INVERSE PROBLEMS FOR THE PAULI HAMILTONIAN IN TWO DIMENSIONS

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Abstract. We prove in two dimensions that the set of Cauchy data for the Pauli Hamiltonian measured on the boundary of a bounded open subset with smooth enough boundary determines uniquely the magnetic field and the electrical potential provided that the electrical potential is small in an appropriate topology. This result has the immediate consequence, in the case that the magnetic potential and electrical potential have compact support, that we can determine uniquely the magnetic field and the electrical potential by measuring the scattering amplitude at a fixed energy provided that the electrical potential is small in an appropriate topology.

Mathematics subject classification (MSC2000): 35R30
Keywords: Pauli Hamiltonian, Cauchy data, Dirichlet-to-Neumann map, inverse scattering

1. Introduction

The Pauli Hamiltonian describes particles in a magnetic field with spin. In two dimensions it is a direct sum of the pair of operators

$$H_{\vec{A},q}u := \sum_{j=1}^{2}\left(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j\right)^2 u \pm Bu - qu \tag{1.1}$$

where $\vec{A}$ denotes the magnetic potential, $B = \mathrm{rot}\vec{A}$ is the magnetic field, and q is the electrical potential (see for instance Chapter 6 of [4]). Thus both direct and inverse problems for the Pauli Hamiltonian consider separately both signs in B in (1.1). For simplicity we will choose the plus sign. All of the results below are also valid for the minus sign in B with minor changes in the arguments.

We first describe the inverse boundary problem we consider. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. We consider the Pauli Hamiltonian given by a real-valued vector field $\vec{A} = (A_1, A_2) \in W^{1,p}(\Omega)$ and an electric potential $q \in L^p(\Omega)$, $p > 2$,

$$H_{\vec{A},q}u := \sum_{j=1}^{2}\left(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j\right)^2 u + Bu - qu = 0 \quad \text{in } \Omega \tag{1.2}$$

Both authors are grateful to the Mathematical Sciences Research Institute (MSRI) for partial support and for providing a very stimulating environment during the inverse problems program in fall 2001. HK is partly supported by KOSEF 98-0701-03-5 and 2000-1-10300-001-1. GU is partly supported by NSF grant DMS-0070488 and a John Simon Guggenheim fellowship.

where $B = \mathrm{rot}\vec{A}$. Fix $\alpha = \frac{p-2}{p}$ throughout this paper. The set of Cauchy data of the solutions of (1.2) is given by

$$\mathcal{C}_{\vec{A},q} := \{(f,g) \in C^{1,\alpha}(\partial\Omega) \times C^{\alpha}(\partial\Omega) : \text{ there exists } u \in C^{1,\alpha}(\overline{\Omega}) \text{ such that } H_{\vec{A},q}u = 0,\ u|_{\partial\Omega} = f,\ ((\nabla - i\vec{A})u)|_{\partial\Omega} \cdot \nu = g\}. \tag{1.3}$$

Here ν denotes the unit normal to $\partial\Omega$. In the case that 0 is not a Dirichlet eigenvalue for $H_{\vec{A},q}$, $\mathcal{C}_{\vec{A},q}$ is the graph of the Dirichlet-to-Neumann (DN) map

$$\Lambda_{\vec{A},q} : C^{1,\alpha}(\partial\Omega) \to C^{\alpha}(\partial\Omega). \tag{1.4}$$

The inverse boundary value problem we consider in this paper is whether we can determine $\vec{A}$ and q from $\mathcal{C}_{\vec{A},q}$.

It was observed in [9, 10] that there is a gauge invariance in the problem. That is, if $\varphi \in C^1(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 1$, $\nabla\varphi|_{\partial\Omega} = 0$, then

$$\mathcal{C}_{\vec{A}+\nabla\varphi,q} = \mathcal{C}_{\vec{A},q}.$$

Therefore we can recover at best the magnetic field, $\mathrm{rot}\vec{A}$, and q from the DN map.

In this paper we prove the following semiglobal identifiability result:

Theorem 1.1. *Let $\vec{A}_j \in W_0^{1,p}(\Omega), j = 1,2$, $rot\vec{A}_1 \in W_0^{1,p}(\Omega)$, $q_1 \in W^{1,p}(\Omega)$, $q_2 \in L^p(\Omega), p > 2$. For each $M > 0$, there exists $\epsilon(M,\Omega,p) > 0$ such that if $\|\vec{A}_1\|_{L^p(\Omega)} \le M$ and $\|q_1\|_{W^{1,p}(\Omega)} \le \epsilon$ and*

$$\mathcal{C}_{\vec{A}_1,q_1} = \mathcal{C}_{\vec{A}_2,q_2}, \tag{1.5}$$

we conclude

$$rot\vec{A}_1 = rot\vec{A}_2 \quad and \quad q_1 = q_2 \quad in\ \Omega. \tag{1.6}$$

$W_0^{1,p}(\Omega)$ denotes the space of $W^{1,p}(\Omega)$-functions whose boundary traces are zero.

Observe that no smallness condition is assumed on the electric potential q_2. We also remark that the only place where we need that the magnetic potential has boundary trace zero is in the proof of Lemma 4.1. If we assume further regularity in the magnetic and electrical potentials then Theorem D of [NSU] allows to extend the magnetic and electrical potentials to $\mathbb{R}^2$ with compact support. More precisely we have:

Theorem 1.2. *Let $\vec{A}_j, q_j \in C^\infty(\overline{\Omega}), j = 1,2$, and $p > 2$. There exists $\epsilon(\Omega,p) > 0$ such that if $\|q_1\|_{W^{1,p}(\Omega)} \le \epsilon$ and*

$$\mathcal{C}_{\vec{A}_1,q_1} = \mathcal{C}_{\vec{A}_2,q_2}, \tag{1.7}$$

we conclude

$$rot\vec{A}_1 = rot\vec{A}_2 \quad and \quad q_1 = q_2 \quad in\ \Omega. \tag{1.8}$$

In [10] Sun proved in two dimensions for the Schrödinger equation in a magnetic field that if $\|\mathrm{rot}\vec{A}_j\|_{W^{1,\infty}(\Omega)}$ $(j = 1,2)$ is small enough and q_j $(j = 1,2)$ are in an open and dense set in an appropriate topology, then we can determine uniquely $\mathrm{rot}\vec{A}_j$ and q_j from the DN map associated to the magnetic potentials and electrical potentials.

We remark that in dimensions $n \geq 3$ a global identifiability result of the magnetic field and electrical potential was proven in [8] for the Schrödinger equation in a magnetic field assuming some smoothness conditions on the coefficients. We also note that even if we assume that $\vec{A}_1 = 0$ in Theorem 1.1, it is unknown whether we can recover q from $\mathcal{C}_q := \mathcal{C}_{\vec{0},q}$, the set of Cauchy data for the standard Schrödinger equation, although several partial results have been proven ([3], [5], [6], [11], [12], [16]).

A particular case of Theorem 1.1 is when the electrical potential in (1.1) is zero. Thus we obtain the following global uniqueness result:

Corollary 1.3. *Let $\vec{A}_j \in W_0^{1,p}(\Omega), j = 1, 2$, $rot\vec{A}_1 \in W_0^{1,p}(\Omega)$, $q_1 = 0$, $q_2 \in L^p(\Omega), p > 2$. For each $M > 0$, there exists $\epsilon(M, \Omega, p) > 0$ such that if $\|\vec{A}_1\|_{L^p(\Omega)} \leq M$*

$$\mathcal{C}_{\vec{A}_1,0} = \mathcal{C}_{\vec{A}_2,q_2}, \tag{1.9}$$

we conclude

$$rot\ A_1 = \ rot\ A_2 \quad and \quad q_1 = q_2 = 0 \quad in\ \Omega. \tag{1.10}$$

As a consequence of Theorem 1.2, a similar result to Corollary 1.3 holds with $\vec{A}_j \in C^\infty(\overline{\Omega}), j = 1, 2$, without the assumption that the magnetic potentials have zero boundary trace.

In the case that the magnetic potentials are zero, we conclude the following semiglobal identifiability result:

Corollary 1.4. *Let $q_1 \in W^{1,p}(\Omega)$, $q_2 \in L^p(\Omega)$, $p > 2$. There exists $\epsilon(\Omega, p) > 0$ such that if $\|q_1\|_{W^{1,p}(\Omega)} \leq \epsilon$ and $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, then $q_1 = q_2$.*

Corollary 1.4 was known previously if q_1 and q_2 are both a-priori close to constant ([16]). As far as we know, even the case $q_1 = 0$ was previously unknown.

It is well known (see for instance chapter 12 of [17], and [7]) that Theorem 1.1 implies a similar result for the inverse scattering problem at fixed energy if we assume that the magnetic potential and electrical potential have compact support.

The scattering amplitude for the Pauli Hamiltonian (1.1) with $\vec{A} \in W^{1,p}(\mathbb{R}^2)$, $q \in L^p(\mathbb{R}^2), p > 2$, $\vec{A}, q$ with compact support, is defined in terms of the outgoing eigenfunctions $\psi_+(\lambda, x, \omega)$ where $\lambda \in \mathbb{R} - 0$, $x \in \mathbb{R}^2$, $\omega \in S^1$. Namely ψ_+ satisfies

$$\psi_+(\lambda, x, \omega) = e^{i\lambda x\cdot\omega} + \frac{a_{\vec{A},q}(\lambda, \theta, \omega)}{|x|^{\frac{1}{2}}} e^{i\lambda|x|} + O(|x|^{-\frac{3}{2}}),$$

where $\theta = \frac{x}{|x|}$. The scattering amplitude, $a_{\vec{A},q}(\lambda, \theta, \omega)$, measures the effect of the magnetic and electrical potential on a plane wave with frequency λ and direction ω of the form $e^{i\lambda x\cdot\omega}$.

The inverse scattering problem at a fixed energy is to determine $\vec{A}, q$ from the scattering amplitude $a_{\vec{A},q}(\lambda_0, \theta, \omega)$ with fixed λ_0. It is easy to see, as in the case of the Cauchy data previously discussed, that we can recover at best the magnetic field and the electrical potential. An immediate consequence of Theorem 1.1 is the following.

Theorem 1.5. *Let $\vec{A}_j$ and q_j have compact supports. Let $\vec{A}_j \in W^{1,p}(\mathbb{R}^2)$, $j = 1, 2$, $rot\vec{A}_1 \in W^{1,p}(\mathbb{R}^2)$, $q_1 \in W^{1,p}(\mathbb{R}^2)$, $q_2 \in L^p(\mathbb{R}^2)$, $p > 2$. For each $M > 0$, there exists $\epsilon(M, \Omega, p) > 0$ such that if $\|\vec{A}_1\|_{L^p(\Omega)} \leq M$ and $\|q_1\|_{W^{1,p}(\Omega)} \leq \epsilon$ and*

$$a_{\vec{A}_1,q_1} = a_{\vec{A}_2,q_2} \quad \textit{for a fixed } \lambda, \tag{1.11}$$

then

$$rot\vec{A}_1 = rot\vec{A}_2 \quad \textit{and} \quad q_1 = q_2 \quad \textit{in } \mathbb{R}^2. \tag{1.12}$$

One can also state colloraries similar to Corollary 1.3 and Corollary 1.4.

The method of proof of Theorem 1.1 is by reducing the problem to a similar one for a second order equation which can be factored in terms of $\overline{\partial}$ and ∂. Recall

$$\overline{\partial} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \quad \partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}).$$

We rewrite (1.2) in the form

$$(\overline{\partial} + \bar{a})(\partial - a)u - \tilde{q}u = 0 \quad \text{in } \Omega \tag{1.13}$$

where

$$a := \frac{1}{2}(A_2 + iA_1), \quad \tilde{q} = \frac{1}{4}q. \tag{1.14}$$

We define the set of Cauchy data associated to (1.13) by

$$\begin{aligned} \mathcal{C}_{a,\tilde{q}} := \{&(f, g) \in C^{1,\alpha}(\partial\Omega) \times C^{\alpha}(\partial\Omega) : u|_{\partial\Omega} = f, \\ &((\partial - a)u)|_{\partial\Omega} = g,\ u \in C^{1,\alpha}(\overline{\Omega}) \text{ a solution of (1.13) } \}. \end{aligned} \tag{1.15}$$

Theorem 1.1 is then a consequence of

Theorem 1.6. *Let $a_j \in W^{1,p}(\Omega)$, $\tilde{q}_1 \in W^{1,p}(\Omega)$, $\tilde{q}_2 \in L^p(\Omega)$, $p > 2$, $j = 1, 2$. For each $M > 0$, there exists $\epsilon(M, \Omega, p) > 0$ such that if $\|a_1\|_{L^p(\Omega)} \leq M$ and $\|\tilde{q}_1\|_{W^{1,p}(\Omega)} \leq \epsilon$ and*

$$\mathcal{C}_{a_1,\tilde{q}_1} = \mathcal{C}_{a_2,\tilde{q}_2}, \tag{1.16}$$

then

$$\tilde{q}_1 = \tilde{q}_2 \quad \textit{and} \quad \overline{\partial}^{-1}\bar{a}_1 + \partial^{-1}a_1 = \overline{\partial}^{-1}\bar{a}_2 + \partial^{-1}a_2 \quad \textit{in } \Omega. \tag{1.17}$$

Of course (1.17) implies that

$$\text{rot}a_1 = \frac{1}{2}(\overline{\partial}\bar{a}_1 + \partial a_1) = \text{rot}a_2 = \frac{1}{2}(\overline{\partial}\bar{a}_2 + \partial a_2) \quad \text{in } \Omega. \tag{1.18}$$

The method of proof of Theorem 1.6 reduces (1.13) to a first order system and follows the lines of [3] to construct complex geometrical optics solutions for (1.13) for all complex frequencies, and uses the scattering transform of Beals and Coifman [2] (see also [13, 14, 15]). An important difference with [3] is that we work directly on the bounded domain Ω rather than the whole space as in [3]. This simplifies several of the arguments.

In section 2 we construct the complex geometric optics solutions for the first order system and consider the corresponding scattering transform. In section 3 we derive the scattering relation of the solutions. Finally in section 4 we prove Theorem 1.6.

2. Complex geometrical optics solutions for the $\overline{\partial}$-system

In this section we reduce the second order equation

$$(\overline{\partial}+\bar{a})(\partial-a)u-qu=0 \quad \text{in } \Omega \tag{2.1}$$

into a system of first order $\overline{\partial}$ type system and construct geometrical optics solutions of the system.

Define the solution operator $\overline{\partial}^{-1}$ of the $\overline{\partial}$-equation by

$$\overline{\partial}^{-1} f(z) := -\frac{1}{\pi}\int_{\Omega}\frac{f(\zeta)}{\zeta-z}d\mu(\zeta), \qquad z\in\Omega \tag{2.2}$$

where $d\mu$ is the Lebesgue measure on $\mathbb{R}^2$. We note that $\overline{\partial}^{-1}$ in this paper is defined as an integral operator on Ω, not on the whole $\mathbb{R}^2$. Then it is well-known that

$$f(z)=\overline{\partial}^{-1}(\overline{\partial}f)(z)+C(f|_{\partial\Omega})(z), \quad z\in\Omega \tag{2.3}$$

where $C(f|_{\partial\Omega})$ is the Cauchy transform of $f|_{\partial\Omega}$, namely,

$$C(f|_{\partial\Omega})(z)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta-z}d\zeta, \quad z\notin\partial\Omega.$$

We then define ∂^{-1} and $\overline{C}$ to be the conjugates of $\overline{\partial}^{-1}$ and C, respectively, i.e.,

$$\partial^{-1}(f) := \overline{\overline{\partial}^{-1}(\bar{f})}, \quad \text{and} \quad \overline{C}(f) := \overline{C(\bar{f})}.$$

Then the following formula immediately follows from (2.3):

$$f(z)=\partial^{-1}(\partial f)(z)+\overline{C}(f|_{\partial\Omega})(z), \quad z\in\Omega. \tag{2.4}$$

We also define

$$\mathcal{C} := \begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix}. \tag{2.5}$$

Let u be a solution of (2.1) and set $w:=(\partial-a)u$. Then the equation (2.1) takes the form

$$\left[\begin{pmatrix} \overline{\partial}+\bar{a} & 0 \\ 0 & \partial-a \end{pmatrix} - \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} w \\ u \end{pmatrix}=0. \tag{2.6}$$

By conjugating the equation (2.6), we have

(2.7)

$$\begin{pmatrix} e^{-\overline{\partial}^{-1}\bar{a}} & 0 \\ 0 & e^{\partial^{-1}a} \end{pmatrix}\left[\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} - \begin{pmatrix} 0 & e^{Ta}q \\ e^{-Ta} & 0 \end{pmatrix}\right]\begin{pmatrix} e^{\overline{\partial}^{-1}\bar{a}} & 0 \\ 0 & e^{-\partial^{-1}a} \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix}=0$$

where

$$Ta(z) := \overline{\partial}^{-1}\bar{a}+\partial^{-1}a.$$

Set

$$D=\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 0 & e^{Ta}q \\ e^{-Ta} & 0 \end{pmatrix}.$$

We are seeking special solutions of the system

$$(D - Q)\psi = 0 \quad \text{in } \Omega \tag{2.8}$$

in the form

$$\psi = \begin{pmatrix} e^{-\overline{\partial}^{-1}\bar{a}} & 0 \\ 0 & e^{\partial^{-1}a} \end{pmatrix} m(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix} \tag{2.9}$$

where $m(z,k)$ is a 2×2 matrix valued function in Ω.

We need a few more definitions (see [3]): For z and k in $\mathbb{C}$, let

$$\tilde{e}_k(z) = \exp(i(z\bar{k} + \bar{z}k)) \quad \text{and} \quad e_k(z) = \exp(i(zk + \bar{z}\bar{k})),$$

and

$$\begin{aligned} \Lambda(z,k) &= \Lambda_k(z) \\ &= \begin{pmatrix} \tilde{e}_k(z) & 0 \\ 0 & e_{-k}(z) \end{pmatrix}. \end{aligned}$$

We then define

$$E_k A = E_k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & \tilde{e}_{-k}a_{12} \\ e_k a_{21} & a_{22} \end{pmatrix}.$$

Then, m defined in (2.9) satisfies

$$D_k m - Qm = 0 \quad \text{in } \Omega \tag{2.10}$$

where D_k is the operator

$$D_k A = E_k^{-1} D E_k A.$$

Let

$$D^{-1} := \begin{pmatrix} \overline{\partial}^{-1} & 0 \\ 0 & \partial^{-1} \end{pmatrix}.$$

We have from (2.10) that

$$DE_k m - E_k Q m = 0 \quad \text{in } \Omega.$$

Applying D^{-1} to the last equation and using (2.3) and (2.4), we get

$$E_k m(z) - D^{-1} E_k Q m(z) = \mathcal{C}(E_k m|_{\partial\Omega})(z), \quad z \in \Omega.$$

Thus we conclude

$$(I - D_k^{-1} Q) m(z) = E_k^{-1} \mathcal{C}(E_k m|_{\partial\Omega})(z), \quad z \in \Omega, \tag{2.11}$$

where I is the 2×2 identity matrix and

$$D_k^{-1} := E_k^{-1} D^{-1} E_k.$$

In order to investigate the invertability of the operator $I - D_k^{-1}Q$, we consider it's null space. We denote by $\mathcal{H}(\Omega)$ the class of holomorphic functions in Ω.

Lemma 2.1. *Suppose that $m \in L^\infty(\Omega)$ and satisfies $(I - D_k^{-1}Q)m = 0$ in Ω. Define ψ by (2.9). Then $\psi_{21}, \psi_{22} \in C^{1,\alpha}(\overline{\Omega})$ are solutions of the equation $(\overline{\partial}+\bar{a})(\partial-a)u-qu = 0$ in Ω and there are $f_{ij} \in \mathcal{H}(\mathbb{C}\setminus\overline{\Omega})\cap(\mathbb{C}\setminus\Omega)$ with $f_{ij}(z) = O(|z|^{-1})$ as $|z| \to \infty$, $i, j = 1, 2$, such that*

$$\begin{cases} \psi_{21}|_{\partial\Omega} & = e^{\partial^{-1}a}e^{-i\bar{z}\bar{k}}\bar{f}_{21}, \\ (\partial - a)\psi_{21}|_{\partial\Omega} & = e^{-\overline{\partial}^{-1}\bar{a}}e^{izk}f_{11}, \end{cases} \tag{2.12}$$

and

$$\begin{cases} \psi_{22}|_{\partial\Omega} & = e^{\partial^{-1}a}e^{-i\bar{z}k}\bar{f}_{22}, \\ (\partial - a)\psi_{22}|_{\partial\Omega} & = e^{-\overline{\partial}^{-1}\bar{a}}e^{iz\bar{k}}f_{12}. \end{cases} \tag{2.13}$$

The converse is also true: If $\psi_{21}, \psi_{22} \in C^{1,\alpha}(\overline{\Omega})$ are solutions of (2.1) and satisfy (2.12) and (2.13), then define ψ by $\psi_{11} := (\partial - a)\psi_{21}$ and $\psi_{12} := (\partial - a)\psi_{22}$, and m by (2.9). Then we have $(I - D_k^{-1}Q)m = 0$ in Ω.

Proof. Suppose that $m \in L^\infty(\Omega)$ and

$$(I - D_k^{-1}Q)m = 0 \quad \text{in } \Omega.$$

Then, $m \in C^{1,\alpha}(\overline{\Omega})$ and satisfies

$$(D_k - Q)m = 0 \quad \text{in } \Omega,$$

and, by (2.11),

$$\mathcal{C}(E_k m|_{\partial\Omega})(z) = 0, \quad z \in \Omega.$$

Thus we obtain

$$C(m_{11}|_{\partial\Omega})(z) = \overline{C}(e_k m_{21}|_{\partial\Omega})(z) = C(\tilde{e}_{-k} m_{12}|_{\partial\Omega})(z) = \overline{C}(m_{22}|_{\partial\Omega})(z) = 0, \quad z \in \Omega. \tag{2.14}$$

(2.14) is equivalent to the fact that there are $f_{ij} \in \mathcal{H}(\mathbb{C}\setminus\overline{\Omega})\cap(\mathbb{C}\setminus\Omega)$ with $f_{ij}(z) = O(|z|^{-1})$ as $|z| \to \infty$, $i, j = 1, 2$, such that

$$\begin{aligned} m_{11}|_{\partial\Omega} &= f_{11}, \\ e_k m_{21}|_{\partial\Omega} &= \overline{f_{21}}, \\ \tilde{e}_{-k} m_{12}|_{\partial\Omega} &= f_{12}, \\ m_{22}|_{\partial\Omega} &= \overline{f_{22}}. \end{aligned}$$

This can be proven using Plemelj's jump formula for the Cauchy integral, namely,

$$\lim_{t\to 0^+} \mathcal{C}(f)(z - t\nu(z)) - \lim_{t\to 0^+} \mathcal{C}(f)(z + t\nu(z)) = f(z), \quad z \in \partial\Omega.$$

Since ψ defined by (2.9), (2.12) and (2.13) follow.

To prove the converse, one can simply reverse the argument above. This completes the proof. ∎

For $a \in L^p(\Omega)$, define $Ta(z) := \overline{\partial}^{-1}\bar{a} + \partial^{-1}a$ as before. Since $\overline{\partial}^{-1}$ and ∂^{-1} are bounded from $L^p(\Omega)$ into $C^\alpha(\overline{\Omega})$ ($\alpha = \frac{p-2}{p}$), there exists a constant $C = C(\Omega, p)$ such that

$$\|Ta\|_{C^\alpha(\overline{\Omega})} \le C\|a\|_p.$$

Therefore, there are constants C and C_1 depending only on Ω and p such that

$$\|\exp(Ta)\|_\infty \le \exp(C\|a\|_p) \tag{2.15}$$

$$\|\exp(Ta)\|_{C^\alpha(\overline{\Omega})} \le C_1\|a\|_p \exp(C\|a\|_p). \tag{2.16}$$

In fact, (2.16) follows from the following estimate:

$$\begin{aligned}|\exp(Ta)(z) - \exp(Ta)(\zeta)| &= |\exp(Ta)(z)||1 - \exp[(Ta)(\zeta) - (Ta)(z)]| \\ &\le C_1 \exp(C\|a\|_p)\|a\|_p|(Ta)(\zeta) - (Ta)(z)| \\ &\le C_1 \exp(C\|a\|_p)\|a\|_p|z - \zeta|^\alpha.\end{aligned}$$

Lemma 2.2. *There exist constants C_1 and C_2 depending only on Ω and p such that if $C_1 \exp(C_2\|a\|_p)\|q\|_p < 1$, then $I - D_k^{-1}Q : L^\infty(\Omega) \to L^\infty(\Omega)$ is invertible.*

Proof. Fix $\alpha = \frac{p-2}{p}$. Since $Q : L^\infty(\Omega) \to L^p(\Omega)$ and $D_k^{-1} : L^p(\Omega) \to C^\alpha(\overline{\Omega})$ are bounded, $D_k^{-1}Q$ is a compact operator on $L^\infty(\Omega)$. Thus it suffices to establish the injectivity of $I - D_k^{-1}Q$. Suppose that $u \in L^\infty(\Omega)$ satisfies

$$u - D_k^{-1}Qu = 0 \quad \text{in } \Omega.$$

Then,

$$E_k u - D^{-1}E_k Qu = 0$$

Considering the first column of the last equation, we have

$$u_{11} = \overline{\partial}^{-1}(e^{Ta}qu_{21}) \tag{2.17}$$

$$e_k(z)u_{21} = \partial^{-1}(e_k e^{-Ta}u_{11}). \tag{2.18}$$

It then follows from (2.15) and (2.17) that

$$\begin{aligned}\|u_{11}\|_\infty &\le C_1\|e^{Ta}qu_{21}\|_p \\ &\le C_1 \exp(C_2\|a\|_p)\|q\|_p\|u_{21}\|_\infty.\end{aligned}$$

We also have from (2.18) that

$$\begin{aligned}\|u_{21}\|_\infty &\le C_3\|e^{-Ta}u_{11}\|_\infty \\ &\le C_3 \exp(C_2\|a\|_p)\|u_{11}\|_\infty.\end{aligned}$$

Note that all the constants in the estimates depend only on Ω and p . Thus if $C_1C_3 \exp(2C_2\|a\|_p)\|q\|_p < 1$, then we have $u_{21} = u_{11} = 0$. In the same way, one can show that $u_{12} = u_{22} = 0$. This completes the proof. □

3. The Scattering Transform

We assume that $I - D_k^{-1}Q : L^\infty(\Omega) \to L^\infty(\Omega)$ is invertible and define

$$m(z,k) := (I - D_k^{-1}Q)^{-1}(I), \tag{3.1}$$

and the scattering transform by

$$S(k) = -\frac{1}{\pi}\mathcal{J}\int_\Omega E_k(Q(z)m(z,k))d\mu(z) \tag{3.2}$$

where the operator $\mathcal{J}$ is defined by

$$\mathcal{J}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & -ia_{12} \\ ia_{21} & 0 \end{pmatrix}.$$

Theorem 3.1. (i) *For each fixed z, $k \to m(z,k)$ is differentiable as a function of k and satisfies*

$$\frac{\partial}{\partial \bar{k}} m(z,k) = m(z,\bar{k})\Lambda_k(z)S(k) \tag{3.3}$$

(ii) *There are constants C_3 and C_4 such that*

$$\|m_{11}(\cdot,k) - 1\|_\infty + \|m_{22}(\cdot,k) - 1\|_\infty \le \frac{C_3}{|k|}(\|a\|_p + 1)\exp(C_4\|a\|_p)\|q\|_p, \tag{3.4}$$

$$\|m_{12}(\cdot,k)\|_\infty + \|m_{21}(\cdot,k)\|_\infty \le \frac{C_3}{|k|}(\|a\|_p + 1)\exp(C_4\|a\|_p). \tag{3.5}$$

Proof. (i) is proved in [3].

Note that m satisfies

$$E_k m - D^{-1}E_k Q m = I.$$

Thus m_{11} and m_{21} satisfy

$$m_{11} = \overline{\partial}^{-1}(e^{Ta}qm_{21}) + 1 \tag{3.6}$$

$$e_k(z)m_{21} = \partial^{-1}(e_k e^{-Ta}m_{11}). \tag{3.7}$$

From (3.6), we have

$$\begin{aligned} \|m_{11} - 1\|_{C^\alpha(\overline{\Omega})} &\le C_1\|e^{Ta}qm_{21}\|_p \\ &\le C_1\exp(C_2\|a\|_p)\|q\|_p\|m_{21}\|_\infty. \end{aligned} \tag{3.8}$$

Since $\partial\overline{\partial}^{-1}$ is a Calderón-Zygmund operator and bounded on L^p $(1 < p < \infty)$, we also have from (3.6)

$$\begin{aligned} \|\partial m_{11}\|_p &\le C_3\|e^{Ta}qm_{21}\|_p \\ &\le C_3\exp(C_2\|a\|_p)\|q\|_p\|m_{21}\|_\infty. \end{aligned} \tag{3.9}$$

Since $e_k(z) = \frac{1}{ik}\partial e_k(z)$, one can see from an integration by parts that

$$\partial^{-1}(e_k e^{-Ta} m_{11})(z)$$
$$= \frac{1}{ik}\left[\overline{C}(e_k e^{-Ta} m_{11})|_{\partial\Omega})(z) + \partial^{-1}(e_k(\partial Ta)e^{-Ta} m_{11})(z) - \partial^{-1}(e_k e^{-Ta}\partial m_{11})(z)\right]$$
$$:= \frac{1}{ik}[I_1 + I_2 + I_3].$$

A standard estimate for the Cauchy transform, (2.16), and (3.8) give

$$\begin{aligned} \|I_1\|_\infty &\le C_4 \|e^{-Ta} m_{11}\|_{C^\alpha(\partial\Omega)} \\ &\le C_4 \|e^{-Ta}\|_{C^\alpha(\overline{\Omega})} \|m_{11}\|_{C^\alpha(\overline{\Omega})} \\ &\le C_5 \exp(C_2\|a\|_p)(\|a\|_p\|q\|_p\|m_{21}\|_\infty + 1). \end{aligned} \tag{3.10}$$

Since ∂T is bounded on L^p ($p < \infty$), we have

$$\begin{aligned} \|I_2\|_\infty &\le C_6 \|(\partial Ta)e^{-Ta} m_{11}\|_p \\ &\le C_6 \|\partial Ta\|_p \|e^{-Ta}\|_\infty \|m_{11}\|_\infty \\ &\le C_7 \|a\|_p \exp(C_2\|a\|_p)(\|q\|_p\|m_{21}\|_\infty + 1). \end{aligned} \tag{3.11}$$

From (3.9), we get

$$\begin{aligned} \|I_3\|_\infty &\le C\|e^{-Ta}\|_\infty \|\partial m_{11}\|_p \\ &\le C_1 \exp(C_2\|a\|_p)\|q\|_p\|m_{21}\|_\infty. \end{aligned} \tag{3.12}$$

It thus follows from (3.7), (3.10), (3.11), and (3.12) that

$$\|m_{21}\|_\infty \le \frac{C_8}{|k|}[\|a\|_p \exp(C_2\|a\|_p)\|q\|_p\|m_{21}\|_\infty + (\|a\|_p + 1)\exp(C_2\|a\|_p)]$$

and hence

$$\|m_{21}\|_\infty \le \frac{C_9}{|k|}(\|a\|_p + 1)\exp(C_2\|a\|_p)$$

if $|k|$ is so large that $C_8\|a\|_p \exp(C_2\|a\|_p)\|q\|_p < \frac{1}{2}|k|$. We can easily conclude from (3.6) that

$$\|m_{11} - 1\|_\infty \le \frac{C_{10}}{|k|}(\|a\|_p + 1)\exp(C_2\|a\|_p)\|q\|_p.$$

Estimates for m_{12} and m_{22} can be derived in the same way. This completes the proof. □

Lemma 3.2. *Suppose that $q \in W^{1,p}(\Omega)$. Let the scattering matrix*

$$S(k) = \begin{pmatrix} 0 & s_1(k) \\ s_2(k) & 0 \end{pmatrix}.$$

Then, there are constants C_1 and C_2 such that

$$|s_1(k)| \le \frac{C_1}{|k|+1}(\|a\|_p + 1)\exp(C_2\|a\|_p)\|q\|_{W^{1,p}}, \tag{3.13}$$

$$|s_2(k)| \le \frac{C_1}{|k|+1}(\|a\|_p + 1)\exp(C_2\|a\|_p)(\|q\|_p + 1). \tag{3.14}$$

Proof. Let
$$m = I + \tilde{m}.$$
By (3.2), we have
$$s_1(k) = -\frac{1}{\pi}\int_\Omega \tilde{e}_k(z)e^{Ta(z)}q(z)(1+\tilde{m}_{22}(z))d\mu(z).$$
If $|k|$ is small, it is easy to see, using (3.4), that
$$|s_1(k)| \le C_1\|a\|_p \exp(C_2\|a\|_p)\|q\|_p.$$
If $|k|$ is large, we get from (3.4)
$$\left|\int_\Omega \tilde{e}_k(z)e^{Ta(z)}q(z)\tilde{m}_{22}(z)d\mu(z)\right| \le \frac{C_1}{|k|}(\|a\|_p+1)\exp(C_2\|a\|_p)\|q\|_p^2.$$
By an integration by parts, we can show that
$$\left|\int_\Omega \tilde{e}_k(z)e^{Ta(z)}q(z)d\mu(z)\right| \le \frac{C_1}{|k|}(\|a\|_p+1)\exp(C_2\|a\|_p)\|q\|_{W^{1,p}}.$$
This gives (3.13).

Since
$$s_2(k) = -\frac{1}{\pi}\int_\Omega e_{-k}(z)e^{-Ta(z)}(1+\tilde{m}_{11}(z))d\mu(z),$$
(3.14) follows from (3.4) and the same argument. This completes the proof. □

4. Proof of Theorem 1.6

In this section we prove Theorem 1.6. We need the following lemma due to Sun (equation (3.44) in [10]).

Lemma 4.1. *If $\mathcal{C}_{a_1,q_1} = \mathcal{C}_{a_2,q_2}$, then $\partial^{-1}a_1 = \partial^{-1}a_2$ on $\partial\Omega$.*

Z. Sun prove this lemma under the assumption that the DN maps are the same. However, exactly the same argument works with the assumption $\mathcal{C}_{a_1,q_1} = \mathcal{C}_{a_2,q_2}$. Also Sun's proof works under the weaker regularity assumptions assumed in this paper.

Lemma 4.2. *Suppose that $\mathcal{C}_{a_1,q_1} = \mathcal{C}_{a_2,q_2}$. If $I - D_k^{-1}Q^{(1)}$ is invertible on $L^\infty(\Omega)$, so is $I - D_k^{-1}Q^{(2)}$.*

Proof. Suppose that
$$(I - D_k^{-1}Q^{(2)})m^{(2)} = 0 \quad \text{in } \Omega.$$
We then define $\psi^{(2)}$ by (2.9). Then, by Lemma 2.1, $\psi_{21}^{(2)}$ and $\psi_{22}^{(2)}$ are solutions of the equation $(\overline{\partial}+\bar{a}_2)(\partial - a_2)u - q_2u = 0$ in Ω and there are $f_{ij} \in \mathcal{H}(\mathbb{C}\setminus\overline{\Omega})\cap(\mathbb{C}\setminus\Omega)$ with $f_{ij}(z) = O(|z|^{-1})$ as $|z|\to\infty$, $i,j = 1,2$, satisfying (2.12) and (2.13). Since $\mathcal{C}_{a_1,q_1} = \mathcal{C}_{a_2,q_2}$, there are solutions, say u_1 and u_2, of $(\overline{\partial}+\bar{a}_1)(\partial - a_1)u - q_1u = 0$ in Ω such that
$$\begin{cases} u_1|_{\partial\Omega} &= e^{\partial^{-1}a_2}e^{-i\bar{z}\bar{k}}\bar{f}_{21},\\ (\partial - a_1)u_1|_{\partial\Omega} &= e^{-\overline{\partial}^{-1}\bar{a}_2}e^{izk}f_{11},\end{cases}$$

and

$$\begin{cases} u_2|_{\partial\Omega} & = e^{\partial^{-1}a_2}e^{-i\bar{z}k}\bar{f}_{22}, \\ (\partial - a_1)u_2|_{\partial\Omega} & = e^{-\overline{\partial}^{-1}\bar{a}_2}e^{iz\bar{k}}f_{12}. \end{cases}$$

Define $m^{(1)}$ by

$$\begin{pmatrix} (\partial - a_1)u_1 & (\partial - a_1)u_2 \\ u_1 & u_2 \end{pmatrix} = \begin{pmatrix} e^{-\overline{\partial}^{-1}\bar{a}_1} & 0 \\ 0 & e^{\partial^{-1}a_1} \end{pmatrix} m^{(1)}(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}.$$

Since $\partial^{-1}a_1 = \partial^{-1}a_2$ on $\partial\Omega$ by Lemma 4.1, we have $m_{11}^{(1)} = f_{11}$, $e_k m_{21}^{(1)} = \bar{f}_{21}$, $\tilde{e}_{-k}m_{12}^{(1)} = f_{12}$, $m_{22}^{(1)} = \bar{f}_{22}$ on $\partial\Omega$ and hence $\mathcal{C}(E_k m^{(1)}|_{\partial\Omega}) = 0$ in Ω. Therefore, by Lemma 2.1, we have

$$(I - D_k^{-1}Q^{(1)})m^{(1)} = 0 \quad \text{in } \Omega.$$

Since $I - D_k^{-1}Q^{(1)}$ is invertible, we have $m^{(1)} = 0$ in Ω. Thus we conclude that $f_{ij} = 0$. It then follows from (2.12) and (2.13) that $\psi_{2j}^{(2)} = \partial\psi_{2j}^{(2)} = 0$ on $\partial\Omega$, $j = 1, 2$. We then have from the unique continuation property of the Schrödinger equation with magnetic potential ([1]) that $\psi_{2j} = 0$ in Ω, and hence $m^{(2)} = 0$. By the Fredholm alternative, we have the invertability of $I - D_k^{-1}Q^{(2)}$ on $L^\infty(\Omega)$. This completes the proof. □

Lemma 4.3. *If $\mathcal{C}_{a_1,q_1} = \mathcal{C}_{a_2,q_2}$ and $I - D_k^{-1}Q^{(1)}$ is invertible on $L^\infty(\Omega)$, then the corresponding scattering matrices coincide, i.e., $S^{(1)}(k) = S^{(2)}(k)$ for all $k \in \mathbb{C}$.*

Proof. Let $m^{(j)} = (I - D_k^{-1}Q^{(j)})^{-1}(I)$, $j = 1, 2$. Then the scattering matrix $S^{(j)}(k)$ is given by

$$\begin{aligned} S^{(j)}(k) &= -\frac{1}{\pi}\mathcal{J}\int_\Omega E_k Q m^{(j)} d\mu(z) \\ &= -\frac{1}{\pi}\mathcal{J}\int_\Omega D E_k m^{(j)} d\mu(z) \\ &= \begin{pmatrix} 0 & \dfrac{1}{2\pi i}\displaystyle\int_{\partial\Omega} \tilde{e}_{-k} m_{12}^{(j)} dz \\ -\dfrac{1}{2\pi i}\displaystyle\int_{\partial\Omega} e_k m_{21}^{(j)} d\bar{z} & 0 \end{pmatrix}. \end{aligned}$$

Thus it suffices to prove that $m^{(1)} = m^{(2)}$ on $\partial\Omega$. This can be proved using (2.11). In fact, define $\psi^{(j)}$, $j = 1, 2$, according to (2.9), namely,

$$\psi^{(j)}(z,k) := \begin{pmatrix} e^{-\overline{\partial}^{-1}\bar{a}_j} & 0 \\ 0 & e^{\partial^{-1}a_j} \end{pmatrix} m^{(j)}(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}.$$

Then, for $k = 1, 2$ and $j = 1, 2$, $\psi_{1k}^{(j)}$ and $\psi_{2k}^{(j)}$ satisfy

$$\begin{aligned} (\overline{\partial} + \bar{a}_j)(\partial - a_j)\psi_{2k}^{(j)} - q_j\psi_{2k}^{(j)} = 0, &\quad \text{in } \Omega \\ (\partial - a_j)\psi_{2k}^{(j)} = \psi_{1k}^{(j)}. & \end{aligned}$$

Since $\mathcal{C}_{a_1,q_1} = \mathcal{C}_{a_2,q_2}$, there exist solutions, say u_1 and u_2, of the equation $(\overline{\partial} + \bar{a}_2)(\partial - a_2)v - q_2 v = 0$ in Ω such that

$$u_k|_{\partial\Omega} = \psi^{(1)}_{2k}|_{\partial\Omega},$$
$$(\partial - a_2)u_k|_{\partial\Omega} = \psi^{(1)}_{1k}|_{\partial\Omega}.$$

Define m by

$$\begin{pmatrix} (\partial - a_2)u_1 & (\partial - a_2)u_2 \\ u_1 & u_2 \end{pmatrix} = \begin{pmatrix} e^{-\overline{\partial}^{-1}\bar{a}_2} & 0 \\ 0 & e^{\partial^{-1}a_2} \end{pmatrix} m(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}.$$

Since $\partial^{-1}a_1 = \partial^{-1}a_2$ on $\partial\Omega$, we have $m(z,k) = m^{(1)}(z,k)$ for all $z \in \partial\Omega$ and $k \in \mathbb{C}$ and hence

$$E_k^{-1}\mathcal{C}(E_k m|_{\partial\Omega}) = E_k^{-1}\mathcal{C}(E_k m^{(1)}|_{\partial\Omega}) = I \quad \text{in } \Omega.$$

It then follows from (2.11) that

$$(I - D_k^{-1}Q^{(2)})m = I = (I - D_k^{-1}Q^{(2)})m^{(2)} \quad \text{in } \Omega.$$

Since $(I - D_k^{-1}Q^{(2)})$ is invertible, we have $m = m^{(2)}$. In particular, $m^{(1)} = m^{(2)}$ on $\partial\Omega$. This completes the proof. □

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. If ϵ is so small that $C_1 \exp(C_2 M)\epsilon < 1$, then by Lemma 2.2, $(I - D_k^{-1}Q^{(1)})$ is invertible on $L^\infty(\Omega)$. Then, $(I - D_k^{-1}Q^{(2)})$ is also invertible on $L^\infty(\Omega)$ by Lemma 4.2. Define $m^{(j)}$, $j = 1, 2$, by

$$(I - D_k^{-1}Q^{(j)})m^{(j)} = I \quad \text{in } \Omega,$$

and let $S^{(j)}(k)$ be the corresponding scattering matrix. Then by Lemma 4.3, we have $S^{(1)}(k) = S^{(2)}(k)$ for all $k \in \mathbb{C}$. Put $S(k) := S^{(1)}(k) = S^{(2)}(k)$ and $m(z,k) := m^{(1)}(z,k) - m^{(2)}(z,k)$. Then, by (3.3), m satisfies

$$\frac{\partial}{\partial \bar{k}} m(z,k) = m(z,\bar{k})\Lambda_k(z)S(k), \quad k \in \mathbb{C}. \tag{4.1}$$

Moreover, by (3.4) and (3.5), we have

$$\|m(\cdot,k)\|_\infty \le \frac{C}{1+|k|}, \quad k \in \mathbb{C}. \tag{4.2}$$

We now show that $m \equiv 0$. In view of (4.1), we obtain

$$\frac{\partial}{\partial \bar{k}}(m_{11} \pm m_{21}) = (m_{12}(z,\bar{k}) \pm m_{22}(z,\bar{k}))e_{-k}s_2,$$
$$\frac{\partial}{\partial \bar{k}}(m_{12} \pm m_{22}) = (m_{11}(z,\bar{k}) \pm m_{21}(z,\bar{k}))\tilde{e}_k s_1.$$

We then obtain using (4.2) that

$$m_{11} \pm m_{21} = \overline{\partial}_k^{-1}((m_{12}(z,\bar{k}) \pm m_{22}(z,\bar{k}))e_{-k}s_2),$$
$$m_{12} \pm m_{22} = \overline{\partial}_k^{-1}((m_{11}(z,\bar{k}) \pm m_{21}(z,\bar{k}))\tilde{e}_k s_1).$$

Here $\overline{\partial}_k^{-1}$ is defined to be

$$\overline{\partial}_k^{-1} f(w) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(k)}{k-w} d\mu(k), \quad w \in \mathbb{C},$$

and ∂_k^{-1} is defined likewise. It is well-known that $\overline{\partial}^{-1} : L^2_{-\delta} \to L^2_{-\delta+1}$ is bounded where L^2_δ is the L^2 space on $\mathbb{C}$ weighted by $(1+|k|^2)^\delta$. Thus it follows from (3.13) and (3.14) that

$$\|m_{11} \pm m_{21}\|_{L^2_\delta} \le C_1(\|a\|_p + 1)\exp(C_2\|a\|_p)(\|q\|_p + 1)\|m_{12} \pm m_{22}\|_{L^2_\delta},$$
$$\|m_{12} \pm m_{22}\|_{L^2_\delta} \le C_1(\|a\|_p + 1)\exp(C_2\|a\|_p)\|q\|_{W^{1,p}}\|m_{11} \pm m_{21}\|_{L^2_\delta}.$$

Here the L^2_δ-norms are in the k-variable. Thus we have

$$\|m_{11} \pm m_{21}\|_{L^2_\delta} \le C(M+1)^2 e^{CM}\epsilon\|m_{11} \pm m_{21}\|_{L^2_\delta}$$

for some constant C. If ϵ is so small that $C(M+1)^2e^{CM}\epsilon < 1$, then we have $m_{11} \pm m_{21} = 0$ and $m_{12} \pm m_{22} = 0$. Hence $m = 0$, or $m^{(1)} = m^{(2)}$. It then follows that $Q^{(1)} = Q^{(2)}$ in Ω, and hence $e^{Ta_1} = e^{Ta_2}$ and $q_1 = q_2$ in Ω. This completes the proof. $\square$

Theorem 1.1 immediately follows from Theorem 1.6 once we observe that

$$\sum_{j=1}^{2}\left(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j\right)^2 = (\overline{\partial} + \bar{a})(\partial - a) + \mathrm{rot} a$$

where $a := \frac{1}{2}(A_2 + iA_1)$.

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