi}$. By Condition μ-SL, there is a conormal distribution $\mathbf{f} \in \mathcal{I}^n(Y) = \mathcal{I}^n(N^*Y)$ such that $\mathbf{f}$ satisfies the linearized conservation law (10) and the principal symbol $\widetilde{f}(y,\eta)$ of $\mathbf{f}$, defined on N^*Y, satisfies $\widetilde{f}(x,s\eta) = \widetilde{f}(x,\eta)s^n$ for $s > 0$. Moreover, by Condition μ-SL there is a family of sources $\mathcal{F}_\varepsilon$, $\varepsilon \in [0,\varepsilon_0)$ such that $\partial_\varepsilon \mathcal{F}_\varepsilon|_{\varepsilon=0} = \mathbf{f}$ and a solution $u_\varepsilon + (\widehat{g},\widehat{\phi})$ of the Einstein equations with the source $\mathcal{F}_\varepsilon$ that depend smoothly on ε and $u_\varepsilon|_{\varepsilon=0} = 0$. Then $\dot{u} = \partial_\varepsilon u_\varepsilon|_{\varepsilon=0} \in \mathcal{I}^{n-3/2}(M_0 \setminus Y; \Lambda_1)$.

Let $(x,\xi) \in N^*Y \cap \Lambda_1$ $(y,\eta) \in T^*M_0$, $y \notin Y$ be a light-like covector such that $(y,\eta) \in \Theta_{x,\xi}$. Since $\dot{u} = (\dot{g},\dot{\phi})$ satisfies the linearized harmonicity condition (46), the principal symbol $\widetilde{a}(y,\eta) = (\widetilde{a}_1(y,\eta),\widetilde{a}_2(y,\eta))$ of $\dot{u}$ satisfies $\widetilde{a}(y,\eta) \in \mathfrak{S}_{y,\eta}$, This shows that the map $R = R(y,\eta,x,\xi)$, given by $R : \widetilde{f}(x,\xi) \mapsto \widetilde{a}(y,\eta)$ that is defined in Lemma 6.1, satisfies $R : \mathfrak{C}_{x,\xi} \to \mathfrak{S}_{y,\eta}$. Since R is one-to-one and the linear spaces $\mathfrak{C}_{x,\xi}$ and $\mathfrak{S}_{y,\eta}$ have the same dimension, we see that

$$R : \mathfrak{C}_{x,\xi} \to \mathfrak{S}_{y,\eta} \tag{79}$$

is a bijection. Hence, when $\mathbf{f} \in \mathcal{I}^n(Y)$ varies so that the linearized conservation law (45) for the principal symbols is satisfied, the principal symbol $\widetilde{a}(y,\eta)$ at (y,η) of the solution $\dot{u}$ of the linearized Einstein equation achieves all values in the $(L+6)$ dimensional space $\mathfrak{S}_{y,\eta}$.

Appendix: Motivation of adaptive source functions using Lagrangian formulation

To motivate the system (56) of partial differential equations, we give in this appendix a non-rigorous discussion.

Following [12, Ch. III, Sect. 6.4, 7.1, 7.2, 7.3] and [3, p. 36] we start by considering the Lagrangians, associated to gravity, scalar fields

$\phi = (\phi_\ell)_{\ell=1}^L$ and non-interacting fluid fields, that is, the number density four-currents $\mathbf{n} = (\mathbf{n}_\kappa(x))_{\kappa=1}^J$ (where each $\mathbf{n}_\kappa$ is a vector field, see [3, p. 33]). We consider also products of vector fields $\mathbf{n}_\kappa(x)$ and $\frac{1}{2}$-density $|\det(g)|^{1/2}$ that denote $\mathbf{p}_\kappa$,

$$\mathbf{p}_\kappa^j(x)\frac{\partial}{\partial x^j} = \mathbf{n}_\kappa^j(x)|\det(g)|^{1/2}\frac{\partial}{\partial x^j}. \tag{80}$$

see [25, p. 53]. Also, $\rho = (-g_{jk}\mathbf{n}_\kappa^j\mathbf{n}_\kappa^k)^{1/2}$ corresponds to the energy density of the fluid. Below, we use the variation of density with respect to the metric,

$$\begin{aligned} \frac{\delta}{\delta g_{jk}}(\sum_{\kappa=1}^J(-g_{nm}\mathbf{p}_\kappa^n\mathbf{p}_\kappa^m)^{1/2}) &= -\sum_{\kappa=1}^J \frac{1}{2}(-g_{nm}\mathbf{p}_\kappa^n\mathbf{p}_\kappa^m)^{-1/2}\mathbf{p}_\kappa^j\mathbf{p}_\kappa^k \\ &= -\sum_{\kappa=1}^J \frac{1}{2}\rho\, \mathbf{n}_\kappa^j\mathbf{n}_\kappa^k\, |\det(g)|^{1/2}. \end{aligned} \tag{81}$$

Due to this, we denote

(82)

$$P \;=\; \sum_{\kappa=1}^J \frac{1}{2}\rho\, \mathbf{n}_j^\kappa \mathbf{n}_k^\kappa dx^j\otimes dx^k, \quad \text{where } \mathbf{n}_k^\kappa = g_{ki}\mathbf{n}_\kappa^i = g_{ki}\mathbf{p}_\kappa^i|\det(g)|^{-1/2}.$$

Below, we consider a model for g, ϕ, and $\mathbf{p}$. We also add in to the model a Lagrangian associated with some scalar valued source fields $S = (S_\ell)_{\ell=1}^L$ and $Q = (Q_k)_{k=1}^K$. We consider action corresponding to the coupled Lagrangians

$$\begin{aligned} &\mathcal{A} = \int_M \Big(L_{grav}(x) + L_{fields}(x) + L_{source}(x)\Big)\, dV_g(x), \\ &L_{grav} = \frac{1}{2}R(g), \\ &L_{fields} = \sum_{\ell=1}^L \Big(-\frac{1}{2}g^{jk}\partial_j\phi_\ell\,\partial_k\phi_\ell - \mathcal{V}(\phi_\ell;S_\ell)\Big) + \\ &\qquad\qquad + \sum_{\kappa=1}^J \Big(-\frac{1}{2}(-g_{jk}\mathbf{p}_\kappa^j\mathbf{p}_\kappa^k)^{\frac{1}{2}}\Big)|\det(g)|^{-\frac{1}{2}}, \\ &L_{source} = \varepsilon\mathcal{H}_\varepsilon(g,S,Q,\mathbf{p},\phi), \end{aligned}$$

where $R(g)$ is the scalar curvature, $dV_g = (-\det(g))^{1/2}dx$ is the volume form on (M,g),

$$\mathcal{V}(\phi_\ell;S_\ell) = \frac{1}{2}m^2\Big(\phi_\ell + \frac{1}{m^2}S_\ell\Big)^2 \tag{83}$$

are energy potentials of the scalar fields ϕ_ℓ that depend on S_ℓ, and $\mathcal{H}_\varepsilon(g,S,Q,\mathbf{p},\phi)$ is a function modeling the measurement device we use. We assume that $\mathcal{H}_\varepsilon$ is bounded and its derivatives with respect to

$S, Q, \mathbf{p}$ are very large (like of order $O((\varepsilon)^{-2})$) and its derivatives with respect of g and ϕ are bounded when $\varepsilon > 0$ is small. We note that the above Lagrangian for the fluid fields is the sum of the single fluid Lagrangians. where for all fluids the master function $\Lambda(s) = s^{1/2}$, that is, the energy density of each fluid is given by $\rho = \Lambda(-g_{jk}\mathbf{n}^j\mathbf{n}^k)$.For fluid Lagrangians, see the discussions in [3, p. 33-37], [12, Ch. III, Sect. 8], [25, p. 53], and [98] and [30, p. 196].

When we compute the critical points of the Lagrangian L and neglect the $O(\varepsilon)$-terms, the equation $\frac{\delta \mathcal{A}}{\delta g} = 0$, together with formulas (81) and (82), give the Einstein equations with a stress-energy tensor T_{jk} defined in (55). The equation $\frac{\delta \mathcal{A}}{\delta \phi} = 0$ gives the wave equations with sources S_ℓ. We assume that $O(\varepsilon^{-1})$ order equations obtained from the equation $(\frac{\delta \mathcal{A}}{\delta S}, \frac{\delta \mathcal{A}}{\delta Q}, \frac{\delta \mathcal{A}}{\delta \mathbf{p}}) = 0$ fix the values of the scalar functions Q and the fields $\mathbf{p}^\kappa$, $\kappa = 1, 2, \dots, J$, and moreover, yield for the sources $S = (S_\ell)_{\ell=1}^L$ equations of the form

$$S_\ell = \mathcal{S}_\ell(g, \phi, \nabla\phi, Q, \nabla Q, P, \nabla^g P) \tag{84}$$

where P is given by (82). Let us aslo write (84) using different notations, as

$$S_\ell = Q_\ell + \mathcal{S}_\ell^{2nd}(g, \phi, \nabla\phi, Q, \nabla Q, P, \nabla^g P).$$

Summarizing, we have obtained, up to the above used approximations, the model (56). However, note that above the field P is not directly controlled but instead, we control $\mathbf{p}$ and the value of the field P is determined by the solution n and formula (82). In this sense P is not controlled, but an observed field.

Above, the function $\mathcal{H}_\varepsilon$ models the way the measurement device works. Due to this we will assume that $\mathcal{H}_\varepsilon$ and thus functions $\mathcal{S}_\ell$ may be quite complicated. The interpretation of the above is that in each measurement event we use a device that fixes the values of the scalar functions Q, $\mathbf{p}$, and gives the equations for S^{2nd} that tell how the sources of the ϕ-fields adapt to these changes so that the physical conservation laws are satisfied.

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