INVERSE PROBLEMS FOR DIFFERENTIAL FORMS ON RIEMANNIAN MANIFOLDS WITH BOUNDARY

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Abstract. Consider a real-analytic orientable connected complete Riemannian manifold M with boundary of dimension $n \geq 2$ and let k be an integer $1 \leq k \leq n$. In the case when M is compact of dimension $n \geq 3$, we show that the manifold and the metric on it can be reconstructed, up to an isometry, from the set of the Cauchy data for harmonic k-forms, given on an open subset of the boundary. This extends a result of [13] when $k = 0$. In the two-dimensional case, the same conclusion is obtained when considering the set of the Cauchy data for harmonic 1-forms. Under additional assumptions on the curvature of the manifold, we carry out the same program when M is complete non-compact. In the case $n \geq 3$, this generalizes the results of [12] when $k = 0$. In the two-dimensional case, we are able to reconstruct the manifold from the set of the Cauchy data for harmonic 1-forms.

1. Introduction and statement of results

The purpose of this paper is to study the inverse problem of the determination of a complete Riemannian manifold (M, g) with boundary from the Cauchy data of harmonic differential forms, given on an open subset of the boundary. We emphasize that the determination of a Riemannian manifold includes, in addition to the reconstruction of the metric, the recovery of the topological and the differentiable structures of M.

Motivated by the problem of electrical impedance tomography, [15, 18], the issue of the reconstruction of a Riemannian manifold from the set of the Cauchy data of harmonic functions is a basic question in the field of inverse problems. Here the Cauchy data can be represented as the graph of the Dirichlet-to-Neumann map Λ_g, which is defined by solving the Dirichlet problem

$$\Delta_g u = 0, \quad u|_{\partial M} = f,$$

with a given $f \in C^\infty(\partial M)$, and setting $\Lambda_g(f) = \partial_\nu u|_{\partial M}$, where ∂_ν is the exterior normal derivative. In particular, the importance of the Dirichlet-to-Neumann map Λ_g is due to the fact that it encodes boundary measurements of voltage and current flux in electrical impedance tomography.

Since the works [12, 13], it is known that a complete real-analytic connected Riemannian manifold of dimension $n \geq 3$ can be recovered, up to an isometry, from the knowledge of the map Λ_g, given on an open subset of the boundary. It is natural to expect that the reconstruction of the manifold is also possible from the Dirichlet-to-Neumann map, associated to the Hodge Laplacian on differential forms. The first step in this direction has been made in [10], where it was shown that the full Taylor series of the metric tensor at the boundary can be recovered from the Dirichlet-to-Neumann map of the Hodge Laplacian on k-forms, $k = 1, \dots, n$. In this work we complete this study by reconstructing the Riemannian manifold from such a Dirichlet-to-Neumann map. Our results are unconditional in the case when M is compact, whereas in the complete non-compact case we require conditions on the curvature of the manifold, as well as a spectral assumption.

In the case when the dimension of M is equal to 2, it was shown in [13] that only the conformal class of the compact real-analytic connected Riemannian manifold can be determined from the knowledge of Λ_g. This obstruction is due to the conformal invariance of the Laplacian on functions, i.e.

$$\Delta_{\sigma g} = \sigma^{-1}\Delta_g, \quad \sigma \in C^\infty(M), \quad \sigma > 0.$$

Furthermore, it was shown in [12] that there exist complete two-dimensional Riemannian manifolds with boundary that are not conformally equivalent, but that have identical Dirichlet-to-Neumann maps Λ_g. The construction in [12] motivated the construction of invisibility cloaks in electrostatics [8]. See the review [9]. We digress to review the construction in [12].

Let (M, g) be a compact 2-dimensional manifold with non-empty boundary, let $x_0 \in M$ and consider manifold

$$\widetilde{M} = M \setminus \{x_0\}$$

with the metric

$$\widetilde{g}_{ij}(x) = \frac{1}{d_M(x, x_0)^2} g_{ij}(x),$$

where $d_M(x, x_0)$ is the distance between x and x_0 on (M, g). Then $(\widetilde{M}, \widetilde{g})$ is a complete, non-compact 2-dimensional Riemannian manifold with the boundary $\partial\widetilde{M} = \partial M$. On the manifolds M and $\widetilde{M}$ we consider the boundary value problems

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases} \quad \text{and} \quad \begin{cases} \Delta_{\widetilde{g}} \widetilde{u} = 0 & \text{in } \widetilde{M}, \\ \widetilde{u} = f & \text{on } \partial\widetilde{M}, \\ \widetilde{u} \in L^\infty(\widetilde{M}). \end{cases}$$

These boundary value problems are uniquely solvable and define the Dirichlet-to-Neumann maps

$$\Lambda_{M,g} f = \partial_\nu u|_{\partial M}, \quad \Lambda_{\widetilde{M},\widetilde{g}} f = \partial_\nu \widetilde{u}|_{\partial\widetilde{M}}.$$

As mentioned earlier in the two dimensional case functions which are harmonic with respect to the metric g stay harmonic with respect to any metric which is conformal to g, one can see that $\Lambda_{M,g} = \Lambda_{\widetilde{M},\widetilde{g}}$. This can be seen using e.g. Brownian motion or capacity arguments. Thus, the boundary measurements for (M, g) and $(\widetilde{M}, \widetilde{g})$ coincide. This gives a counter example for the inverse electrostatic problem on Riemannian surfaces - even the topology of possibly non-compact Riemannian surfaces can not be determined using boundary measurements (see Figure 1).

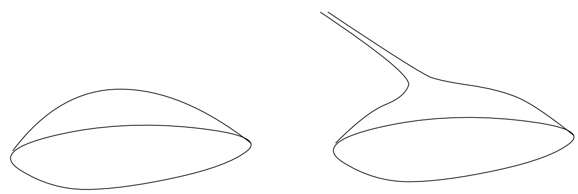

FIGURE 1. Blowing up a metric at a point, after [12]. The electrostatic boundary measurements on the boundary of the surfaces, one compact and the other noncompact but complete, coincide.

In this paper, we exploit the observation that the conformal invariance of the Laplacian can be broken by passing to forms of degree 1. Working on the level of 1-forms, in the compact real-analytic case, we reconstruct the manifold up to an isometry from the corresponding Dirichlet-to-Neumann map. In the complete non-compact case, we do the same under the assumption that the Gaussian curvature of the manifold should be bounded from below, as well as a spectral assumption.

The plan of the paper is as follows. The remainder of this section is devoted to the description of the precise assumptions, used throughout the paper, as well as to the statements of the main results. In Section 2, we construct an example of two complete two-dimensional manifolds in the same conformal class, which can be told apart using the Cauchy data of harmonic 1-forms. The proofs of our results start in Section 3, where, following [12] closely, we address the case when M is compact of dimension $n \geq 3$. The two-dimensional compact case is analyzed in Section 4. Here an important role is played by the result that we can recover the Taylor series of the metric tensor at the boundary from the Dirichlet-to-Neumann map, associated to the Hodge Laplacian on 1-forms. The final Section 5 establishes the results in the complete case.

1.1. **Notation.** Let (M, g) be a real-analytic orientable connected complete Riemannian manifold of dimension $n \geq 2$ with a compact real-analytic boundary ∂M and let the metric g be real-analytic up to the boundary. Let T^*M be the cotangent bundle on M and let $\Lambda^k T^*M$, $k = 0, 1, \ldots, n$, be the bundles of the exterior differential k-forms. Denote by $C^\infty(M, \Lambda^k T^*M)$ the space of smooth exterior differential forms of degree k. Here the smoothness is understood up to the boundary of M. The metric tensor g induces the volume form $\mu = \mu_g \in C^\infty(M, \Lambda^n T^*M)$ and the Hodge star isomorphism

$$* : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{n-k} T^*M), \quad \omega \wedge *\eta = g(\omega, \eta)\mu.$$

Here in local coordinates, $\mu = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n$, provided that $dx^1, \dots, dx^n$ is a positive basis of T_x^*M.

Let $d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ be the exterior differential. Then the codifferential operator is defined by

$$\delta : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k-1} T^*M), \quad \delta\omega = (-1)^{nk+n+1} * d * \omega,$$

and the Hodge-Laplace operator is given by

$$\Delta_g^{(k)} = \Delta^{(k)} : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^k T^*M), \quad \Delta^{(k)}\omega = (d\delta + \delta d)\omega.$$

To study differential forms on the boundary of M, we consider the inclusion map $i : \partial M \to M$ and its pull-back,

$$i^* : C^\infty(M, \Lambda^k T^*M) \to C^\infty(\partial M, \Lambda^k T^*M).$$

Then we define the tangential trace of a k-form as

$$\mathbf{t} : C^\infty(M, \Lambda^k T^*M) \to C^\infty(\partial M, \Lambda^k T^*M), \quad \mathbf{t}\omega = i^*\omega, \quad k = 0, 1, \dots, n-1,$$

and the normal trace as

$$\mathbf{n} : C^\infty(M, \Lambda^k T^*M) \to C^\infty(\partial M, \Lambda^{n-k} T^*M), \quad \mathbf{n}\omega = i^*(*\omega), \quad k = 1, 2, \dots, n.$$

The L^2-inner product of differential k-forms is given by

$$(\omega, \eta)_{L^2} = \int_M \omega \wedge *\bar{\eta}, \quad \omega, \eta \in C_0^\infty(M, \Lambda^k T^*M),$$

where $\bar{\eta}$ is the complex conjugate of η. The space $L^2(M, \Lambda^k T^*M)$ is defined as the completion of the space $C_0^\infty(M, \Lambda^k T^*M)$ of compactly supported differential k-forms on M in the corresponding L^2-norm. When $\omega \in C_0^\infty(M, \Lambda^k T^*M)$, let

$$\|\omega\|_s = \left(\sum_{j=0}^{s} \|\nabla^{(j)}\omega\|_{L^2}^2\right)^{1/2},$$

where $\nabla^{(j)}$ is the j-th order covariant derivative of a k-form. The completion of the space $C_0^\infty(M, \Lambda^k T^*M)$ with respect to this norm is the standard Sobolev space of sections of $\Lambda^k T^*M$, denoted by $H^s(M, \Lambda^k T^*M)$.

The space $C_0^\infty(M, \Lambda^k T^*M)$ is dense in $H^s(M, \Lambda^k T^*M)$ for $s \geq 0$, and the tangential and normal traces extend to continuous surjections

$$\mathbf{t} : H^s(M, \Lambda^k T^*M) \to H^{s-1/2}(\partial M, \Lambda^k T^*M),$$
$$\mathbf{n} : H^s(M, \Lambda^k T^*M) \to H^{s-1/2}(\partial M, \Lambda^{n-k} T^*M),$$

as soon as $s > 1/2$. The operators d and δ extend to continuous mappings

$$d : H^s(M, \Lambda^k T^*M) \to H^{s-1}(M, \Lambda^{k+1}T^*M),$$
$$\delta : H^s(M, \Lambda^k T^*M) \to H^{s-1}(M, \Lambda^{k-1}T^*M),$$

for $s \geq 1$.

We set

$$\langle \mathbf{t}\omega, \mathbf{n}\eta \rangle = \int_{\partial M} \mathbf{t}\omega \wedge \mathbf{n}\bar{\eta}, \quad \omega \in C^\infty(M, \Lambda^k T^*M),\ \eta \in C^\infty(M, \Lambda^{k+1}T^*M).$$

From [17, Proposition 2.1.2] we recall the Stokes' formula

$$(d\omega, \eta)_{L^2} - (\omega, \delta\eta)_{L^2} = \langle \mathbf{t}\omega, \mathbf{n}\eta \rangle,\ \omega \in H^1(M, \Lambda^k T^*M),\ \eta \in H^1(M, \Lambda^{k+1}T^*M). \tag{1.1}$$

The elements of the space

$$\mathcal{H}^k(M) = \{\omega \in H^1(M, \Lambda^k T^*M) : d\omega = 0, \delta\omega = 0\}$$

are called harmonic fields. Notice that the space $\mathcal{H}^k(M)$ is infinite dimensional for $1 \leq k \leq n-1$, see [17, Theorem 3.4.2]. Two subspaces are distinguished in $\mathcal{H}^k(M)$,

$$\mathcal{H}^k_D(M) = \{\omega \in \mathcal{H}^k(M) : \mathbf{t}\omega = 0\} \quad \text{and}$$
$$\mathcal{H}^k_N(M) = \{\omega \in \mathcal{H}^k(M) : \mathbf{n}\omega = 0\},$$

which are called the Dirichlet and Neumann harmonic fields, respectively. According to [17, Theorem 2.2.2], $\mathcal{H}^k_D(M)$ and $\mathcal{H}^k_N(M)$ are finite-dimensional, when M is compact.

1.2. **The case of a compact manifold.** Here we assume that the manifold M is compact with a non-empty boundary. Then the set of the Cauchy data of harmonic k-forms is given by

$$\mathcal{C}_g^{(k)} = \{(\mathbf{t}\omega, \mathbf{n}\omega, \mathbf{n}d\omega, \mathbf{t}\delta\omega) : \omega \in C^\infty(M, \Lambda^k T^*M), \Delta^{(k)}\omega = 0\}. \tag{1.2}$$

Here we notice that

$$\mathcal{C}_g^{(0)} = \{(\mathbf{t}\omega, \mathbf{n}d\omega) : \omega \in C^\infty(M, \Lambda^0 T^*M), \Delta^{(0)}\omega = 0\},$$

and

$$\mathcal{C}_g^{(n)} = \{(\mathbf{n}\omega, \mathbf{t}\delta\omega) : \omega \in C^\infty(M, \Lambda^n T^*M), \Delta^{(n)}\omega = 0\}.$$

According to [6, 10], the natural Dirichlet data for a harmonic k-form ω is

$$(\mathbf{t}\omega, \mathbf{n}\omega),$$

and the natural Neumann data is

$$(\mathbf{n}d\omega, \mathbf{t}\delta\omega).$$

Specifically, it is known [6, Theorem 1] that the problem

$$\begin{aligned} \Delta^{(k)}\omega &= 0 \quad \text{in} \quad M, \\ \mathbf{t}\omega &= f_1 \quad \text{on} \quad \partial M, \\ \mathbf{n}\omega &= f_2 \quad \text{on} \quad \partial M. \end{aligned} \tag{1.3}$$

has a unique solution $\omega \in C^\infty(M, \Lambda^k T^*M)$ if $f_1 \in C^\infty(\partial M, \Lambda^k T^*M)$ and $f_2 \in C^\infty(\partial M, \Lambda^{n-k} T^*M)$. Thus, the natural Dirichlet-to-Neumann map for the k-form Hodge Laplacian is given by

$$\begin{aligned} \Lambda_g^{(k)} &: C^\infty(\partial M, \Lambda^k T^*M) \times C^\infty(\partial M, \Lambda^{n-k} T^*M) \\ &\to C^\infty(\partial M, \Lambda^{n-k-1} T^*M) \times C^\infty(\partial M, \Lambda^{k-1} T^*M), \\ &\Lambda_g^{(k)}(f_1, f_2) = (\mathbf{n} d\omega, \mathbf{t}\delta\omega), \end{aligned} \tag{1.4}$$

where ω is the solution to (1.3). Notice that the set of the Cauchy data (1.2) for the harmonic k-forms is equal to the graph of the Dirichlet-to-Neumann map (1.4).

Another approach to the Dirichlet and Neumann boundary conditions for harmonic differential forms is given in [19, Ch 5, Section 9],

$$\begin{aligned} &\text{the Dirichlet boundary conditions}: \quad (\mathbf{t}\omega, \mathbf{t}\delta\omega), \\ &\text{the Neumann boundary conditions}: \quad (\mathbf{n}\omega, \mathbf{n}d\omega). \end{aligned}$$

It follows from [17, Lemma 3.4.7] that the problem

$$\begin{aligned} \Delta^{(k)}\omega &= 0 \quad \text{in} \quad M, \\ \mathbf{t}\omega &= g_1 \quad \text{on} \quad \partial M, \\ \mathbf{t}\delta\omega &= g_2 \quad \text{on} \quad \partial M, \end{aligned} \tag{1.5}$$

is solvable in $C^\infty(M, \Lambda^k T^*M)$, if and only if

$$\langle g_2, \mathbf{n}\lambda \rangle = 0, \quad \forall \lambda \in \mathcal{H}_D^k(M).$$

The solution of (1.5) is unique up to an arbitrary Dirichlet field $\lambda \in \mathcal{H}_D^k(M)$. In this case, in order to define the Dirichlet-to-Neumann map one has to specify the solution of (1.5), for instance by requiring that

$$(\omega, \lambda)_{L^2} = 0, \quad \forall \lambda \in \mathcal{H}_D^k(M). \tag{1.6}$$

Notice that (1.6) gives a finite number of linear conditions. Let

$$W = \{(g_1, g_2) \in C^\infty(\partial M, \Lambda^k T^*M) \times C^\infty(\partial M, \Lambda^{k-1} T^*M) : \langle g_2, \mathbf{n}\lambda \rangle = 0, \\ \forall \lambda \in \mathcal{H}_D^k(M)\}.$$

Then the Dirichlet-to-Neumann map is defined by

$$\begin{aligned} &\widetilde{\Lambda}_g^{(k)} : W \to C^\infty(\partial M, \Lambda^{n-k} T^*M) \times C^\infty(\partial M, \Lambda^{n-k-1} T^*M), \\ &\widetilde{\Lambda}_g^{(k)}(g_1, g_2) = (\mathbf{n}\omega, \mathbf{n}d\omega), \end{aligned}$$

where ω is the solution of (1.5) satisfying (1.6).

Proposition 1.1. *The knowledge of* $\widetilde{\Lambda}_g^{(k)}$ *determines the set of the Cauchy data* (1.2) *modulo a finite dimensional space.*

Proof. Let

$$U = \{(\mathbf{t}\omega, \mathbf{n}\omega, \mathbf{n}d\omega, \mathbf{t}\delta\omega) : \\ \omega \in C^\infty(M, \Lambda^k T^*M), \Delta^{(k)}\omega = 0, \langle \mathbf{t}\delta\omega, \mathbf{n}\lambda\rangle = 0, \forall \lambda \in \mathcal{H}_D^k(M)\}.$$

Then we can write

$$\mathcal{C}_g^{(k)} = U \oplus \widetilde{U},$$

where $\widetilde{U}$ is finite-dimensional. Furthermore,

$$\mathcal{C}_g^{(k)} = \{(\mathbf{t}\omega, \mathbf{n}\omega, \mathbf{n}d\omega, \mathbf{t}\delta\omega) \in U : (\omega, \lambda)_{L^2} = 0, \forall \lambda \in \mathcal{H}_D^k(M)\} \oplus \widetilde{U} \oplus \widehat{U},$$

$\widehat{U}$ is finite-dimensional. The claim follows. □

Proposition 1.1 implies that the knowledge of $\widetilde{\Lambda}_g^{(k)}$ determines $\Lambda_g^{(k)}$ modulo a smoothing operator. Hence, it follows from [10] that in case $n \geq 3$, the knowledge of $\widetilde{\Lambda}_g^{(k)}$ allows us to recover the Taylor series at the boundary of the metric g in the boundary normal coordinates.

Finally yet another definition of the Dirichlet-to-Neumann operator is given in [1]. For any $0 \leq k \leq n-1$, the Dirichlet-to-Neumann operator

$$\widehat{\Lambda}_g^{(k)} : C^\infty(\partial M, \Lambda^k T^*M) \to C^\infty(\partial M, \Lambda^{n-k-1} T^*M)$$

is defined as follows. Given $g_1 \in C^\infty(\partial M, \Lambda^k T^*M)$, the boundary problem

$$\begin{aligned} \Delta^{(k)}\omega = 0 \quad &\text{in} \quad M, \\ \mathbf{t}\omega = g_1 \quad &\text{on} \quad \partial M, \\ \mathbf{t}\delta\omega = 0 \quad &\text{on} \quad \partial M, \end{aligned} \tag{1.7}$$

is solvable in $C^\infty(M, \Lambda^k T^*M)$. The solution of (1.7) is unique up to an arbitrary Dirichlet field $\lambda \in \mathcal{H}_D^k(M)$. Defining

$$\widehat{\Lambda}_g^{(k)} g_1 = \mathbf{n}d\omega,$$

it is clear that $\widehat{\Lambda}_g^{(k)}$ is independent of the choice of a solution of (1.7).

In [1], an explicit formula is obtained which expresses the Betti numbers of the manifold M in terms of $\widehat{\Lambda}_g^{(k)}$.

The following is our first main result.

Theorem 1.2. *Let M_1 and M_2 be compact real-analytic orientable connected Riemannian manifolds with real-analytic boundaries ∂M_1 and ∂M_2, respectively. Moreover, let the metrics g_1 and g_2 be real-analytic up to the boundary and $\dim M_1 = \dim M_2 = n \geq 3$. Let $\Gamma_1 = \Gamma_2 = \Gamma$ be a non-empty open subset of ∂M_1 and ∂M_2. If for some integer $1 \leq k \leq n$,*

$$\mathcal{C}_{g_1}^{(k)}|_\Gamma = \mathcal{C}_{g_2}^{(k)}|_\Gamma, \tag{1.8}$$

then M_1 and M_2 are isometric.

Here the sets $\Gamma_1 \subset \partial M_1$ and $\Gamma_2 \subset \partial M_2$ are identified by a real-analytic diffeomorphism.

Notice that if the manifolds M_1 and M_2 are isometric then the bundles of the exterior differential k-forms $\Lambda^k T^* M_1$ and $\Lambda^k T^* M_2$ are naturally isometric for any $k = 1, \dots, n$. It follows therefore from Theorem 1.2 that the Cauchy data (1.8) determine also the exterior differential form bundle structure over the manifold.

In the two dimensional case, it is well-known [14] that there is an obstruction to reconstructing the metric from the set of Cauchy data for harmonic 0-forms, due to the conformal invariance of the corresponding Laplacian. It turns out that the conformal invariance can be broken by passing to forms of higher degree, and the reconstruction becomes possible. Thus, we obtain the following result.

Theorem 1.3. *Let M_1 and M_2 be compact real-analytic orientable connected Riemannian manifolds of dimension two with real-analytic boundaries ∂M_1 and ∂M_2, respectively. Moreover, let the metrics g_1 and g_2 be real-analytic up to the boundary. Let $\Gamma_1 = \Gamma_2 = \Gamma$ be a non-empty open subset of ∂M_1 and ∂M_2. If*

$$\mathcal{C}_{g_1}^{(1)}|_\Gamma = \mathcal{C}_{g_2}^{(1)}|_\Gamma,$$

then M_1 and M_2 are isometric.

1.3. **The case of a complete non-compact manifold.** Let (M, g) be a complete non-compact real-analytic orientable connected Riemannian manifold with compact real-analytic boundary ∂M. Let the metric g be real-analytic up to the boundary and $n = \dim(M) \geq 2$.

Let $f_1 \in C^\infty(\partial M, \Lambda^k T^* M)$ and $f_2 \in C^\infty(\partial M, \Lambda^{n-k} T^* M)$. Then consider the problem

$$\begin{aligned} &\Delta^{(k)}\omega = 0 \quad \text{in} \quad M, \\ &\mathbf{t}\omega = f_1 \quad \text{on} \quad \partial M, \\ &\mathbf{n}\omega = f_2 \quad \text{on} \quad \partial M. \end{aligned} \tag{1.9}$$

Let us make the following hypotheses.

Assumption (A1). Let $n \geq 3$.

- If $k = 1$, then the Ricci curvature tensor is bounded from below.

- If $k = 2, \dots, n$, then the Riemann curvature tensor is bounded on M.

Assumption (A2). Let $n = 2$. Then the Gaussian curvature of M is bounded from below.

In the two-dimensional case, similarly to the compact situation, we shall only be concerned with 1-forms.

To discuss the solvability of the problem (1.9) in the case when the complete manifold M is non-compact, we shall need the following result.

Proposition 1.4. *Assume that the assumption* (A1) *holds. Then the operator* $\Delta^{(k)} = \Delta^{(k)}_F$, $k = 1, \dots, n$, *equipped with the domain*

$$\mathcal{D}(\Delta^{(k)}_F) = \mathcal{K} \cap \{\omega \in L^2(M, \Lambda^k T^* M) : \Delta^{(k)}\omega \in L^2(M, \Lambda^k T^* M)\}, \tag{1.10}$$

where

$$\mathcal{K} = \{\omega \in H^1(M, \Lambda^k T^* M) : \mathbf{t}\omega = 0,\ \mathbf{n}\omega = 0\},$$

is a non-negative self-adjoint operator on $L^2(M, \Lambda^k T^* M)$. *Assuming that the assumption* (A2) *holds, we get the same conclusion for the operator* $\Delta^{(1)}$.

Proof. The quadratic form

$$\mathcal{D} : \mathcal{K} \times \mathcal{K} \to \mathbb{R}, \quad \mathcal{D}(\omega, \omega) = \|d\omega\|^2_{L^2} + \|\delta\omega\|^2_{L^2},$$

is densely defined and non-negative in $L^2(M, \Lambda^k T^* M)$.

Let us show that $\mathcal{D}$ is closed, i.e. $\mathcal{K}$ is complete with respect to the norm

$$\|\omega\|_{\mathcal{D}} = \sqrt{\mathcal{D}(\omega, \omega) + \|\omega\|^2_{L^2}}.$$

To this end, we shall use the following result from [17, Theorem 2.1.5]: as the boundary ∂M is compact, for all $\omega \in H^1(M, \Lambda^k T^* M)$ and $\mathbf{t}\omega = 0$, we have

$$\|\omega\|^2_{H^1} = \|\omega\|^2_{L^2} + (\mathcal{R}\omega, \omega)_{L^2} + \mathcal{D}(\omega, \omega) + \int_{\partial M} \mathcal{S}\omega \wedge *\bar{\omega}, \tag{1.11}$$

where $\mathcal{R} \in \mathrm{End}(\Lambda^k T^* M)$ is the bundle endomorphism, determined by the Riemannian curvature tensor on M and $\mathcal{S} \in \mathrm{End}(\Lambda^k T^* M|_{\partial M})$ is the bundle endomorphism, determined by the second fundamental form of ∂M.

The compactness of ∂M implies that the bundle endomorphism $\mathcal{S}$ is bounded, and

$$\Big| \int_{\partial M} \mathcal{S}\omega \wedge *\omega \Big| \le C_{\mathcal{S}} \|\omega\|^2_{L^2(\partial M)}, \quad C_{\mathcal{S}} > 0.$$

Furthermore, the compactness of ∂M yields that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\|\omega\|^2_{L^2(\partial M)} \le \epsilon \|\omega\|^2_{H^1(M)} + C_\epsilon \|\omega\|^2_{L^2(M)},$$

see [17, Corollary 2.1.6].

If the Riemann curvature tensor is bounded on M, then the bundle endomorphism $\mathcal{R}$ is also bounded, and

$$\left|(\mathcal{R}\omega,\omega)_{L^2}\right| \le C_{\mathcal{R}}\|\omega\|^2_{L^2(M)}, \quad C_{\mathcal{R}}>0.$$

In the case of 1-forms, $(\mathcal{R}\omega,\omega)_{L^2} = -\int_M \mathrm{Ric}(\omega,\omega)\mu$, where Ric is the Ricci tensor, viewed as the bilinear form

$$\mathrm{Ric}: \Lambda^1T^*M\times\Lambda^1T^*M\to\mathbb{R}.$$

When $n=2$, the Ricci curvature tensor is given by

$$\mathrm{Ric}^{ij}=Kg^{ij},$$

where K is the Gaussian curvature of M.

Hence, the assumptions of the proposition and (1.11) imply that

$$\|\omega\|_{H^1}\le C|\!|\!|\omega|\!|\!|_{\mathcal{D}}, \quad C>0. \tag{1.12}$$

Thus, the fact that the quadratic form D is closed follows from (1.12) and the fact that the tangential $\mathbf{t}$ and normal $\mathbf{n}$ traces

$$\mathbf{t}: H^1(M,\Lambda^kT^*M)\to H^{1/2}(\partial M,\Lambda^kT^*M), \quad k=0,1,\dots,n-1,$$
$$\mathbf{n}: H^1(M,\Lambda^kT^*M)\to H^{1/2}(\partial M,\Lambda^{n-k}T^*M), \quad k=1,2,\dots,n,$$

are continuous, see [17, Theorem 1.3.7]. The operator $\Delta_F^{(k)}$ is the non-negative self-adjoint operator, associated to the closed form $\mathcal{D}$, and the form domain of $\Delta_F^{(k)}$ is the space $\mathcal{K}$.

□

Remark 1.5. *The Hodge Laplace operator $\Delta^{(0)}$, acting on 0-forms and equipped with the domain, given by* (1.10)*, is a non-negative self-adjoint operator on $L^2(M)$. This follows from the proof of Proposition* 1.4*, since in this case the bundle endomorphism $\mathcal{R}$ in* (1.11) *vanishes, see* [17, Theorem 2.1.5].

In what follows, assume that the assumption (A1) holds when $n\ge 3$, and the assumption (A2) holds when $n=2$. Denote by $\mathrm{spec}_d(\Delta_F^{(k)})$, $\mathrm{spec}_{ess}(\Delta_F^{(k)})$ and $\mathrm{spec}(\Delta_F^{(k)})$, the discrete spectrum, the essential spectrum, and the spectrum of $\Delta_F^{(k)}$, respectively.

Remark 1.6. *Let us show that $0\notin spec_d(\Delta_F^{(k)})$. Take $\omega\in\mathcal{D}(\Delta_F^{(k)})$ with $\Delta^{(k)}\omega = 0$. Thanks to the boundary conditions, it follows that $d\omega=\delta\omega=0$. By an application of* [17, Theorem 3.4.4] *, we get $\omega=0$.*

In what follows, assume that $0\notin\mathrm{spec}_{ess}(\Delta_F^{(k)})$. To solve the problem (1.9), let us first observe that there is at most one solution $\omega\in(L^2\cap C^\infty)(M,\Lambda^kT^*M)$. Indeed, if $\omega\in(L^2\cap C^\infty)(M,\Lambda^kT^*M)$, and $\Delta^{(k)}\omega=0$, $\mathbf{t}\omega=0$, $\mathbf{n}\omega=0$, then by [17, Theorem 2.1.5], $\omega\in\mathcal{D}(\Delta_F^{(k)})$. Remark 1.6 implies that $\omega=0$.

To construct such a solution, let $F \in C^\infty(M, \Lambda^k T^*M)$ be an arbitrary form vanishing outside a bounded set, such that

$$\mathbf{t}F = f_1, \quad \mathbf{n}F = f_2.$$

Since $0 \notin \operatorname{spec}(\Delta_F^{(k)})$, there exists a unique $v \in \mathcal{D}(\Delta_F^{(k)})$ such that

$$\Delta_F^{(k)} v = -\Delta^{(k)} F.$$

Observe that by elliptic regularity, v is C^∞-smooth up to the boundary of M. We let $\omega = v + F$ be the unique solution of (1.9) in $(L^2 \cap C^\infty)(M, \Lambda^k T^*M)$.

Therefore, the Dirichlet-to-Neumann map for the k-form Laplacian is determined by

$$\begin{aligned}\Lambda_g^{(k)} &: C^\infty(\partial M, \Lambda^k T^*M) \times C^\infty(\partial M, \Lambda^{n-k} T^*M)\\ &\to C^\infty(\partial M, \Lambda^{n-k-1} T^*M) \times C^\infty(\partial M, \Lambda^{k-1} T^*M),\\ &\Lambda_g^{(k)}(f_1, f_2) = (\mathbf{n}d\omega, \mathbf{t}\delta\omega),\end{aligned}$$

where $\omega \in (L^2 \cap C^\infty)(M, \Lambda^k T^*M)$ is the solution to (1.9). Let Γ be a non-empty open subset of ∂M and $f_1 \in C^\infty(\partial M, \Lambda^k T^*M)$, $f_2 \in C^\infty(\partial M, \Lambda^{n-k} T^*M)$ be supported on Γ. Then we define the Dirichlet-to-Neumann map related to Γ by

$$\Lambda_{g,\Gamma}^{(k)}(f_1, f_2) = (\mathbf{n}d\omega|_\Gamma, \mathbf{t}\delta\omega|_\Gamma).$$

The following is our main result in the complete non-compact case.

Theorem 1.7. *Let M_1 and M_2 be complete non-compact real-analytic orientable connected Riemannian manifolds,* $\dim(M_1) = \dim(M_2) = n \geq 2$, *with compact real-analytic boundaries ∂M_1 and ∂M_2, respectively. Let the metrics g_1 and g_2 be real-analytic up to the boundary, and let $\Gamma_1 = \Gamma_2 = \Gamma$ be a non-empty open subset of ∂M_1 and ∂M_2.*

- *If $n \geq 3$, assume that* (A1) *holds. Suppose also that $0 \notin spec_{ess}(\Delta_{F,M_1}^{(k)})$ and $0 \notin spec_{ess}(\Delta_{F,M_2}^{(k)})$, and that for some integer $1 \leq k \leq n$,*

$$\Lambda_{g_1,\Gamma}^{(k)} = \Lambda_{g_2,\Gamma}^{(k)}.$$

- *If $n = 2$, assume that* (A2) *holds. Suppose also that $0 \notin spec_{ess}(\Delta_{F,M_1}^{(1)})$ and $0 \notin spec_{ess}(\Delta_{F,M_2}^{(1)})$, and that*

$$\Lambda_{g_1,\Gamma}^{(1)} = \Lambda_{g_2,\Gamma}^{(1)}.$$

Then M_1 and M_2 are isometric.

2. Examples in dimension two

The purpose of this section is to illustrate Theorem 1.7 by exhibiting an explicit example of two complete non-compact two-dimensional Riemannian manifolds, which can be distinguished from the knowledge of the set of the Cauchy data for harmonic 1-forms, while the sets of the Cauchy data for harmonic 0-forms, produced by both manifolds, are identical.

Let $M = S^1_\theta \times [0,+\infty)_z$. Here $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. We shall consider two complete metrics on M, belonging to the same conformal class,

$$g_1 = e^{-2z}(d\theta)^2 + (dz)^2,$$

and

$$g_2 = (1+f(z))g_1, \quad f(z) = ze^{-z}.$$

The Gaussian curvature of the manifold (M,g_1) is $K_{g_1} = -1$. We notice that (M,g_1) is isometric to the following surface of revolution in $\mathbb{R}^3$, equipped with the standard metric,

$$(z,\theta) \mapsto (h(z), e^{-z}\cos\theta, e^{-z}\sin\theta), \tag{2.1}$$

where

$$h(z) = \int_0^z \sqrt{1-e^{-2t}}dt.$$

Explicitly, the surface of revolution (2.1) is the familiar pseudosphere, obtained by rotating the curve

$$x(y) = -\sqrt{1-y^2} - \log y + \log(1+\sqrt{1-y^2}), \quad 0<y<1, \tag{2.2}$$

about the x-axis, see Figure 2.

To compute the Gaussian curvature K_{g_2} of (M,g_2), we use that

$$\Delta^{(0)}_{g_1}(\frac{1}{2}\log(1+f(z))) + K_{g_1} = K_{g_2}(1+f(z)),$$

see [2]. Now

$$\Delta^{(0)}_{g_1} = -e^{2z}\partial_\theta^2 - \partial_z^2 + \partial_z, \tag{2.3}$$

and a direct computation gives that

$$K_{g_2} = \frac{e^{-2z}(z+1-3z^2) + e^{-z}(3-6z) - 2}{2(1+ze^{-z})^3}.$$

In particular, $|K_{g_2}|$ is uniformly bounded on M. Furthermore, since $K_{g_1} \neq K_{g_2}$, the manifolds (M,g_1) and (M,g_2) are not isometric.

Let us now verify that zero is not in the spectrum of the Hodge Laplacian on 1-forms $\Delta^{(1)}_{g_1}$, provided with the domain (1.10). Writing $\omega(\theta,z) = \omega_1(\theta,z)d\theta +$

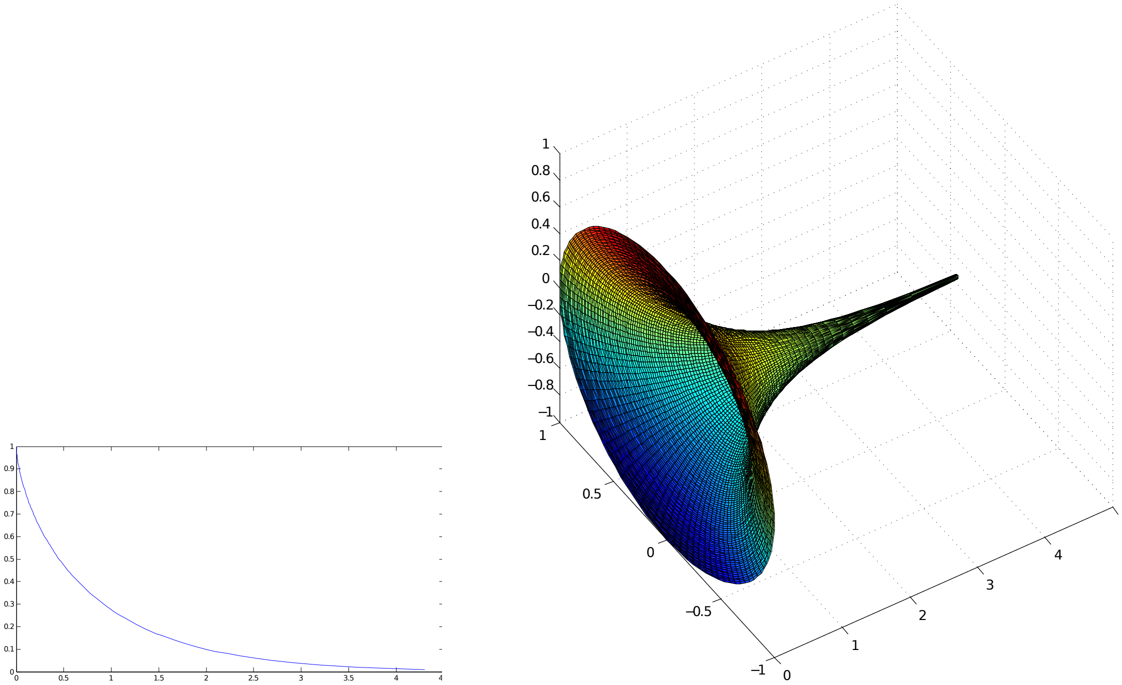

FIGURE 2. The surface of revolution on the right is obtained by rotating the tractrix (2.2), on the left, about the x-axis.

$\omega_2(\theta,z)dz$ for a 1-form ω, a straightforward computation, using the formula (4.4) given below, gives that

$$\Delta_{g_1}^{(1)} = \begin{pmatrix} -\partial_z^2 - e^{2z}\partial_\theta^2 - \partial_z & 2\partial_\theta \\ -2e^{2z}\partial_\theta & -\partial_z^2 - e^{2z}\partial_\theta^2 + \partial_z \end{pmatrix}.$$

This operator is considered in the Hilbert space

$$\mathcal{H} = L^2(S^1 \times \mathbb{R}_+; e^z d\theta dz) \times L^2(S^1 \times \mathbb{R}_+; e^{-z} d\theta dz), \tag{2.4}$$

equipped with the inner product

$$\langle \omega, \eta \rangle_{L^2} = \int_0^{2\pi} \int_0^{+\infty} (\omega_1 \bar{\eta}_1 e^z + \omega_2 \bar{\eta}_2 e^{-z}) d\theta dz.$$

It follows from Proposition 1.4 that the form domain of the operator $\Delta_{g_1}^{(1)}$ is given by

$$\mathcal{K} = \{\omega \in H^1(M, \Lambda^1 T^* M) : \mathbf{t}w = 0, \mathbf{n}w = 0\}.$$

Let us now obtain an explicit description of the space $\mathcal{K}$. Notice first that the boundary conditions take the form $\omega_1(\theta,0) = \omega_2(\theta,0) = 0$. To make the condition that $\omega \in H^1(M, \Lambda^1 T^* M)$ explicit, we shall use (1.11). We get

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{+\infty} (|\omega_1|^2 e^z + |\omega_2|^2 e^{-z}) d\theta dz < \infty,\\
&\int_0^{2\pi}\int_0^{+\infty} (e^{3z}|\partial_\theta \omega_1|^2 + e^z|\partial_z\omega_1|^2) d\theta dz < \infty,\\
&\int_0^{2\pi}\int_0^{+\infty} (e^{z}|\partial_\theta \omega_2|^2 + e^{-z}|\partial_z\omega_2|^2) d\theta dz < \infty.
\end{aligned} \tag{2.5}$$

It is now convenient to perform a conjugation by the unitary operator

$$U = \begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix},$$

in order to pass to the unweighted space $(L^2(S^1 \times \mathbb{R}_+; d\theta dz))^2$. We have

$$U \Delta^{(1)}_{g_1} U^{-1} = \begin{pmatrix} -\partial_z^2 + 1/4 - e^{2z}\partial_\theta^2 & 2e^z \partial_\theta \\ -2e^z\partial_\theta & -\partial_z^2 + 1/4 - e^{2z}\partial_\theta^2 \end{pmatrix},$$

and the conditions (2.5), when expressed in terms of

$$u_1 = e^{z/2}\omega_1 \in L^2(S^1 \times \mathbb{R}_+, d\theta dz), \quad u_2 = e^{-z/2}\omega_2 \in L^2(S^1 \times \mathbb{R}_+, d\theta dz),$$

become

$$\int_0^{+\infty}\int_0^{2\pi} (e^{2z}|\partial_\theta u_j|^2 + |\partial_z u_j|^2) dz d\theta < \infty, \quad j = 1, 2. \tag{2.6}$$

Taking a Fourier decomposition with respect to the variable $\theta \in S^1$, we write

$$U \Delta^{(1)}_{g_1} U^{-1} = \bigoplus_{k \in \mathbb{Z}} P_k, \quad P_k = \begin{pmatrix} -\partial_z^2 + 1/4 + e^{2z}k^2 & 2ike^z \\ -2ike^z & -\partial_z^2 + 1/4 + e^{2z}k^2 \end{pmatrix}. \tag{2.7}$$

It follows from (2.6) that the form domain of the operator P_k, $k \neq 0$, is given by

$$\{(u_1, u_2) \in (L^2(\mathbb{R}_+))^2 : \int_0^{+\infty} (|\partial_z u_j|^2 + e^{2z}|u_j|^2) dz < \infty, u_j(0) = 0, j = 1, 2\}. \tag{2.8}$$

Let us check that this space is compactly embedded in $(L^2(\mathbb{R}_+))^2$. Indeed, let $(f_l)_{l=1}^\infty$ be such that

$$\int_0^{+\infty} (|\partial_z f_l|^2 + e^{2z}|f_l|^2) dz \le 1, \quad l = 1, 2, \dots.$$

Then

$$\begin{aligned}
\|f_l - f_m\|^2_{L^2} &\le \int_0^R |f_l - f_m|^2 dz + e^{-R}\int_R^{+\infty} e^z |f_l - f_m|^2 dz\\
&\le \int_0^R |f_l - f_m|^2 dz + 4e^{-R}, \quad R > 0.
\end{aligned}$$

Since the embedding $H^1((0,R)) \to L^2((0,R))$ is compact for any $R > 0$, the claim follows. Hence, the contribution to the continuous spectrum of the operator $\Delta_{g_1}^{(1)}$ with the Dirichlet boundary conditions comes only from $k = 0$ in (2.7). Here we have the operator P_0, whose spectrum is $[1/4, +\infty)$ and is purely absolutely continuous. We conclude that zero is not in the spectrum of the self-adjoint realization of $\Delta_{g_1}^{(1)}$, constructed in Proposition 1.4. See also [5].

Let us now also show that zero is also not in the spectrum of $\Delta_{g_2}^{(1)}$, provided with the domain (1.10). To compute the coordinate expression for $\Delta_{g_2}^{(1)}$, we shall take advantage of the fact that the metric g_2 is a conformal multiple of g_1 and use the following identity,

$$\Delta_{g_2}^{(1)}\omega = \tau\Delta_{g_1}^{(1)}\omega - i_{\nabla_{g_1}\tau}d\omega + d\tau \wedge \delta_{g_1}\omega,$$

where

$$\tau(z) = \frac{1}{1+f(z)}$$

and i_X is the contraction of a differential form with a vector field X, see [2]. We get

$$\Delta_{g_2}^{(1)} = \tau(z)\Delta_{g_1}^{(1)} - \tau'(z)\begin{pmatrix} \partial_z & -\partial_\theta \\ e^{2z}\partial_\theta & \partial_z - 1 \end{pmatrix}.$$

Since $*_{g_2} = *_{g_1}$ on 1-forms, it follows that the operator $\Delta_{g_2}^{(1)}$ should be considered in the Hilbert space $\mathcal{H}$ in (2.4). As $1/2 < \tau(z) \le 1$, an explicit computation shows that the form domain of $\Delta_{g_2}^{(1)}$ is the same as the form domain of $\Delta_{g_1}^{(1)}$, i.e.

$$\{(\omega_1(\theta,z), \omega_2(\theta,z)) \in \mathcal{H} : (2.5)\text{ holds, } \omega_1(\theta,0) = \omega_2(\theta,0) = 0\}.$$

Conjugating by the unitary operator U, as before, we get

$$\begin{aligned} U\Delta_{g_2}^{(1)}U^{-1} &= \tau(z)\begin{pmatrix} -\partial_z^2 + 1/4 - e^{2z}\partial_\theta^2 & 2e^z\partial_\theta \\ -2e^z\partial_\theta & -\partial_z^2 + 1/4 - e^{2z}\partial_\theta^2 \end{pmatrix} \\ &\quad - \tau'(z)\begin{pmatrix} \partial_z - 1/2 & -e^z\partial_\theta \\ e^z\partial_\theta & \partial_z - 1/2 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} -\partial_z(\tau(z)\partial_z) + \frac{\tau(z)}{4} + \frac{\tau'(z)}{2} - \tau(z)e^{2z}\partial_\theta^2 & (2\tau(z) + \tau'(z))e^z\partial_\theta \\ -(2\tau(z) + \tau'(z))e^z\partial_\theta & -\partial_z(\tau(z)\partial_z) + \frac{\tau(z)}{4} + \frac{\tau'(z)}{2} - \tau(z)e^{2z}\partial_\theta^2 \end{pmatrix}.$$

Taking the Fourier decomposition with respect to the variable $\theta \in S^1$, we have

$$U\Delta_{g_2}^{(1)}U^{-1} = \bigoplus_{k\in\mathbb{Z}} L_k,$$

where L_k is defined by

$$\begin{pmatrix} -\partial_z(\tau(z)\partial_z) + \frac{\tau(z)}{4} + \frac{\tau'(z)}{2} + \tau(z)e^{2z}k^2 & (2\tau(z) + \tau'(z))e^z ik \\ -(2\tau(z) + \tau'(z))e^z ik & -\partial_z(\tau(z)\partial_z) + \frac{\tau(z)}{4} + \frac{\tau'(z)}{2} + \tau(z)e^{2z}k^2 \end{pmatrix}.$$

The form domain of the operator L_k, $k \neq 0$, is given by (2.8), and it is compactly embedded in $(L^2(\mathbb{R}_+))^2$. Hence, the spectrum of L_k, $k \neq 0$, is discrete. When $k = 0$, we have

$$L_0 = \left(-\partial_z(\tau(z)\partial_z) + \frac{\tau(z)}{4} + \frac{\tau'(z)}{2} \right) \otimes I.$$

Since the potential $\tau(z)/4 + \tau'(z)/2 \to 1/4$ as $z \to +\infty$, it follows from [4, Chapter 8] that the essential spectrum of L_0 is contained in $[1/4, +\infty)$. We conclude that 0 is not in the spectrum of $\Delta^{(1)}_{g_2}$.

An application of Theorem 1.7 shows therefore that the set of the Cauchy data for harmonic 1-forms allows us to distinguish between the Riemannian manifolds (M, g_1) and (M, g_2).

We shall conclude this example by pointing out that the manifolds (M, g_1) and (M, g_2) are undistinguishable on the level of the Cauchy data for harmonic 0-forms. Recall from Remark 1.5 that the Hodge Laplace operator $\Delta^{(0)}_{g_1}$ and $\Delta^{(0)}_{g_2}$, equipped with the domain, given by (1.10), are non-negative self-adjoint operators on $L^2(M)$. Using (2.3), we see as above that zero is not in the spectrum of $\Delta^{(0)}_{g_1}$. As

$$\Delta^{(0)}_{g_2} = (1 + f(z))^{-1} \Delta^{(0)}_{g_1}, \tag{2.9}$$

the same conclusion holds for the spectrum of $\Delta^{(0)}_{g_2}$. It follows that given $v \in C^\infty(\partial M)$, the boundary value problem

$$\begin{aligned} \Delta^{(0)}_{g_j} w_j &= 0, \quad \text{on } M, \\ \mathbf{t} w_j &= v, \end{aligned} \tag{2.10}$$

has a unique solution $w_j \in L^2(M, \mu_{g_j})$, $j = 1, 2$. Since $\mu_{g_2} = (1 + f(z))\mu_{g_1}$, we have $L^2(M, \mu_{g_2}) = L^2(M, \mu_{g_1})$ as spaces, with equivalent norms. In view of (2.9), we get $w_1 = w_2 = w$. Moreover, $\mathbf{n}_{g_2} dw = \mathbf{n}_{g_1} dw$. It follows that the manifolds (M, g_1) and (M, g_2) produce the same set of the Cauchy data of harmonic 0-forms.

Remark 2.1. *In the paper* [12]*, the case of the Laplacian on* 0*-forms on a complete non-compact manifold was considered. The construction of the solution to* (2.10) *in* [12] *is based on the maximum principle and leads to a solution in* $L^\infty(M)$*, whereas in the present paper* L^2*-methods are employed. The definition of the Dirichlet-to-Neumann map in* [12] *is consequently different from the definition used in this paper. Nevertheless, it is clear that the sets of the Cauchy data of the manifolds* (M, g_1) *and* (M, g_2)*, defined using the* L^∞*-approach, are identical.*

3. The case of a compact manifold. Proof of Theorem 1.2

3.1. **Reconstruction near the boundary.** Near Γ let us consider boundary normal coordinates given by a local coordinate chart $(x^1,\dots,x^n)$, where $x^n \geq 0$ is the distance to Γ and $x' = (x^1,\dots,x^{n-1})$ is a local chart on Γ. In these coordinates the metric g has the form, see e.g. [14],

$$g = \sum_{i,j=1}^{n-1} g_{ij}(x)dx^i dx^j + (dx^n)^2.$$

It was proved in [10] that for any integer k, $1 \leq k \leq n$, the set of Cauchy data $\mathcal{C}_g^k|_\Gamma$ determines all the normal derivatives $\partial_{x^n}^l g_{ij}(x',0)$, $l = 0,1,2,\dots$, of the metric tensor g at Γ.

By considering the Taylor expansion of the metric g near a given point on Γ, we extend the metric to a boundary collar $\Gamma \times (-r, 0]$, for $r > 0$ small enough, so that the extended metric remains real-analytic. This metric is also uniquely determined. We introduce the real-analytic manifold $\widetilde{M}$ obtained by attaching to M a boundary collar $\Gamma \times (-r, 0]$, equipped with this metric.

3.2. **Green's forms.** It is known [6, p.139] that for any $k = 1,\dots,n$, there is a Green's form $G(x,y)$, which is a double form of degree k, satisfying

$$\begin{aligned} &\Delta_x^{(k)} G(x,y) = \delta_{x,y} \quad \text{in} \quad \widetilde{M}, \\ &\mathbf{t}_x G(x,y) = 0 \quad \text{on} \quad \partial\widetilde{M}, \\ &\mathbf{n}_x G(x,y) = 0 \quad \text{on} \quad \partial\widetilde{M}, \end{aligned}$$

where $y \in \widetilde{M} \setminus \partial\widetilde{M}$ and

$$\delta_{x,y} = \sum_{\substack{i_1<\dots<i_k \\ j_1<\dots<j_k}} \delta(y-x) dx^{i_1} \wedge \dots \wedge dx^{i_k} \cdot dy^{j_1} \wedge \dots \wedge dy^{j_k}$$

is the delta double current, supported at $x = y$. See [16] for the notions of a double form and a double current.

Throughout the following discussion the degree k, $k = 1,\dots,n$, will be kept fixed. We have, $G(x,y) = G(y,x)$. Furthermore, as $x \to y$, $G(x,y)$ has the following asymptotic behavior (see [6, p.136])

$$G(x,y) \sim \frac{1}{(n-2)s_n d_{\widetilde{M}}(x,y)^{n-2}} \Gamma_{(i_1,\dots,i_k),(j_1,\dots,j_k)}(y) dx^{i_1} \wedge \dots \wedge dx^{i_k} \cdot dy^{j_1} \wedge \dots \wedge dy^{j_k}, \tag{3.1}$$

where $d_{\widetilde{M}}(x,y)$ is the geodesic distance from x to y in $\widetilde{M}$, s_n is the area of the unit n-sphere and

$$\Gamma_{(i_1,\dots,i_k),(j_1,\dots,j_k)}(y) = \begin{vmatrix} g_{i_1 j_1}(y) & \dots & g_{i_k j_1}(y) \\ \dots & \dots & \dots \\ g_{i_1 j_k}(y) & \dots & g_{i_k j_k}(y) \end{vmatrix}.$$

Here $\Gamma_{(i_1,\dots,i_k),(j_1,\dots,j_k)}(y)$ is positive real-analytic function on $\widetilde{M}$, as it is a Gram determinant. The asymptotic relation (3.1) can be differentiated with respect to x any number of times.

Lemma 3.1. *Let $y \in \widetilde{M} \setminus \partial\widetilde{M}$. Then the Green's form $G(x,y)$ is real-analytic for $x \in \widetilde{M} \setminus (\{y\} \cup \partial\widetilde{M})$.*

This lemma follows from [7, Theorem 4.1, p. 108], since $G(x,y)$ is a solution of an elliptic system of partial differential equations on $\widetilde{M} \setminus (\{y\} \cup \partial\widetilde{M})$ with real-analytic coefficients.

By the symmetry of the Green's forms, we have the following

Corollary 3.2. *Let $x \in \widetilde{M} \setminus \partial\widetilde{M}$. Then the Green's form $G(x,y)$ is real-analytic for $y \in \widetilde{M} \setminus (\{x\} \cup \partial\widetilde{M})$.*

We shall also need the following fact which follows from [6, Theorem 1].

Lemma 3.3. *Let $y \in \widetilde{M} \setminus \partial\widetilde{M}$. Then the Green's form $G(x,y)$ is C^∞-smooth up to the boundary of $\partial\widetilde{M}$, away from y.*

3.3. **Reconstruction of the Green's forms in the collar neighborhood of the boundary.** Recall that near Γ we consider a coordinate chart $(x^1,\dots,x^n)$ where x^n is the distance to Γ and $x' = (x^1,\dots,x^{n-1})$ is a chart on Γ. We write a k-form as

$$\omega = \sum_{i_1<\dots<i_k} \omega_{i_1,\dots,i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

The restriction of ω to Γ is given by

$$\omega|_\Gamma = \sum_{i_1<\dots<i_k} \omega_{i_1,\dots,i_k}|_\Gamma dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

The restriction of the normal derivative of ω to Γ is defined by

$$\partial_{x^n}\omega|_\Gamma = \sum_{i_1<\dots<i_k} \partial_{x^n}\omega_{i_1,\dots,i_k}|_\Gamma dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

Observe, as explained in [10], that given the induced metric on Γ and the first normal derivative of the metric on Γ, the knowledge of the Dirichlet data $(\mathbf{t}\omega, \mathbf{n}\omega)$ on Γ implies the knowledge of $\omega|_\Gamma$ and the knowledge of the Neumann data $(\mathbf{t}\delta\omega, \mathbf{n}d\omega)$ on Γ determines $\partial_{x^n}\omega|_\Gamma$.

For future references, let us also recall that the Hodge Laplacian on k-forms is a second order elliptic partial differential system, with a scalar principal symbol, which in local coordinates given by

$$\Delta^{(k)} = -(g^{jk}(x)\partial_{x^j}\partial_{x^k}) \otimes I + B_j(x)\partial_{x^j} + C(x), \tag{3.2}$$

where $(g^{jk}) = (g_{jk})^{-1}$, I is $d \times d$ identity matrix, and $B_j(x)$, $C(x)$ are real-analytic functions with values in the set of $d \times d$-matrices, $d = \begin{pmatrix} n \\ k \end{pmatrix}$.

Denote $U = \widetilde{M}_1 \setminus M_1 = \widetilde{M}_2 \setminus M_2$. Strictly speaking, here the manifolds $\widetilde{M}_1 \setminus M_1$ and $\widetilde{M}_2 \setminus M_2$ are identified by a real-analytic isometry.

Lemma 3.4. *The Green's forms $G_j(x,y)$ satisfy*

$$G_1(x,y) = G_2(x,y), \quad (x,y) \in U \times U \setminus \{x = y\}.$$

Proof. Let $y \in U$ and consider the following Dirichlet problem on M_2,

$$\begin{aligned}
&\Delta^{(k)}_{g_2}\omega(x) = 0, \quad x \in M_2,\\
&\mathbf{t}\omega(x) = \mathbf{t}_x G_1(x,y), \quad x \in \Gamma,\\
&\mathbf{n}\omega(x) = \mathbf{n}_x G_1(x,y), \quad x \in \Gamma,\\
&\mathbf{t}\omega(x) = 0, \quad x \in \partial M_2 \setminus \Gamma,\\
&\mathbf{n}\omega(x) = 0, \quad x \in \partial M_2 \setminus \Gamma.
\end{aligned}$$

Notice that in this problem the boundary data are continuous. By [17, Theorem 3.4.10] the problem above is uniquely solvable, and the hypothesis that $\mathcal{C}_{g_2}|_\Gamma = \mathcal{C}_{g_1}|_\Gamma$ implies that

$$\begin{aligned}
&\mathbf{n}d\omega(x) = \mathbf{n}_x dG_1(x,y), \quad x \in \Gamma,\\
&\mathbf{t}\delta\omega(x) = \mathbf{t}_x \delta G_1(x,y), \quad x \in \Gamma.
\end{aligned}$$

Let

$$\widetilde{\omega}(x) = \begin{cases} \omega(x), & x \in M_2,\\ G_1(x,y), & x \in U. \end{cases}$$

Now since the Cauchy data of $\omega(x)$ and $G_1(x,y)$ coincide on Γ, it follows from (3.2) that $\widetilde{\omega}$ satisfies

$$\begin{aligned}
&\Delta^{(k)}_{g_2}\widetilde{\omega}(x) = \delta_{x,y}, \quad x \in \widetilde{M}_2,\\
&\mathbf{t}\widetilde{\omega}(x) = 0, \quad x \in \partial\widetilde{M}_2,\\
&\mathbf{n}\widetilde{\omega}(x) = 0, \quad x \in \partial\widetilde{M}_2.
\end{aligned}$$

Thus, $\widetilde{\omega}(x) - G_2(x,y)$ solves the homogeneous Dirichlet problem on $\widetilde{M}_2$. Since the manifold $\widetilde{M}_2$ is compact, the solution to the Dirichlet problem is unique, cf. [6, Theorem 1], i.e. $\widetilde{\omega}(x) = G_2(x,y)$ for $x \in \widetilde{M}_2 \setminus \{y\}$.

□

3.4. **Embedding of the manifold into a Sobolev space.** Following [12] we will prove Theorem 1.2 via certain embeddings of $\widetilde{M}_j$ into a Sobolev space given by the Green's forms. Let $\widetilde{U} \subset\subset U$ be a relatively compact subset of U. Then for some $s < 1 - n/2$, we define the maps

$$\mathcal{G}_j : \widetilde{M}_j \to H^s(\widetilde{U}, \Lambda^k T^* \widetilde{U}), \quad x \mapsto G_j(x, \cdot). \tag{3.3}$$

Proposition 3.5. *The maps $\mathcal{G}_j$ are $C^1(\widetilde{M}_j, H^s(\widetilde{U}, \Lambda^k T^* \widetilde{U}))$, for $s < 1 - n/2$.*

Proof. Let V be a relatively compact subset of U such that

$$\widetilde{U} \subset\subset V \subset\subset U.$$

Assume first that $x \in \widetilde{M}_j \setminus V$. Then it is known [20, Chapter 7] that

$$G_j(x, y) \in C^\infty(\widetilde{M}_j \setminus V \times \overline{\widetilde{U}}, \Lambda^k T^*(\widetilde{M}_j \setminus V \times \overline{\widetilde{U}})).$$

Hence,

$$\mathcal{G}_j|_{\widetilde{M}_j \setminus V} \in C^\infty(\widetilde{M}_j \setminus V, H^l(\widetilde{U}, \Lambda^k T^* \widetilde{U})), \quad \text{for any } l. \tag{3.4}$$

Let now $x \in V$. It is easily seen that the following map is C^1,

$$V \ni x \mapsto \delta_{x,y} \in H^{s-2}(V, \Lambda^k T^* V),$$

since we have assumed that $s < 1 - n/2$. As $G_j(x, y) = G_j(x, y)$, we get

$$\Delta_y^{(k)} G_j(x, y) = \delta_{x,y}, \quad y \in V,$$

which implies that $G_j(x, \cdot) \in H^s(\widetilde{U}, \Lambda^k T^* \widetilde{U})$. To estimate $\|G_j(x, \cdot)\|_{H^s(\widetilde{U}, \Lambda^k T^* \widetilde{U})}$, let us take a properly supported parametrix E_y of $\Delta_y^{(k)}$ in V so that

$$E_y \Delta_y^{(k)} - 1 = R_y,$$

where R_y is smoothing in V. Hence,

$$\|G_j(x, \cdot)\|_{H^s(\widetilde{U}, \Lambda^k T^* \widetilde{U})} \le \|E_y \delta_{x,y}\|_{H^s(\widetilde{U}, \Lambda^k T^* \widetilde{U})} + \|R_y G_j(x, \cdot)\|_{H^s(\widetilde{U}, \Lambda^k T^* \widetilde{U})}.$$

As

$$E_y : H^{s-2}_{comp}(V, \Lambda^k T^* V) \to H^s_{comp}(V, \Lambda^k T^* V)$$

is bounded, we get

$$\|E_y \delta_{x,y}\|_{H^s(\widetilde{U}, \Lambda^k T^* \widetilde{U})} \le C \|\delta_{x,y}\|_{H^{s-2}(V, \Lambda^k T^* V)}.$$

The asymptotic behavior of the Green's forms (3.1) implies that

$$G_j(x, \cdot) \in C^1(V, L^1_{loc}(V, \Lambda^k T^* V)).$$

This and the fact that R_y is smoothing yield that

$$R_y G(x, \cdot) \in C^1(V, L^2(\widetilde{U}, \Lambda^k T^* \widetilde{U})).$$

Hence,

$$\|R_y G_j(x,\cdot)\|_{H^s(\widetilde U,\Lambda^k T^*\widetilde U)} \le \|R_y G_j(x,\cdot)\|_{L^2(\widetilde U,\Lambda^k T^*\widetilde U)}.$$

This shows that the function

$$V \ni x \mapsto G_j(x,\cdot) \in H^s(\widetilde U, \Lambda^k T^*\widetilde U) \tag{3.5}$$

is C^1. Combining (3.4) and (3.5), we get the claim.

□

Lemma 3.6. *The map $\mathcal{G}_j : \widetilde{M_j} \to H^s(\widetilde U, \Lambda^k T^*\widetilde U)$ defined by* (3.3) *is an embedding. Moreover, $\mathcal{G}_j$ is real-analytic on $\widetilde{M_j} \setminus \overline{\widetilde U}$.*

Proof. As the manifold $\widetilde{M_j}$ is compact, the fact that $\mathcal{G}_j$ is an embedding is equivalent to the fact that $\mathcal{G}_j$ is an injective immersion.

Let us first show that $\mathcal{G}_j$ is immersion. Suppose that there is a point $x_0 \in \widetilde{M_j}$ such that the derivative of $\mathcal{G}_j$,

$$\begin{aligned} D\mathcal{G}_j(x_0) &: T_{x_0}\widetilde{M_j} \to H^s(\widetilde U, \Lambda^k T^*\widetilde U), \\ D\mathcal{G}_j(x_0)v &= v^i \frac{\partial}{\partial x^i} G_j(x,\cdot)|_{x_0}, \end{aligned}$$

where $v = v^i(\partial/\partial x^i) \in T_{x_0}\widetilde{M_j}$, is not injective. Thus, there is a vector $0 \neq v \in T_{x_0}\widetilde{M_j}$ such that $v^i(\partial/\partial x^i)G_j(x_0,y) = 0$ for all $y \in \widetilde U$. By the real-analyticity of the Green's forms, we have that $v^i(\partial/\partial x^i)G_j(x_0,y) = 0$ for all $y \in \widetilde{M_j} \setminus \{x_0\}$. Since this contradicts the asymptotic relation (3.1) when y tends to x_0, we get that $\mathcal{G}_j$ is immersion.

Let us now show that $\mathcal{G}_j$ is injective. Assume that this is not the case, then there are $x_1 \neq x_2$ in $\widetilde{M_j}$ such that

$$G_j(x_1,y) = G_j(x_2,y) \tag{3.6}$$

for all $y \in \widetilde U$. By analyticity, (3.6) holds for all $y \in \widetilde{M_j} \setminus \{x_1,x_2\}$. Now the asymptotic (3.1) implies that $G_j(x_1,\cdot)$ is singular only at $y = x_1$ and $G_j(x_2,\cdot)$ is singular only at $y = x_2$. Thus, $x_1 = x_2$.

□

In the rest of this section we shall prove the following result which implies Theorem 1.2.

Theorem 3.7. *Assume that the Green's forms $G_j(x,y)$ satisfy*

$$G_1(x,y) = G_2(x,y), \quad (x,y) \in \widetilde U \times \widetilde U \setminus \{x=y\}, \tag{3.7}$$

where $\widetilde U \subset\subset U$. Thus,

$$\mathcal{G}_1(\widetilde{M_1}) = \mathcal{G}_2(\widetilde{M_2}) \subset H^s(\widetilde U, \Lambda^k T^*\widetilde U)$$

and the map

$$(\mathcal{G}_2)^{-1}\mathcal{G}_1 : \widetilde{M}_1 \to \widetilde{M}_2$$

is an isometry.

Before proving Theorem 3.7 let us introduce some notation. Let

$$N(\varepsilon_0) = \{x \in \widetilde{M}_1 : d_{\widetilde{M}_1}(x, \partial\widetilde{M}_1) \le \varepsilon_0\},$$
$$C(\varepsilon_0) = \{x \in \widetilde{M}_1 : d_{\widetilde{M}_1}(x, \partial\widetilde{M}_1) > \varepsilon_0\},$$

where $\varepsilon_0 > 0$ is arbitrary and small enough so that $C(\varepsilon_0)$ is connected. Let $x_0 \in \widetilde{U} \cap C(\varepsilon_0)$ and $B_1 \subset C(\varepsilon_0)$ be the largest connected open set containing x_0 such that $\mathcal{G}_1(x) \in \mathcal{G}_2(\widetilde{M}_2)$ for all $x \in B_1$. The existence of the set B_1 follows from (3.7). Thus, as $\mathcal{G}_2$ is injective, we can define the map

$$J = (\mathcal{G}_2)^{-1}\mathcal{G}_1 : B_1 \to \widetilde{M}_2.$$

Now let $D_1 \subset B_1$ be the largest connected open set containing x_0 for which J is a local isometry, i.e., $g_1 = J^*g_2$ and J is a real-analytic local diffeomorphism. The existence of the set D_1 follows from the construction of the manifold U.

In order to prove Theorem 3.7 we will show that $N(\varepsilon_0) \cup D_1 = \widetilde{M}_1$. Assume the contrary, i.e. $\widetilde{M}_1 \setminus (N(\varepsilon_0) \cup D_1) \neq \emptyset$. Let $x_1 \in \widetilde{M}_1 \setminus (N(\varepsilon_0) \cup D_1)$ be a point closest to x_0. Clearly $x_1 \in \partial D_1$.

Lemma 3.8. *There exists $x_2 \in \widetilde{M}_2^{int}$ such that*

$$\mathcal{G}_2(x_2) = \mathcal{G}_1(x_1) = u \in H^s(\widetilde{U}, \Lambda^k T^*\widetilde{U}),$$

and there exists a sequence $p_i \in D_1$ such that

$$\lim_{i\to\infty} p_i = x_1, \quad \lim_{i\to\infty} J(p_i) = x_2.$$

Proof. As $x_1 \in \partial D_1$, there is a sequence $p_i \in D_1$ such that $p_i \to x_1$ when $i \to \infty$. Since J is a local isometry on D_1, there is a sequence $q_i \in \widetilde{M}_2$ such that $\mathcal{G}_2(q_i) = \mathcal{G}_1(p_i)$. Since $\widetilde{M}_2$ is compact, the sequence q_i has a convergent subsequence. If there is a convergent subsequence of q_i which converges to an interior point of $\widetilde{M}_2$, then by continuity of $\mathcal{G}_1$ and $\mathcal{G}_2$, we get the claim of the lemma.

Assume now that for all convergent subsequences of q_i, we have

$$q_{i_k} \to q_0 \in \partial\widetilde{M}_2, \quad k \to \infty. \tag{3.8}$$

Lemma 3.3 implies that for a fixed point $y \in \widetilde{U}$,

$$G_2(q_{i_k}, y) \to G_2(q_0, y) = 0, \quad k \to \infty.$$

As $G_1(p_{i_k}, y) = G_2(q_{i_k}, y)$ for $y \in \widetilde{U}$, we have $G_1(x_1, y) = 0$ for all $y \in \widetilde{U}$. By analyticity, $G_1(x_1, y) = 0$ for all $y \neq x_1$. But this contradicts the asymptotic behavior (3.1) of the Green's form near $y = x_1$. Hence, (3.8) cannot be valid.

□

Now we assume that $\widetilde{U} \subset\subset U$ is chosen so that $x_j \notin \overline{\widetilde{U}}$, $j = 1, 2$. Then $\mathcal{G}_j$ is an analytic embedding in a neighborhood of x_j. Notice that $\widetilde{U} \subset D_1$.

Lemma 3.9. *We have*

$$D\mathcal{G}_1(x_1)(T_{x_1}\widetilde{M}_1) = D\mathcal{G}_2(x_2)(T_{x_2}\widetilde{M}_2) \subset H^s(\widetilde{U}, \Lambda^k T^*\widetilde{U}).$$

Proof. By Lemma 3.8, there is a sequence $(p_i)_{i=1}^{\infty}$, $p_i \in D_1$, such that $p_i \to x_1$ when $i \to \infty$ and $J(p_i) \to x_2$ when $i \to \infty$. Now by the definition of the set D_1, the maps $\mathcal{G}_1$ and $\mathcal{G}_2 J$ coincide in D_1 and hence so do their differentials, i.e.

$$D\mathcal{G}_1|_{D_1} = D\mathcal{G}_2|_{J(D_1)}.$$

Thus,

$$D\mathcal{G}_1(p_j)(T_{p_i}\widetilde{M}_1) = D\mathcal{G}_2(J(p_j))(T_{J(p_i)}\widetilde{M}_2).$$

Since by the choice of s, the differentials $D\mathcal{G}_1$ and $D\mathcal{G}_2$ are continuous on $\widetilde{M}_1$ and $\widetilde{M}_2$, we get

$$D\mathcal{G}_1(x_1)(T_{x_1}\widetilde{M}_1) = D\mathcal{G}_2(x_2)(T_{x_2}\widetilde{M}_2).$$

□

We set

$$\mathcal{V} = D\mathcal{G}_1(x_1)(T_{x_1}\widetilde{M}_1) = D\mathcal{G}_2(x_2)(T_{x_2}\widetilde{M}_2) \subset H^s(\widetilde{U}, \Lambda^k T^*\widetilde{U}).$$

Since $D\mathcal{G}_j$ are injective, we have

$$\dim \mathcal{V} = \dim T_{x_j}\widetilde{M}_j = n.$$

Let

$$P : H^s(\widetilde{U}, \Lambda^k T^*\widetilde{U}) \to \mathcal{V}$$

be the orthogonal projection, with respect to the Hilbert space structure of H^s. Consider the map

$$P\mathcal{G}_j : \widetilde{M}_j \to \mathcal{V}, \quad x \mapsto PG_j(x, \cdot).$$

The derivative of this map at the point x_j is

$$DP\mathcal{G}_j(x_j) = P(D\mathcal{G}_j(x_j)) = D\mathcal{G}_j(x_j),$$

which is bijective. By the inverse function theorem,

$$P\mathcal{G}_j : \text{neigh}(x_j, \widetilde{M}_j) \to \text{neigh}(u, \mathcal{V})$$

is a real-analytic diffeomorphism. Writing

$$\mathcal{G}_j(x) = P\mathcal{G}_j(x) + (1-P)\mathcal{G}_j(x),$$

we see that the set $\mathcal{G}_j(\widetilde{M}_j)$ can be represented as follows,

$$(P\mathcal{G}_j(\widetilde{M}_j), (1-P)\mathcal{G}_j(\widetilde{M}_j)).$$

Thus, locally the set

$$\{\mathcal{G}_j(x) : x \in \text{neigh}(x_j, \widetilde{M}_j)\}$$

can be represented as a graph

$$\{(v, (1-P)\mathcal{G}_j(P\mathcal{G}_j)^{-1}v) : v \in \text{neigh}(u, \mathcal{V})\}$$

of the real-analytic function $(1-P)\mathcal{G}_j(P\mathcal{G}_j)^{-1}$ in $\text{neigh}(u, \mathcal{V})$.

Let us now show that x_1 is an interior point of B_1. To this end, we shall prove that for any point $\widetilde{x}_1 \in \text{neigh}(x_1, \widetilde{M}_1)$, there is a unique point $\widetilde{x}_2 \in \text{neigh}(x_2, \widetilde{M}_2)$ such that

$$\mathcal{G}_1(\widetilde{x}_1) = \mathcal{G}_2(\widetilde{x}_2). \tag{3.9}$$

As the sets $\mathcal{G}_j(\text{neigh}(x_j, \widetilde{M}_j))$ are the graphs of analytic functions, (3.9) is equivalent to the fact that $P\mathcal{G}_2(\widetilde{x}_2) = P\mathcal{G}_1(\widetilde{x}_1)$. Thus,

$$\widetilde{x}_2 = (P\mathcal{G}_2)^{-1}(P\mathcal{G}_1)(\widetilde{x}_1)$$

and (3.9) follows.

Therefore, in $\text{neigh}(x_1, \widetilde{M}_1)$, the map

$$J = (P\mathcal{G}_2)^{-1}(P\mathcal{G}_1) \tag{3.10}$$

is real-analytic.

To summarize, we have proved that the map

$$J : D_1 \cup \text{neigh}(x_1, \widetilde{M}_1) \to \widetilde{M}_1$$

is real-analytic. Here we assume that the set $\text{neigh}(x_1, \widetilde{M}_1)$ is connected. Moreover, when restricted to $\widetilde{U} \subset D_1$, J is a local isometry. An application of [11, Lemma 3', p. 256] allows us to conclude that J is a local isometry on $D_1 \cup \text{neigh}(x_1, \widetilde{M}_1)$. Therefore, x_1 is an interior point of D_1. This contradicts our assumption and, thus, $N(\varepsilon_0) \cup D_1 = \widetilde{M}_1$. As ε_0 can be chosen arbitrarily small and by the choice of D_1, we have that $\mathcal{G}_1(\widetilde{M}_1) = \mathcal{G}_2(\widetilde{M}_2)$. Thus, $J : \widetilde{M}_1 \to \widetilde{M}_2$ is bijective, and, therefore, J is an isometry. This proves Theorem 3.7 and, hence, Theorem 1.2.

Remark. The point of the following remark is to provide an alternative approach to showing that J is an isometry in a neighborhood of x_1, by relying upon the asymptotic behavior of the Green's forms. Here we follow an idea of [12].

Let us observe that (3.9) and (3.10) give that

$$G_2(J(x),J(y)) = G_1(x,y), \quad x \in \mathrm{neigh}(x_1,\widetilde{M}_1),\ y \in \widetilde{U}.$$

Here we used that $J(y) = y$ in $\widetilde{U}$. As J is real-analytic in $D_1 \cup \mathrm{neigh}(x_1,\widetilde{M}_1)$ and Green's forms are real-analytic, we get

$$G_2(J(x),J(y)) = G_1(x,y), \quad x,y \in \mathrm{neigh}(x_1,\widetilde{M}_1),\ x \neq y. \tag{3.11}$$

Using the asymptotic behavior (3.1) of the Green's forms when x is near y, we shall prove that J is an isometry in the neighborhood of y. In order to do that let us recall the relation between the Riemannian metric tensor and distance function. For any $\xi \in T_y\widetilde{M}_1$, there is a unique geodesic $\mu : \mathbb{R} \to \widetilde{M}_1$ such that $\mu(0) = y$, $\dot\mu(0) = \xi$, and

$$d_{\widetilde{M}_1}(\mu(0),\mu(t)) = \int_0^t \sqrt{g_1(\dot\mu(s),\dot\mu(s))}ds, \quad \text{for } |t| \text{ small enough.}$$

Thus,

$$\lim_{t\to 0} \frac{d_{\widetilde{M}_1}(\mu(0),\mu(t))}{t} = g_1(\xi,\xi)^{1/2}. \tag{3.12}$$

It follows from (3.1) that as $x \to y$, we have

$$G_2(J(x),J(y)) \sim \frac{1}{(n-2)s_n d_{\widetilde{M}_2}(J(x),J(y))^{n-2}} \Gamma^{g_2}_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(J(y))$$
$$(L_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(y))^2 dx^{i_1} \wedge \cdots \wedge dx^{i_k} \cdot dy^{j_1} \wedge \cdots \wedge dy^{j_k},$$

where $L_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(y)$ are certain $k \times k$-minors of the Jacobian of $J(y)$. There are indices $(i_1,\ldots,i_k),(j_1,\ldots,j_k)$ such that $L_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(x_1) \neq 0$. Thus, shrinking the neighborhood of x_1, if necessary, we may assume that

$$L_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(y) \neq 0, \quad \forall y \in \mathrm{neigh}(x_1,\widetilde{M}_1).$$

Hence, (3.11) implies that

$$\lim_{x\to y} \frac{d_{\widetilde{M}_2}(J(x),J(y))}{d_{\widetilde{M}_1}(x,y)} = f(y), \tag{3.13}$$

where

$$f(y) = \left(\frac{\Gamma^{g_2}_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(J(y))(L_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(y))^2}{\Gamma^{g_1}_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(y)} \right)^{1/(n-2)}$$

is positive real-analytic function in $\mathrm{neigh}(x_1,\widetilde{M}_1)$.

Combining (3.12) and (3.13), we get

$$\frac{g_2(J(y))(J'(y)\xi, J'(y)\xi)}{g_1(y)(\xi,\xi)} = f(y)^2, \tag{3.14}$$

for any $y \in \text{neigh}(x_1, \widetilde{M}_1)$ and $\xi \in T_y\widetilde{M}_1$. As both right and left-hand sides of (3.14) are real-analytic in $\text{neigh}(x_1, \widetilde{M}_1) \cup D_1$, (3.14) holds also on D_1. Since

$$J^* g_2 = g_1, \quad \text{on} \quad \widetilde{U} \subset D_1,$$

we have $f(y) \equiv 1$ on $\text{neigh}(x_1, \widetilde{M}_1)$. Hence, J is an isometry in a neighborhood of x_1.

4. The case of a compact 2-dimensional manifold. Proof of Theorem 1.3

Let (M, g) be a smooth compact connected Riemannian manifold of dimension $n = 2$ with a smooth non-empty boundary. Let (x^1, x^2) be the boundary normal coordinates defined locally near a point at the boundary. Here x^1 is a local coordinate for ∂M and $x^2 \geq 0$ is the distance to the boundary. In these coordinates, the metric has the following form

$$g = g_{11}(x^1, x^2)(dx^1)^2 + (dx^2)^2.$$

Recall that the Hodge star isomorphism is defined by

$$\omega \wedge *\omega = g(\omega, \omega)\mu, \tag{4.1}$$

where $\mu \in C^\infty(M, \Lambda^2 T^* M)$ is the volume form given in the boundary normal coordinates by

$$\mu = \sqrt{g_{11}} dx^1 \wedge dx^2,$$

assuming that dx^1, dx^2 is a positive basis of $T^*_x M$. Thus, (4.1) implies that

$$dx^1 \wedge *dx^1 = g^{11}\sqrt{g_{11}} dx^1 \wedge dx^2,$$

and therefore,

$$* \, dx^1 = \sqrt{g^{11}} dx^2, \quad *dx^2 = -\sqrt{g_{11}} dx^1, \quad *(dx^1 \wedge dx^2) = \sqrt{g^{11}}. \tag{4.2}$$

Here $g^{11} = g_{11}^{-1}$.

4.1. **The case of 0-forms.**

Let $\widetilde{g}$ be a smooth Riemannian metric on M in the same conformal class as g, i.e. there is a smooth real-valued function φ on M such that

$$\widetilde{g} = e^{2\varphi} g.$$

Then it is known [2, Chapter 1.J] that under the conformal change, the Hodge star operator on k-forms satisfies

$$*_{\widetilde{g}} = e^{(2-2k)\varphi} *_g . \tag{4.3}$$

It follows from (4.3) that the Hodge Laplacian on 0-forms is conformally invariant, i.e.

$$\Delta^{(0)}_{\widetilde{g}} = e^{-2\varphi} \Delta^{(0)}_g.$$

Moreover, $\mathcal{C}_{\widetilde{g}}^{(0)} = \mathcal{C}_g^{(0)}$.

When computing the set of the Cauchy data $\mathcal{C}_g^{(0)}$ in boundary normal coordinates, we get, when $\omega \in C^\infty(M)$,

$$\mathbf{t}\omega = \omega|_{x^2=0},$$
$$\mathbf{n}_g d\omega = -\sqrt{g_{11}(x^1,0)}\partial_{x^2}\omega|_{x^2=0}dx^1.$$

Notice that the set of the Cauchy data $\mathcal{C}_g^{(0)}$ is the graph of the Dirichlet-to-Neumann map as defined in [14]. As explained in [14], $\mathcal{C}_g^{(0)}$ does not contain any information about the metric along the boundary. However, if one considers the Dirichlet-to-Neumann map as a function valued map by setting

$$\omega|_{x^2=0} \mapsto \partial_{x^2}\omega|_{x^2=0},$$

then it contains the information about the restriction of the metric to the boundary, see [14].

4.2. **The case of 1-forms.** Given a 1-form $\omega = \omega_1 dx^1 + \omega_2 dx^2$, we shall identify ω with the vector $(\omega_1, \omega_2)^{\mathrm{t}}$. An explicit computation using (4.2) shows that in boundary normal coordinates, the Hodge Laplacian on 1-forms has the form

$$\Delta^{(1)} = D_{x^2}^2 I + g^{11} D_{x^1}^2 I + iE(x)D_{x^2} + iF(x)D_{x^1} + Q(x), \tag{4.4}$$

where $D_{x^j} = \frac{1}{i}\partial_{x^j}$, $j = 1, 2$, and

$$E(x) = \frac{\partial_{x^2} g^{11}}{2g^{11}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$F(x) = \begin{pmatrix} -\frac{3}{2}\partial_{x^1} g^{11} & \frac{\partial_{x^2} g^{11}}{g^{11}} \\ -\partial_{x^2} g^{11} & -\frac{\partial_{x^1} g^{11}}{2} \end{pmatrix}, \quad Q(x) = \begin{pmatrix} -\frac{\partial^2_{x^1} g^{11}}{2} & \frac{1}{2}\partial^2_{x^1 x^2} \log g^{11}, \\ -\frac{\partial^2_{x^1 x^2} g^{11}}{2} & \frac{1}{2}\partial^2_{x^2} \log g^{11} \end{pmatrix}.$$

Here I is the 2×2 identity matrix.

Our next goal is to show that the knowledge of the Cauchy data for harmonic 1-forms determines the Taylor series at the boundary of the metric g in the boundary normal coordinates. Let us point out that the reconstruction of the Taylor series in this case becomes possible, thanks to the special structure of the lower order terms in $\Delta^{(1)}$. We shall need the following lemma.

Lemma 4.1. *There exists a matrix-valued pseudodifferential operator $A(x, D_{x^1})$ of order one in x^1 depending smoothly on x^2 such that*

$$\Delta^{(1)} = (D_{x^2} I + iE(x) - iA(x, D_{x^1}))(D_{x^2} I + iA(x, D_{x^1})), \tag{4.5}$$

modulo a smoothing operator. Here $A(x, D_{x^1})$ is unique modulo a smoothing term, if we require that its principal symbol satisfies

$$\sigma(A((x^1,0), D_{x^1})) = -\sqrt{g^{11}(x^1,0)}|\xi_1|I.$$

Proof. The existence of the factorization (4.5) follows closely [14, Proposition 1.1], where the case of the Laplacian on functions is considered.

Combining (4.4) and (4.5), we see that

$$A^2 + i[D_{x^2}I, A] - E(x)A = g^{11}D_{x^1}^2 I + iF(x)D_{x^1} + Q(x), \tag{4.6}$$

modulo a smoothing operator. Let us write the full symbol of $A(x, D_{x^1})$ as follows,

$$A(x, \xi_1) \sim \sum_{j=-\infty}^{1} a_j(x, \xi_1),$$

with a_j taking values in 2×2 matrices with entries homogeneous of degree j in ξ_1. Thus, (4.6) implies that

$$\sum_{l=-\infty}^{2} \Big(\sum_{\substack{j+k-\alpha=l \\ \alpha\ge 0,\ j,k\le 1}} \frac{1}{\alpha!}\partial_{\xi_1}^{\alpha} a_j D_{x^1}^{\alpha} a_k\Big) - E(x) \sum_{j=-\infty}^{1} a_j(x,\xi_1) + \sum_{j=-\infty}^{1} \partial_{x^2} a_j(x,\xi_1) \tag{4.7}$$

$$= g^{11}\xi_1^2 I + iF(x)\xi_1 + Q(x).$$

Equating the terms homogeneous of degree two in (4.7), we get

$$a_1^2(x, \xi_1) = g^{11}(x)\xi_1^2 I,$$

so we should choose $a_1(x, \xi_1)$ to be the scalar matrix given by

$$a_1(x, \xi_1) = -\sqrt{g^{11}(x)}|\xi_1| I.$$

Equating the terms homogeneous of degree one in (4.7), we have an equation for $a_0(x, \xi_1)$,

$$2a_1a_0 = iF(x)\xi_1 + E(x)a_1 - \partial_{x^2}a_1 - \partial_{\xi_1}a_1 D_{x^1}a_1, \tag{4.8}$$

which is uniquely solvable for a_0 given our choice of a_1. We shall now consider terms homogeneous of degree zero in (4.7),

$$2a_1a_{-1} = Q(x) + E(x)a_0 - \partial_{x^2}a_0 - \sum_{\substack{j+k-\alpha=0 \\ \alpha\ge 0,\ 0\le j,k\le 1}} \frac{1}{\alpha!}\partial_{\xi_1}^{\alpha} a_j D_{x^1}^{\alpha} a_k), \tag{4.9}$$

which is uniquely solvable for a_{-1} given our choice of a_1 and a_0.

Proceeding recursively with respect to the degree of homogeneity in (4.7), we choose, for $m \ge 1$,

$$a_{-m-1} = \frac{1}{2}a_1^{-1}(Ea_{-m} - \partial_{x^2}a_{-m} - \sum_{\substack{j+k-\alpha=-m \\ \alpha\ge 0, -m\le j,k\le 1}} \frac{1}{\alpha!}\partial_{\xi_1}^{\alpha} a_j D_{x^1}^{\alpha} a_k).$$

This completes the proof. □

Proposition 4.2. *The knowledge of the subset of the set of Cauchy data on 1-forms given by*

$$\{(\mathbf{t}\omega, \mathbf{n}d\omega) : \omega \in C^\infty(M, \Lambda^1 T^*M), \Delta^{(1)}\omega = 0, \mathbf{n}\omega = 0\},$$

determines the full Taylor series at the boundary of the metric g in the boundary normal coordinates.

Proof. We shall first show how to recover the restriction of the metric g to the boundary. To this end, we shall compute the Cauchy data of a harmonic 1-form $\omega = \omega_1 dx^1 + \omega_2 dx^2$ in boundary normal coordinates. Using (4.2), we get

$$\begin{aligned}
\mathbf{t}\omega &= \omega_1(x_1, 0)dx^1, \\
\mathbf{n}\omega &= -\omega_2(x_1, 0)\sqrt{g_{11}(x^1, 0)}dx^1, \\
\mathbf{n}d\omega &= \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right)|_{x^2=0}\sqrt{g^{11}(x^1, 0)}, \\
\mathbf{t}\delta\omega &= -\left(\frac{\partial}{\partial x^1}(\omega_1\sqrt{g^{11}}) + \frac{\partial}{\partial x^2}(\omega_2\sqrt{g_{11}})\right)|_{x^2=0}\sqrt{g^{11}(x^1, 0)}.
\end{aligned}$$

The same argument as in [14, Proposition 1.2] and in [10] shows that the operator $A(x, D_{x^1})$ in the factorization (4.5) of the Hodge Laplacian $\Delta^{(1)}$ has the following meaning: given a harmonic 1-form $\omega = \omega_1 dx^1 + \omega_2 dx^2$, we have

$$\begin{pmatrix} \partial_{x^2}\omega_1|_{x^2=0} \\ \partial_{x^2}\omega_2|_{x^2=0} \end{pmatrix} = A((x^1, 0), D_{x^1}) \begin{pmatrix} \omega_1|_{x^2=0} \\ \omega_2|_{x^2=0} \end{pmatrix},$$

modulo a smoothing operator. We shall represent the operator A by a 2×2-matrix of scalar operators A_{jk}, $1 \le j, k \le 2$. Taking an arbitrary harmonic 1-form ω with $\mathbf{n}\omega = 0$, we get

$$\partial_{x^2}\omega_1|_{x^2=0} = A_{11}((x^1, 0), D_{x^1})\omega_1|_{x^2=0},$$

modulo smoothing. Thus,

$$\sqrt{g^{11}(x^1, 0)}\partial_{x^2}\omega_1|_{x_2=0} = \sqrt{g^{11}(x^1, 0)}A_{11}((x^1, 0), D_{x^1})\omega_1|_{x^2=0},$$

modulo smoothing, and, therefore, the knowledge of the Cauchy data for harmonic 1-forms ω with $\mathbf{n}\omega = 0$ implies the knowledge of the full symbol of the operator

$$\sqrt{g^{11}(x^1, 0)}A_{11}((x^1, 0), D_{x^1}). \tag{4.10}$$

The factorization (4.5) yields that the principal symbol of $A_{11}((x^1, 0), D_{x^1})$ is $-\sqrt{g^{11}(x^1, 0)}|\xi_1|$. Hence, from the knowledge of the principal symbol of the operator (4.10), we determine the metric $g_{11}(x^1, 0)$ along the boundary. Having found the metric along the boundary, we recover the full symbol of $A_{11}((x^1, 0), D_{x^1})$.

We shall determine the normal derivatives of the metric at the boundary from the knowledge of the full symbol of the operator $A_{11}((x^1, 0), D_{x^1})$. It follows from

(4.8) that the homogeneous symbol of order zero $a_{0,11}$ of the operator A_{11} is given by

$$a_{0,11} = -\frac{1}{2\sqrt{g^{11}}|\xi_1|}((Ea_1)_{11} + \partial_{x^2}\sqrt{g^{11}}|\xi_1|) + T_0 = -\frac{\partial_{x^2}g^{11}}{2g^{11}} + T_0, \tag{4.11}$$

where

$$T_0 = -\frac{1}{2\sqrt{g^{11}}|\xi_1|}(iF_{11}\xi_1 - (\partial_{\xi_1}a_1 D_{x^1}a_1)_{11})$$

contains only the metric $g_{11}(x^1, 0)$ and its tangential derivatives. Thus, using (4.11), we can determine the normal derivative $\partial_{x^2}g_{11}(x^1, 0)$. Notice that (4.8) implies that $a_0((x^1, 0), \xi_1)$ contains only the metric along the boundary $g_{11}(x^1, 0)$, its tangential derivatives and the first normal derivative $\partial_{x^2}g_{11}(x^1, 0)$.

It follows from (4.9) that the homogeneous symbol of order -1 of the operator A_{11} has the form

$$a_{-1,11} = -\frac{1}{2\sqrt{g^{11}}|\xi_1|}(-\partial_{x^2}a_{0,11}) + T_{-1}, \tag{4.12}$$

where

$$T_{-1} = -\frac{1}{2\sqrt{g^{11}}|\xi_1|}(Q_{11} + (Ea_0)_{11} - \sum_{\substack{j+k-\alpha=-0 \\ \alpha\ge 0,\ 0\le j,k\le 1}} \frac{1}{\alpha!}(\partial_{\xi_1}^{\alpha}a_j D_{x^1}^{\alpha}a_k)_{11})$$

contains only the metric along the boundary, its tangential derivatives and its first normal derivative $\partial_{x^2}g_{11}(x^1, 0)$. Substituting $a_{0,11}$ into (4.12), we get

$$a_{-1,11} = -\frac{1}{2\sqrt{g^{11}}|\xi_1|}\left(\frac{\partial_{x^2}^2 g^{11}}{2g^{11}}\right) + \widetilde{T}_{-1}, \tag{4.13}$$

where $\widetilde{T}_{-1}$ contains only the metric along the boundary, its tangential derivatives and its first normal derivative. Hence, from (4.13), we can recover the second normal derivative of the metric along the boundary. Notice that a_{-1} contains only the metric, its tangential derivatives and its normal derivatives up to order 2 along the boundary.

Let us now proceed by induction with respect to the degree of homogeneity in (4.7). Let $m \ge 1$ and suppose we have shown that

$$a_{-j,11} = -\frac{1}{(2\sqrt{g^{11}}|\xi_1|)^j}\left(\frac{\partial_{x^2}^{j+1} g^{11}}{2g^{11}}\right) + T_{-j}, \quad 1 \le j \le m, \tag{4.14}$$

where T_{-j} is an expression involving only $\partial_{x^2}^l g_{11}$, $l = 0, \dots, j$, and their tangential derivatives along ∂M, and, furthermore, also suppose that we have shown that a_{-j} contains only $\partial_{x^2}^l g_{11}$, $l = 0, \dots, j+1$, and their tangential derivatives along ∂M.

We shall show how to determine the normal derivative $\partial_{x^2}^{m+2} g_{11}(x^1, 0)$ from the knowledge of the term $a_{-m-1,11}$ of the full symbol of A_{11}. We have

$$a_{-m-1,11} = -\frac{1}{2\sqrt{g^{11}}|\xi_1|}((Ea_{-m})_{11} - \partial_{x^2} a_{-m,11} - \sum_{\substack{j+k-\alpha=-m \\ \alpha\geq 0, -m\leq j,k\leq 1}} \frac{1}{\alpha!}(\partial_{\xi_1}^{\alpha} a_j D_{x^1}^{\alpha} a_k)_{11}) \tag{4.15}$$

It follows from (4.14) that the only term in the right hand side of (4.15) which contains the normal derivative $\partial_{x^2}^{m+2} g_{11}(x^1, 0)$ is

$$\frac{1}{2\sqrt{g^{11}}|\xi_1|}\partial_{x^2} a_{-m,11}.$$

We conclude that the representation (4.14) holds also for a_{-m-1} and, thus, the derivative $\partial_{x^2}^{m+2} g_{11}(x^1, 0)$ is determined.

□

We are now ready to complete the **proof of Theorem 1.3**. To this end, let us recall the asymptotics of the Green's 1-form $G(x, y)$, as $x \to y$, $y \in \widetilde{M}^{int}$,

$$G(x, y) \sim -\frac{1}{s_2} \log d_{\widetilde{M}}(x, y) \det(g) dx^1 \wedge dx^2 \cdot dy^1 \wedge dy^2, \tag{4.16}$$

see [6, p.136]. We refer to (3.1) for the notation. Given Proposition 4.2 and the asymptotics (4.16), the proof of Theorem 1.3 is obtained by repeating the proof of Theorem 1.2.

4.3. **The case of 2-forms.** Let g and $\widetilde{g}$ be two smooth Riemannian metrics on M in the same conformal class, i.e. there is a smooth real-valued function φ on M such that

$$\widetilde{g} = e^{2\varphi} g.$$

Then using (4.3), for the Hodge Laplacian on 2-forms, we have

$$\begin{aligned} &\Delta_{\widetilde{g}}^{(2)} \omega = \Delta_g^{(2)}(e^{-2\varphi}\omega), \quad \omega \in C^{\infty}(M, \Lambda^2 T^* M), \\ &\mathbf{n}_{\widetilde{g}} \omega = \mathbf{n}_g(e^{-2\varphi}\omega), \\ &\mathbf{t}\delta_{\widetilde{g}} \omega = \mathbf{t}\delta_g(e^{-2\varphi}\omega). \end{aligned}$$

Hence,

$$\mathcal{C}_{\widetilde{g}}^{(2)} = \mathcal{C}_g^{(2)}. \tag{4.17}$$

Remark. Let $\omega = \omega_{12} dx^1 \wedge dx^2$ be a 2-form. An explicit computation using (4.2) shows that in boundary normal coordinates, the Hodge Laplacian on 2-forms has the form

$$\Delta^{(2)} = D_{x^2}^2 + g^{11} D_{x^1}^2 + iE(x) D_{x^2} + iF(x) D_{x^1} + Q(x), \tag{4.18}$$

where

$$E(x) = -\frac{\partial_{x^2} g^{11}}{2g^{11}}, \quad F(x) = -\frac{3}{2}\partial_{x^1} g^{11}, \quad Q(x) = -\frac{\partial_{x^1}^2 g^{11}}{2} - \frac{\partial_{x^2}^2 \log g^{11}}{2}.$$

Factorizing the Laplacian $\Delta^{(2)}$ similarly to the case of 1-forms, one can then determine the principal symbol of the corresponding Dirichlet-to-Neumann map, and show that it does not carry any information about the metric on the boundary of the manifold.

5. The case of a complete manifold. Proof of Theorem 1.7

In order to prove Theorem 1.7 we shall follow the proof of Theorem 1.2, valid in the compact case. It follows from [10], when $n \geq 3$ and $k = 1, \dots, n$, and from Proposition 4.2, when $n = 2$ and $k = 1$, that the knowledge of the Dirichlet-to-Neumann map $\Lambda_{g,\Gamma}^{(k)}$ determines all the normal derivatives $\partial_{x^n}^l g_{ij}(x', 0)$, $l = 0, 1, 2, \dots$, of the metric tensor g at Γ, in boundary normal coordinates.

Recall that we work under the assumption (A1), when $n \geq 3$, and the assumption (A2), when $n = 2$. As in the proof of Theorem 1.2, we construct the real-analytic manifold $\widetilde{M}$ by attaching to M a boundary collar $\Gamma \times (-r, 0]$, with the extension of the metric g. Here r is small enough, so that the extended metric remains real-analytic. In the same way as in Proposition 1.4, given the Hodge Laplacian on $\widetilde{M}$, we shall consider the non-negative self-adjoint realization $\Delta_{F,\widetilde{M}}^{(k)}$.

Since the manifolds $\widetilde{M}$ and M differ by a compact region, the following result is essentially well known, see [3].

Proposition 5.1. *If* $0 \notin spec_{ess}(\Delta_{F,M}^{(k)})$, *then* $0 \notin spec_{ess}(\Delta_{F,\widetilde{M}}^{(k)})$.

Recall now from the assumptions of Theorem 1.7 that $0 \notin \mathrm{spec}(\Delta_{F,\widetilde{M}}^{(k)})$. Thus, by Proposition 5.1, we may introduce the inverse operator

$$(\Delta_{F,\widetilde{M}}^{(k)})^{-1} : L^2(\widetilde{M}, \Lambda^k T^* \widetilde{M}) \to \mathcal{D}(\Delta_{F,\widetilde{M}}^{(k)}),$$

given by

$$(\Delta_{F,\widetilde{M}}^{(k)})^{-1} w(x) = \int_{\widetilde{M}} G(x,y) \wedge *w(y).$$

Here the Schwartz kernel $G(x, y)$ is the corresponding Green's form, which satisfies

$$\begin{aligned}
\Delta_x^{(k)} G(x,y) &= \delta_{x,y} \quad \text{in} \quad \widetilde{M}, \\
\mathbf{t}_x G(x,y) &= 0 \quad \text{on} \quad \partial\widetilde{M}, \\
\mathbf{n}_x G(x,y) &= 0 \quad \text{on} \quad \partial\widetilde{M},
\end{aligned}$$

where $y \in \widetilde{M} \setminus \partial\widetilde{M}$ and

$$\delta_{x,y} = \sum_{\substack{i_1<\cdots<i_k \\ j_1<\cdots<j_k}} \delta(y-x) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \cdot dy^{j_1} \wedge \cdots \wedge dy^{j_k}$$

is the delta double current, supported at $x = y$.

Proposition 5.2. *The Green's form $G(x, y)$ has the following properties.*

(i) $G(x, y) = G(y, x)$.

(ii) *Let $y \in \widetilde{M} \setminus \partial\widetilde{M}$. Then $G(x, y)$ is real-analytic for $x \in \widetilde{M} \setminus (\{y\} \cup \partial\widetilde{M})$.*

(iii) *Let $y \in \widetilde{M} \setminus \partial\widetilde{M}$. Then $G(x, y)$ is C^∞-smooth up to the boundary of $\partial\widetilde{M}$, away from y.*

(iv) *Let $n \geq 3$. Then as $x \to y \in \widetilde{M}^{int}$, $G(x, y)$ has the following asymptotic behavior*

$$G(x,y) \sim \frac{1}{(n-2)s_n d_{\widetilde{M}}(x,y)^{n-2}} \Gamma_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(y) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \cdot dy^{j_1} \wedge \cdots \wedge dy^{j_k}, \tag{5.1}$$

where $d_{\widetilde{M}}(x, y)$ is the geodesic distance from x to y in $\widetilde{M}$, s_n is the area of the unit n-sphere and $\Gamma_{(i_1,\ldots,i_k),(j_1,\ldots,j_k)}(y)$ is a positive real-analytic function on $\widetilde{M}$. The asymptotic relation (5.1) can be differentiated with respect to x any number of times.

(v) *Let $n = 2$. Then the asymptotics of the Green's 1-form $G(x, y)$, as $x \to y$, $y \in \widetilde{M}^{int}$, has the following form,*

$$G(x,y) \sim -\frac{1}{s_2} \log d_{\widetilde{M}}(x,y) det(g) dx^1 \wedge dx^2 \cdot dy^1 \wedge dy^2. \tag{5.2}$$

The asymptotic relation (5.2) can be differentiated with respect to x any number of times.

Proof. The property (i) is clear, and (ii) follows from [7, Theorem 4.1, p. 108], since $G(x, y)$ is a solution of an elliptic system of partial differential equations on $\widetilde{M} \setminus (\{y\} \cup \partial\widetilde{M})$ with real-analytic coefficients. The property (iii) follows from [17, Theorem 3.4.6].

Let us show (iv) and (v). Consider $\Omega \subset\subset \widetilde{M}^{int}$. Let $G^\Omega(x, y)$ be a Green's form on a compact manifold $\overline{\Omega}$ defined as in Section 3.2. Then

$$(\Delta_x^{(k)} + \Delta_y^{(k)})(G(x,y) - G^\Omega(x,y)) = 0, \quad (x,y) \in \Omega \times \Omega,$$

and therefore,

$$G(x,y) = G^\Omega(x,y) + h(x,y), \quad h(x,y) \text{ is } C^\infty - \text{smooth double form on } \Omega \times \Omega.$$

The proof is complete.

□

As before, let us set $U = \widetilde{M_1} \setminus M_1 = \widetilde{M_2} \setminus M_2$. In analogy with Lemma 3.4, we have the following result.

Lemma 5.3. *The Green's forms $G_j(x, y)$ satisfy*

$$G_1(x, y) = G_2(x, y), \quad (x, y) \in U \times U \setminus \{x = y\}.$$

Let $\widetilde{U} \subset\subset U$ be a relatively compact subset of U. In order to prove Theorem 1.7, as in the compact case, we shall make use of the maps

$$\mathcal{G}_j : \widetilde{M_j} \to H^s(\widetilde{U}, \Lambda^k T^* \widetilde{U}), \quad x \mapsto G_j(x, \cdot), \tag{5.3}$$

defined for some $s < 1 - n/2$. Proposition 3.5 continues to be valid in the case when $\widetilde{M_j}$ are complete, and therefore, the maps $\mathcal{G}_j$ are $C^1(\widetilde{M_j}, H^s(\widetilde{U}, \Lambda^k T^* \widetilde{U}))$, for $s < 1 - n/2$. Furthermore, arguing as in the proof of Lemma 3.6, we see that the maps (5.3) are injective immersions, which suffices for us to proceed with the proof of Theorem 1.7 along the lines of the argument presented in the proof of Theorem 1.2. The only modification required in the complete non-compact case concerns the proof of Lemma 3.8. We shall therefore give a proof of this result in this case.

Proof of Lemma 3.8 in the non-compact case. As $x_1 \in \partial D_1$, there is a sequence $p_i \in D_1$ such that $p_i \to x_1$ when $i \to \infty$. Hence,

$$d_{\widetilde{M_1}}(x_0, p_i) \to d_{\widetilde{M_1}}(x_0, x_1) \quad i \to \infty.$$

As every sequence in $\mathbb{R}$ has a monotone subsequence, we can assume that

$$d_{\widetilde{M_1}}(x_0, p_i) \le d_{\widetilde{M_1}}(x_0, p_{i+1}) \le d_{\widetilde{M_1}}(x_0, x_1), \quad i = 1, 2, \dots.$$

Let μ_i be the shortest curve from x_0 to p_i in $\widetilde{M_1}$, i.e. assume that

$$\text{length}_{g_1}(\mu_i) = d_{\widetilde{M_1}}(x_0, p_i).$$

The fact that x_1 is the closest point of $\widetilde{M_1} \setminus (N(\varepsilon_0) \cup D_1))$ implies that μ_i is contained in D_1. Since J is a local isometry on D_1, there is a sequence $q_i \in \widetilde{M_2}$ such that $q_i = J(p_i)$ and, moreover,

$$d_{\widetilde{M_2}}(q_i, x_0) \le \text{length}_{g_2}(J(\mu_i)) = \text{length}_{g_1}(\mu_i) \le d_{\widetilde{M_1}}(x_0, x_1).$$

Thus, the sequence q_i is bounded and the remainder of the proof of Lemma 3.8 proceeds similarly to the compact case.

This completes the proof of Theorem 1.7.

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