Reconstruction of the Potential from Partial Cauchy Data for the Schrödinger Equation

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Abstract

In this paper we prove in dimension $d \geq 3$ that the knowledge of the partial Cauchy data for the Schrödinger equation on any open subset Γ of the boundary determines uniquely the potential q provided that q is known in a neighborhood of the boundary. We also derive a formula for calculating the potential q from the partial Dirichlet to Neumann map associated to q. We apply the methods to identify locations of small volume fraction perturbations of the potential (or the index of refraction) of inhomogeneous media by making local measurements on the boundary.

1 Introduction

It was shown in [7] that in dimension $d \geq 3$ the complete Cauchy data (on all the boundary) uniquely determines a potential q. It is an open problem whether partial Cauchy data on an open set of the boundary still determines uniquely the potential. Kohn and Vogelius [5] proved that we can uniquely identify a *real-analytic potential* if we know the restriction of the Dirichlet to Neumann map on any open subset Γ of the boundary for all functions supported in Γ.

Recently, Bukhgeim and Uhlmann [4] showed, by using Carleman estimates to construct complex geometrical optics solutions vanishing on the complement of Γ, that this is possible for the general case $q \in L^\infty$ from knowledge of the Cauchy data measured on *particular subsets* Γ of the boundary, depending on the geometry of the domain.

The goal of this paper is to prove that in dimension $d \geq 3$ the knowledge of the partial Cauchy data for the Schrödinger equation $\Delta - q$ on *any open subset* Γ of the boundary determines uniquely the potential q provided that q is *known in a neighborhood of the boundary*. We also derive a formula for calculating the potential q from the partial Dirichlet-to-Neumann map (DN) associated to q and Γ by extending Nachman's reconstruction procedure [6] to this case. We also use in the reconstruction procedure

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the complex geometrical optics solutions constructed in [4]. It turns out that if the potential is a-priori known in a neighborhood of the boundary both the proof of the global uniqueness result and the derivation of a reconstruction formula are *surprisingly simple*. Note that this assumption is very realistic in most of the applications. In fact we apply this reconstruction to identify small volume fraction perturbations of the potential by measuring the local DN map on Γ. An important physical application is to reconstruct small volume fraction perturbations of the index of refraction or sound speed of a medium by measuring at fixed frequency the corresponding DN map on part of the boundary.

The paper is organized as follows. In section 2 we use the geometrical optics solutions for the Schrödinger equation to prove a global uniqueness result. The main idea for proving this global uniqueness result is to write an identity similar to that proven by Alessandrini [1] and notice that we can approximate the geometrical optics solutions in the interior of the domain by functions that are solutions to the Schrödinger equation and vanishing on the complementary of Γ as it is the case for a constant potential [2].

In section 3 we derive a formula for calculating the potential q from the partial DN map associated to q and Γ.

In section 4 we consider for the Schrödinger equation the inverse problem of identifying locations of small volume fraction perturbations of the potential and the index of refraction. Applying the results in the previous sections we reduce this inverse problem arising in applications to calculations of inverse Fourier transforms.

2 Global Uniqueness

Let $\Omega \subset \mathbf{R}^d$ be a bounded domain with $\mathcal{C}^2$ boundary. We denote by ν the unit-outer normal to $\partial\Omega$. Let Γ be a smooth open subset of the boundary $\partial\Omega$ and Γ_c denotes $\partial\Omega \setminus \overline{\Gamma}$. We assume throughout that $\mathbf{R}^d \setminus \overline{\Omega}$ is connected.

Introduce the trace space

$$\widetilde{H}^{\frac{1}{2}}(\Gamma) = \Big\{ f \in H^{\frac{1}{2}}(\partial\Omega), f \equiv 0 \text{ on } \Gamma_c \Big\}.$$

Here and in the sequel we identify f defined only on Γ with its extension by 0 to all $\partial\Omega$. We define $H^{-\frac{1}{2}}(\Gamma)$ as the dual of $\widetilde{H}^{\frac{1}{2}}(\Gamma)$. We define the set of partial Cauchy data for real-valued $q \in L^\infty(\Omega)$ by

$$C_q = \Big\{ (u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}|_\Gamma) \,|\, u \in H^1(\Omega), (\Delta - q)u = 0 \text{ on } \Omega, u|_{\Gamma_c} = 0 \Big\}.$$

We have $C_q \subset \widetilde{H}^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$. If 0 is not a Dirichlet eigenvalue of $\Delta - q$, then in fact C_q is a graph, namely

$$C_q = \Big\{ (f, \Lambda_q(f)) \in \widetilde{H}^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \Big\},$$

where $\Lambda_q(f) = \frac{\partial u}{\partial \nu}|_\Gamma$ with $u \in H^1(\Omega)$ the solution of

$$\left\{ \begin{array}{l} (\Delta - q)u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = f \in \widetilde{H}^{\frac{1}{2}}(\Gamma). \end{array} \right.$$

Λ_q is the *local Dirichlet to Neumann map* in this case.

Our main result in this section is

Theorem 1 *Let $d \geq 3$ and q_i real-valued in $L^\infty(\Omega), i = 1,2$. Assume $q_1 = q_2$ almost everywhere in a neighborhood of the boundary $\partial\Omega$ and $C_{q_1} = C_{q_2}$. Then $q_1 = q_2$ almost everywhere in Ω.*

We use in the proof two Lemmas. The first one is a generalization of Alessandrini's identity [1].

Lemma 1 *Let $\Omega' \subset\subset \Omega$ containing the support of $q_1 - q_2$, Ω' bounded with $\mathcal{C}^2$ boundary. Let $u_i \in H^1(\Omega)$ satisfy*

$$(1) \qquad \begin{cases} (\Delta - q_i)u_i = 0 \ \text{in } \Omega \\ u_i|_{\Gamma_c} = 0, \quad i = 1,2. \end{cases}$$

Assume $q_1 = q_2$ almost everywhere in a neighborhood of the boundary $\partial\Omega$ and $C_{q_1} = C_{q_2}$. Then

$$(2) \qquad \int_{\Omega'} (q_1 - q_2)u_1 u_2 \, dx = 0.$$

Proof. Using Green's theorem we have that

$$(3) \qquad \int_{\Omega'} (q_1 - q_2)u_1 u_2 \, dx = \int_\Gamma (\frac{\partial u_1}{\partial \nu} u_2 - u_1 \frac{\partial u_2}{\partial \nu}) \, ds,$$

where ds denotes surface measure.

Let $v_1 \in H^1(\Omega)$ be solution of $(\Delta - q_1)v_1 = 0$ in Ω such that $v_1|_{\Gamma_c} = 0$ and $v_1|_\Gamma = u_2|_\Gamma$. From $C_{q_1} = C_{q_2}$ it follows that

$$(4) \qquad v_1|_{\Gamma_c} = 0, v_1|_\Gamma = u_2|_\Gamma \Rightarrow \frac{\partial v_1}{\partial \nu}|_\Gamma = \frac{\partial u_2}{\partial \nu}|_\Gamma.$$

Then by Green's theorem again

$$(5) \qquad 0 = \int_{\Omega'} (q_1 - q_1)u_1 v_1 \, dx = \int_\Gamma (\frac{\partial u_1}{\partial \nu} v_1 - u_1 \frac{\partial v_1}{\partial \nu}) \, ds.$$

Then from (3), (4), and (5) we deduce the identity (2). □

The second Lemma states that the set of solutions of the Schrödinger equation with boundary data 0 on Γ_c is dense in $L^2(\Omega')$ in the set of all solutions.

Lemma 2 *[2] Let $q \in L^\infty(\Omega)$. Let $\Omega' \subset\subset \Omega$, Ω' open set with $\mathcal{C}^2$ boundary and $\Omega \setminus \overline{\Omega'}$ is connected. We define*

$$\widetilde{N}(\Omega) = \Big\{ v \in H^2(\Omega) \,|\, (\Delta - q)v = 0 \ \text{in } \Omega, v = 0 \ \text{on } \ \Gamma_c \Big\}$$

and

$$N(\Omega) = \Big\{ v \in H^2(\Omega) \,|\, (\Delta - q)v = 0 \ \text{in } \Omega \Big\}.$$

Then $\widetilde{N}(\Omega)$ is dense, in the $L^2(\Omega')$ norm, in $N(\Omega)$.

Proof. Assume there is an element in $N(\Omega)$, which we denote v, such that $\int_{\Omega'} vw dx = 0, \forall\, w \in \widetilde{N}(\Omega)$. Let G_0 be the Dirichlet Green's function in Ω:

$$\begin{cases} (\Delta_x - q)G_0(x,y) = \delta_y(x) \ \text{ in } \ \Omega, \\ G_0(x,y) = 0 \ \text{ for } \ x \in \partial\Omega. \end{cases}$$

Define $\widetilde{H}^{\frac{3}{2}}(\Gamma) = \Big\{ p \in H^{\frac{3}{2}}(\partial\Omega), p \equiv 0 \text{ on } \Gamma_c \Big\}$.

For any $w \in \widetilde{N}(\Omega)$ integration by parts yields

$$w(x) = \int_\Gamma \frac{\partial G_0(x,y)}{\partial \nu(y)} \, w(y) \, ds(y), x \in \Omega,$$

where $w|_\Gamma \in \widetilde{H}^{\frac{3}{2}}(\Gamma)$ since w belongs to $H^2(\Omega)$. Conversely, for any $p \in \widetilde{H}^{\frac{3}{2}}(\Gamma)$, the function

$$w(x) = \int_\Gamma \frac{\partial G_0(x,y)}{\partial \nu(y)} \, p(y) \, ds(y), x \in \Omega,$$

satisfies $(\Delta - q)w = 0$ in Ω. Also, since for any $x \in \Gamma_c$, $G_0(x,y) = 0$ for any $y \in \Omega$ and therefore $\dfrac{\partial G_0}{\partial \nu(y)}(x,y) = 0, \forall \; x \in \Gamma_c, \forall \; y \in \Gamma$, we have $w = 0$ on Γ_c. Therefore $w \in \widetilde{N}(\Omega)$ can be represented as follows

$$w(x) = \int_\Gamma \frac{\partial G_0(x,y)}{\partial \nu(y)} \, p(y) \, ds(y), x \in \Omega,$$

for some $p \in \widetilde{H}^{\frac{3}{2}}(\Gamma)$. Thus

$$\int_{\Omega'} v(y) \frac{\partial G_0(x,y)}{\partial \nu(x)} \, dy = 0, \;\; \forall \; x \in \Gamma.$$

Define $u(x) := \displaystyle\int_{\Omega'} v(y) G_0(x,y) \, dy$. The function $u \in H^2(\Omega)$ satisfies $u|_{\partial\Omega} = 0, \dfrac{\partial u}{\partial \nu}|_\Gamma = 0$, and

$$(\Delta - q)u = \begin{cases} v & \text{in } \Omega', \\ 0 & \text{in } \Omega \setminus \overline{\Omega'}. \end{cases}$$

By the unique continuation principle it follows that $u \equiv 0$ in $\Omega \setminus \overline{\Omega'}$, and so $u = \dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega'$. Since $(\Delta - q)u = v$ in Ω', it follows by multiplying this equation by v and integrating by parts over Ω' that $\displaystyle\int_{\Omega'} v^2 dx = 0$. Thus, $v \equiv 0$ in Ω', and, by again the unique continuation principle applied to v, $v \equiv 0$ in Ω, which proves the density lemma. $\square$

Proof of Theorem 1. We extend $q_i = 0$ in $\mathbf{R}^d \setminus \overline{\Omega}$. Using [7], for $-1 < \delta < 0$, we can construct solutions of $(\Delta - q_i)v_i = 0$ on $\mathbf{R}^d$ of the form

$$v_i = e^{x \cdot \rho_i}(1 + \psi_{q_i}(x, \rho_i)), i = 1, 2 \tag{6}$$

for $|\rho_i|$ sufficiently large with $\psi_{q_i}(\cdot, \rho_i) \in L^2_\delta(\mathbf{R}^d)$. Moreover

$$||\psi_{q_i}(\cdot, \rho_i)||_{L^2_\delta(\mathbf{R}^d)} \leq \frac{C}{|\rho_i|}. \tag{7}$$

Here $L^2_\delta(\mathbf{R}^d)$ denotes the weighted L^2-space

$$L^2_\delta(\mathbf{R}^d) = \Big\{ f \in L^2_{loc}(\mathbf{R}^d) | \int_{\mathbf{R}^d} (1 + |x|^2)^\delta |f(x)|^2 \, dx \; < +\infty \Big\}.$$

We choose

$$\rho_1 = \frac{\eta}{2} + i(\frac{k+l}{2})$$
$$\rho_2 = -\frac{\eta}{2} + i(\frac{k-l}{2})$$

where $\eta, k, l \in \mathbf{R}^d$ such that $\eta \cdot k = \eta \cdot l = k \cdot l = 0$, and $|\eta|^2 = |k|^2 + |l|^2$.

Using Lemma 2 we can approximate any $v_i \in N(\Omega)$ by elements of $\widetilde{N}(\Omega)$. Therefore by Lemma 1

$$\int_{\Omega'} (q_1 - q_2) v_1 v_2 \, dx = 0 \tag{8}$$

$\forall\ v_i \in H^1(\Omega)$ solution of $(\Delta - q_i)v_i = 0$ in Ω. Substituting (6) into (8), letting $|l| \to +\infty$, and using (7) we conclude

$$\widehat{(q_1 - q_2)}(k) = 0 \quad \forall\, k \in \mathbf{R}^d$$

proving the theorem. □

Theorem 1 has an immediate consequence in Electric Impedance Tomography. Let $\gamma \in \mathcal{C}^2(\overline{\Omega})$ be a strictly positive function on $\overline{\Omega}$. If we assume that there are no sources or sinks of current in Ω, the equation for the potential with Dirichlet data supported on Γ is

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = f \in \widetilde{H}^{\frac{1}{2}}(\Gamma). \end{cases} \tag{9}$$

The local Dirichlet to Neumann map is defined in this case as follows:

$$\begin{aligned} \Lambda_\gamma : \widetilde{H}^{\frac{1}{2}}(\Gamma) &\to H^{-\frac{1}{2}}(\Gamma) \\ f &\mapsto (\gamma \frac{\partial u}{\partial \nu})|_\Gamma. \end{aligned}$$

As a direct consequence of Theorem 1 we obtain

Corollary 1 *Let $d \geq 3$ and $\gamma_i \in \mathcal{C}^2(\overline{\Omega}), i = 1, 2$. Assume $\gamma_1 = \gamma_2$ in a neighborhood of the boundary $\partial\Omega$ and*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \forall\, f \in \widetilde{H}^{\frac{1}{2}}(\Gamma).$$

Then $\gamma_1 = \gamma_2$.

Proof. As it is well known by now we can reduce the conductivity problem (9) to a Schrödinger equation using the transform $w = \gamma^{\frac{1}{2}} u$. If u solves (9), then w satisfies $(\Delta - q)w = 0$ in Ω with $q = \dfrac{\Delta \gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}}$. It is also easy to see that

$$\Lambda_q(f) = \gamma^{-\frac{1}{2}}|_{\partial\Omega} \Lambda_\gamma(\gamma^{-\frac{1}{2}}|_{\partial\Omega} f) + \frac{1}{2}(\gamma^{-1}\frac{\partial \gamma}{\partial \nu})|_{\partial\Omega} f.$$

Now if we assume that γ is known in a neighborhood of the boundary $\partial\Omega$ than we know $\Lambda_q(f)$ for all $f \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$ and therefore the proof of Corollary 1 immediately follows from Theorem 1. □

Another immediate consequence is to an inverse problem for the acoustic equation at fixed frequency. The propagation of time-harmonic waves with frequency ω in a medium with index of refraction or sound speed $n(x) > 0$ is given by:

$$(\Delta + \omega^2 n(x))u = 0 \quad \text{in } \Omega,$$

The set of Cauchy data is defined by

$$C_n = \Big\{(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}|_\Gamma) \mid u \in H^1(\Omega), (\Delta + \omega^2 n(x))u = 0 \text{ on } \Omega, u|_{\Gamma_c} = 0\Big\}.$$

Then we have

Corollary 2 *Let n_i be a strictly positive function in $L^\infty(\Omega)$, $i = 1, 2$ and $n_i = n_0$, a fixed positive constant, in a neighborhood of the boundary $\partial\Omega$. Then if $C_{n_1} = C_{n_2}$ we conclude $n_1 = n_2$.*

We conclude this section by observing that using the identity (2), the density Lemma 2 and the complex geometrical optics solutions (6) it is easy to prove a stability result (a logarithmic dependence continuous result) by rewriting Alessandrini's proof [1] for this case.

3 Reconstruction

Let $q_0 \in L^\infty(\Omega)$ be a known function. Assume that $q = q_0$ almost everywhere in a neighborhood of $\partial\Omega$. Denote $\Omega' \subset\subset \Omega$, Ω' bounded open with $\mathcal{C}^2$ boundary containing the support of $q - q_0$. In this section we derive a formula for calculating q from the local Dirichlet to Neumann map $\Lambda_q : \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$.

Assume that $\Lambda_q : \widetilde{H}^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\Gamma)$ is known then for any $u, v \in H^1(\Omega)$ satisfying

$$(\Delta - q)u = 0 \text{ in } \Omega, (\Delta - q_0)v = 0 \text{ in } \Omega$$
$$u|_{\Gamma_c} = v|_{\Gamma_c} = 0$$

we have

$$\int_\Omega (q - q_0)uv \, dx = \int_\Gamma u(\Lambda_q - \Lambda_{q_0})v \, ds, \tag{10}$$

where Λ_{q_0} denotes the local Dirichlet to Neumann map associated to the potential q_0.

Extend q and q_0 by 0 in $\mathbf{R}^d$. Let $\rho \in \mathbf{C}^d \setminus \{0\}$ with $\rho \cdot \rho = 0$. Suppose $u_\rho \in H^2_{loc}(\mathbf{R}^d \setminus \Omega)$ is a solution of

$$\begin{cases} \Delta u_\rho = 0 \quad \text{in } \mathbf{R}^d \setminus \overline{\Omega}, \\ (\Delta - q)u_\rho = 0 \quad \text{in } \Omega, u_\rho|_{\Gamma_c} = 0, u_\rho = 0 \text{ on } \partial\Omega, [\frac{\partial u_\rho}{\partial \nu}] = 0 \text{ on } \partial\Omega \setminus \Gamma_c, \end{cases} \tag{11}$$

subject to the radiation condition

$$e^{-x\cdot\rho}u_\rho - 1 \in L^2_\delta = \Big\{f : \int_{\mathbf{R}^d} (1 + |x|^2)^\delta |f(x)|^2 \, dx \ < +\infty\}, -1 < \delta < 0\Big\}. \tag{12}$$

Here $[f]$ denotes the jump of f. Let Δ^D_ρ denote the operator $\Delta + 2\rho \cdot \nabla$ in $\mathbf{R}^d \setminus \overline{\Omega}$ with *a homogeneous Dirichlet boundary condition* on $\partial\Omega$. Let

$$G^D_\rho \in \mathcal{D}'(\mathbf{R}^d \setminus \overline{\Omega} \times \mathbf{R}^d \setminus \overline{\Omega})$$

denote the Schwartz kernel of the operator $(\Delta_\rho^D)^{-1}$ satisfying the same condition at infinity as $G_\rho(x,y) = G_\rho(x-y)$ where $G_\rho \in \mathcal{D}'(\mathbf{R}^d \times \mathbf{R}^d)$ denotes the Schwartz kernel of the operator $(\Delta_\rho)^{-1}$. The distribution G_ρ is given by

$$G_\rho(x) = \int_{\mathbf{R}^d} \frac{e^{ix\cdot\xi}}{\xi^2 + 2i\rho\cdot\xi} d\xi.$$

We can find G_ρ^D by solving the integral equation

$$G_\rho^D(x,y) = G_\rho(x,y) + \int_{\partial\Omega} G_\rho(x,y)\frac{\partial G_\rho^D}{\partial \nu_y}(x,y)ds(y),$$

for any x outside Ω. We can find the solution of this integral equation by solving the integral equation of the first kind

$$-G_\rho(x) = \int_{\partial\Omega} G_\rho(x,y)\frac{\partial G_\rho^D}{\partial \nu_y}(x,y), \quad x \in \partial\Omega.$$

This equation can be easily be shown to be uniquely solvable.

We have that

$$g_\rho^D(x,y) = e^{x\cdot\rho} G_\rho^D(x,y)$$

is an exterior Dirichlet Green's function for Δ, namely

$$\begin{cases} \Delta_x g_\rho^D(x,y) = \delta_y & \text{in } \mathbf{R}^d \setminus \overline{\Omega} \\ g_\rho^D(x,y) = 0 & \forall\, x \in \partial\Omega, y \in \mathbf{R}^d \setminus \overline{\Omega}. \end{cases}$$

Using Green's theorem in $\mathbf{R}^d \setminus \overline{\Omega}$ and the radiation condition (12) we conclude that

$$u_\rho(x) = e^{x\cdot\rho} + \int_\Gamma \frac{\partial g_\rho^D}{\partial \nu(y)}(x,y) u_\rho(y)\, ds(y), \quad \forall\, x \in \mathbf{R}^d \setminus \overline{\Omega}.$$

Therefore

$$\frac{\partial u_\rho}{\partial \nu}|_\Gamma(x) = \rho\cdot\nu(x)e^{x\cdot\rho} + \text{ p.v. } \int_\Gamma \frac{\partial^2 g_\rho^D}{\partial\nu(x)\partial\nu(y)}(x,y)u_\rho(y)\, ds(y), \quad \forall\, x \in \Gamma,$$

where the latter integral is interpreted as a singular integral.

Since

$$\frac{\partial u_\rho}{\partial \nu}|_\Gamma(x) = \Lambda_q(u_\rho|_\Gamma)$$

we obtain that $u_\rho|_\Gamma$ solves the (hyper-singular) integral equation on the open surface Γ:

$$\Lambda_q(u_\rho|_\Gamma)(x) - \text{ p.v. } \int_\Gamma \frac{\partial^2 g_\rho^D}{\partial\nu(x)\partial\nu(y)}(x,y)u_\rho|_\Gamma(y)\, ds(y) = \rho\cdot\nu(x)e^{x\cdot\rho}, \tag{13}$$

$\forall\, x \in \Gamma$.

The following holds.

Lemma 3 *Assume that* 0 *is not a Dirichlet eigenvalue of* $(\Delta - q)$ *in* Ω.

a) *Suppose* u_ρ *is a solution of (11)-(12). Then* $u_\rho|_\Gamma$ *solves (13).*

b) *Conversely, if* $u_\rho|_\Gamma \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$ *is a solution of (13) then*

$$u_\rho(x) = \begin{cases} e^{x\cdot\rho} + \displaystyle\int_\Gamma \frac{\partial g_\rho^D}{\partial\nu(y)}(x,y)u_\rho(y)\; ds(y), & \forall\, x \in \mathbf{R}^d \setminus \overline{\Omega}, \\ \displaystyle\int_\Gamma \frac{\partial G_0(x,y)}{\partial\nu(y)}\; u_\rho(y)\; ds(y), x \in \Omega, & \end{cases}$$

is a solution of (11)-(12).

c) *There is a unique solution* $u_\rho|_\Gamma \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$ *of (13).*

Proof. **a)** is already proven. **b)** immediately follows from the representation formulas of u_ρ in Ω and $\mathbf{R}^d \setminus \overline{\Omega}$ and properties of the Green's functions g_ρ^D and G_0. Let us proof **c)**. It is easily seen that g_ρ^D has the same singularity near $x = y$ that of the Green's function of classical potential theory

$$g_0(x,y) = \frac{1}{(d-2)\omega_d}|x-y|^{2-d}, \quad (\omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}).$$

Let us define the double layer potentials

$$N_\rho(f) = \text{ p.v. } \int_\Gamma \frac{\partial^2 g_\rho^D}{\partial\nu(x)\partial\nu(y)}(x,y)f|_\Gamma(y)\; ds(y), \quad \forall\, f \in \widetilde{H}^{\frac{1}{2}}(\Gamma),$$

and

$$N_0(f) = \text{ p.v. } \int_\Gamma \frac{\partial^2 g_0}{\partial\nu(x)\partial\nu(y)}(x,y)f|_\Gamma(y)\; ds(y), \quad \forall\, f \in \widetilde{H}^{\frac{1}{2}}(\Gamma).$$

We rewrite

$$\Lambda_q + N_\rho = \Lambda_q + N_0 + (N_\rho - N_0).$$

We have that $N_\rho - N_0 : \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ is compact. Moreover, $\Lambda_q + N_0 : \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ is of Fredholm type since the imbedding $L^2(\Omega) \hookrightarrow H^1(\Omega)$ is compact. Therefore (13) is an inhomogeneous integral equation of Fredholm type for $u_\rho|_\Gamma$.

Now assume that 0 is not a Dirichlet eigenvalue of $(\Delta - q)$ in Ω. Let $w_\rho \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$ be a solution to the homogeneous equation $(\Lambda_q + N_\rho)(w_\rho) = 0$ on Γ.

Define

$$\psi_\rho = \begin{cases} \psi_\rho^i & \text{in } \Omega, \\ \psi_\rho^e & \text{in } \mathbf{R}^d \setminus \overline{\Omega}, \end{cases}$$

where ψ_ρ^i is the solution to the interior problem

$$\begin{cases} (\Delta - q)\psi_\rho^i = 0 & \text{in } \Omega, \\ \psi_\rho^i = w_\rho \in \widetilde{H}^{\frac{1}{2}}(\Gamma) & \text{on } \partial\Omega, \end{cases}$$

and

$$\psi_\rho^e(x) = \int_\Gamma \frac{\partial g_\rho^D}{\partial\nu(y)}(x,y)w_\rho(y)\; ds(y), \quad \forall\, x \in \mathbf{R}^d \setminus \overline{\Omega}.$$

From the properties of the Dirichlet Green's function g_ρ^D it follows that ψ_ρ^e satisfies

$$\begin{cases} \Delta\psi_\rho^e = 0 \quad \text{in } \mathbf{R}^d \setminus \overline{\Omega}, \\ e^{-x\cdot\rho}\psi_\rho^e \in L_\delta^2, \\ \dfrac{\partial \psi_\rho^e}{\partial\nu}|_\Gamma = \Lambda_q(w_\rho) = \dfrac{\partial\psi_\rho^i}{\partial\nu}|_\Gamma \\ \psi_\rho^e = \psi_\rho^i \quad \text{on } \partial\Omega, \end{cases}$$

and therefore ψ_ρ solves

$$\begin{cases} \Delta\psi_\rho = 0 \quad \text{in } \mathbf{R}^d \setminus \overline{\Omega}, \\ (\Delta - q)\psi_\rho = 0 \quad \text{in } \Omega, \\ e^{-x\cdot\rho}\psi_\rho \in L_\delta^2, \\ [\psi_\rho] = 0 \quad \text{across } \partial\Omega, \\ [\dfrac{\partial\psi_\rho}{\partial\nu}] = \widetilde{\Lambda}_q(\psi_\rho) - \text{ p.v. } \displaystyle\int_{\partial\Omega} \frac{\partial^2 g_\rho^D}{\partial\nu(x)\partial\nu(y)}(x,y)\psi_\rho(y)\, ds(y) \quad \text{across } \partial\Omega, \end{cases}$$

where $[f]$ denotes the jump of f across $\partial\Omega$ and $\widetilde{\Lambda}_q : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ denotes the DN map associated to q on all $\partial\Omega$.

Recall that $G_\rho \in \mathcal{D}'(\mathbf{R}^d \times \mathbf{R}^d)$ is the Schwartz kernel of the operator $(\Delta + 2\rho\cdot\nabla)^{-1}$ and define $g_\rho(x,y) = e^{x\cdot\rho}G_\rho(x,y)$ as a Green's function for Δ. Following [6], we can prove that ψ_ρ has the following integral representation

$$\psi_\rho(x) = \int_{\partial\Omega} g_\rho(x,y)\Big(\widetilde{\Lambda}_q(\psi_\rho)(y) - \text{ p.v. } \int_{\partial\Omega} \frac{\partial^2 g_\rho^D}{\partial\nu(y)\partial\nu(z)}(y,z)\psi_\rho(z)\, ds(z)\Big)\, ds(y),$$

$\forall\, x \in \mathbf{R}^d \setminus \overline{\Omega}$.

Now following [6] we have the identity

$$\int_{\partial\Omega} g_\rho(x,y) \text{ p.v. } \int_{\partial\Omega} \frac{\partial^2 g_\rho^D}{\partial\nu(y)\partial\nu(z)}(y,z)f(z)\, ds(z)\, ds(y) = \int_{\partial\Omega} \frac{\partial g_\rho}{\partial\nu(y)}(x,y)f(y)\, ds(y)$$

for all $f \in H^{\frac{1}{2}}(\Omega)$ from which we conclude, as in [6], that $\psi_\rho|_{\partial\Omega}$ solves the integral equation

$$(\frac{I}{2} + S_\rho\widetilde{\Lambda}_q - B_\rho)(\psi_\rho|_{\partial\Omega}) = 0, \tag{14}$$

where

$$S_\rho(f)(x) = \int_{\partial\Omega} g_\rho(x,y)f(y)\, ds(y) \text{ and } B_\rho(f)(x) = \int_{\partial\Omega} \frac{\partial g_\rho}{\partial\nu(y)}(x,y)f(y)\, ds(y)$$

are a single and double layer potential respectively. From [6] it immediately follows that the trivial solution is the unique solution to (14) and so, $\psi_\rho|_{\partial\Omega} = w_\rho = 0$. Thus **c)** holds and we can write

$$u_\rho|_\Gamma = (\Lambda_q + N_\rho)^{-1}(\rho_1\cdot\nu(x)e^{x\cdot\rho_1}|_\Gamma)|_\Gamma. \tag{15}$$

The proof of **c)** is complete. □

Lemma 4 *Let $\rho \in \mathbf{C}^d \setminus \{0\}$ with $\rho \cdot \rho = 0$. Let $u_\rho|_\Gamma \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$ be the solution of (13). Then*

$$u_\rho(x) = \int_\Gamma \frac{\partial G_0(x,y)}{\partial \nu(y)} \; u_\rho(y) \; ds(y) = e^{x\cdot\rho}(1 + \theta_q(x,\rho)) \quad \textit{in } \Omega',$$

where

$$||\theta_q(.,\rho)||_{L^2(\Omega')} \leq \frac{C}{|\rho|}.$$

Here the constant C is independent of ρ.

Proof. For simplicity, but without loss of generality, let us assume that $q_0 = 0$. Let $\rho \in \mathbf{C}^d$ with $\rho \cdot \rho = 0$ and $\rho = \tau(\xi + i\eta)$ with $\xi, \eta \in \mathbf{R}^d, |\xi| = |\eta| = 1$. Define $\theta_q(x,\rho) = e^{-x\cdot\rho} u_\rho(x) - 1$. Then we have

$$\Delta_\rho \theta_q = q\chi(\Omega)\theta_q \quad \text{in } \mathbf{R}^d \setminus \Gamma_c,$$

where $\chi(\Omega)$ is the characteristic function of Ω. In a small neighborhood ω_ρ of the boundary $\partial\Omega$ let n denote the normal coordinate and $\varphi(\frac{n}{|\rho|})$ a smooth cut-off function which vanishes on ω_ρ. Because Ω' is compactly supported in Ω then $\varphi(\frac{n}{|\rho|})\theta_q = \theta_q$ on Ω' for $|\rho|$ sufficiently large. Moreover, if $\widetilde{\theta}_q = \varphi(\frac{n}{|\rho|})\theta_q$ in $\mathbf{R}^d$ then

$$\Delta_\rho \widetilde{\theta}_q = q\chi(\Omega)\widetilde{\theta}_q + \frac{1}{|\rho|}\nabla\widetilde{\theta}_q \cdot \nabla\varphi + \widetilde{\theta}_q(\frac{1}{|\rho|^2}\Delta\varphi + 2\frac{\rho}{|\rho|}\cdot\nabla\varphi) \quad \text{in } \mathbf{R}^d.$$

By the classical estimate from [7] this together with the radiation condition $\widetilde{\theta}_q \in L^2_\delta$ implies that

$$||\widetilde{\theta}_q||_{L^2_\delta(\mathbf{R}^d)} \leq \frac{C}{\tau},$$

for some constant C independent of τ and thus, the desired estimate holds. □

Now let

$$\rho_1 = \tfrac{\eta}{2} + i(\tfrac{k+l}{2})$$
$$\rho_2 = -\tfrac{\eta}{2} + i(\tfrac{k-l}{2})$$

where $\eta, k, l \in \mathbf{R}^d$ such that $\eta \cdot k = \eta \cdot l = k \cdot l = 0$, and $|\eta|^2 = |k|^2 + |l|^2$. By applying Lemma 4, we take

$$u = u_\rho = e^{x\cdot\rho_1}(1 + \theta_q(x,\rho_1)) \quad \text{in } \Omega',$$

and

$$v = v_\rho = e^{x\cdot\rho_2}(1 + \theta_{q_0}(x,\rho_2)) \quad \text{in } \Omega',$$

to obtain from (10) that

$$(\widehat{q - q_0})(-k) = \lim_{|l|\to+\infty} \int_\Gamma u_\rho(\Lambda_q - \Lambda_{q_0})v_\rho \; ds. \tag{16}$$

The boundary values of the solutions $u_\rho|_\Gamma$ are recovered from Λ_q by (15).

We have proved the following reconstruction formula.

Theorem 2 *Let $q_0 \in L^\infty(\Omega)$ be a given function. Assume that 0 is not a Dirichlet eigenvalue of $(\Delta - q)$ in Ω and $q = q_0$ almost everywhere in a neighborhood of $\partial\Omega$. Then*

$$(\widehat{q - q_0})(-k)$$
$$= \lim_{|l|\to+\infty} \int_\Gamma (\Lambda_q + N_{\rho_1})^{-1}(\rho_1 \cdot \nu(x) e^{x\cdot\rho_1}|_\Gamma)(x)(\Lambda_q - \Lambda_{q_0})(\Lambda_{q_0} + N_{\rho_2})^{-1}(\rho_2 \cdot \nu(x) e^{x\cdot\rho_2}|_\Gamma)(x) \; ds(x).$$

4 Application: reconstruction of the locations of small volume fraction perturbations of the potential

In this section we apply the reconstruction procedure described in Section 3 for identifying the locations of small volume fraction perturbations of the potential. Assume that $\Omega \subset \mathbf{R}^d, d \geq 3$ contains a finite number of inhomogeneities, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbf{R}^d$ is a bounded, smooth domain containing the origin. The total collection of inhomogeneities is $\mathcal{B}_\alpha = \cup_{j=1}^m (z_j + \alpha B_j)$. The points $z_j \in \Omega, j = 1, \ldots, m$, which determine the location of the inhomogeneities, are assumed to satisfy the following inequalities:

$$|z_j - z_l| \geq c_0 > 0, \forall\, j \neq l \quad \text{and} \quad \operatorname{dist}(z_j, \partial\Omega) \geq c > 0, \forall\, j, \tag{17}$$

where c is a positive constant. Assume that $\alpha > 0$, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their distance to $\mathbf{R}^d \setminus \overline{\Omega}$ is larger than c. Let $\Gamma \subset \partial\Omega$ be a given open subset of $\partial\Omega$. Let $q(x) \in \mathcal{C}^0(\Omega)$ denote the unperturbed potential. We assume that $q(x)$ is known on a neighborhood of the boundary $\partial\Omega$. Let $q_j(x) \in \mathcal{C}^0(\overline{z_j + \alpha B_j})$ denote the potential of the j-th inhomogeneity, $z_j + \alpha B_j$. Introduce the perturbed potential

$$q_\alpha(x) = \begin{cases} q(x), & x \in \Omega \setminus \overline{\mathcal{B}_\alpha}, \\ q_j(x), & x \in z_j + \alpha B_j,\ j = 1 \ldots m. \end{cases} \tag{18}$$

Consider the Schrödinger equation in the presence of the inhomogeneities $\mathcal{B}_\alpha$

$$\begin{cases} (\Delta - q_\alpha) u_\alpha = 0 \text{ in } \Omega \\ u_\alpha|_{\partial\Omega} = f \in \widetilde{H}^{\frac{1}{2}}(\Gamma), \end{cases}$$

and define the local Dirichlet to Neumann map associated to q_α by $\Lambda_{q_\alpha}(f) := \frac{\partial u_\alpha}{\partial \nu}|_\Gamma$ for all $f \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$. Let u denote the solution to the Schrödinger equation with the Dirichlet boundary condition $u = f$ on $\partial\Omega$ in the absence of any inhomogeneities and Λ_q be the local DN map associated to q.

In this section we apply Theorem 2 to identify efficiently the locations $\{z_j\}_{j=1}^m$ of the small inhomogeneities $\mathcal{B}_\alpha$ from the knowledge of the difference between the local DN maps $\Lambda_{q_\alpha} - \Lambda_q$ on Γ.

Let v be any function in $\tilde{N}(\Omega)$. As in [3], the following asymptotic formula can be derived :

$$\int_\Gamma \Big(\frac{\partial u_\alpha}{\partial \nu} v - \frac{\partial v}{\partial \nu} u_\alpha \Big)\ ds \;=\; \alpha^d \sum_{j=1}^m (q(z_j) - q_j(z_j)) |B_j| u(z_j) v(z_j) + o(\alpha^d), \tag{19}$$

where $|B_j|$ stands for the volume of the set B_j and the remainder $o(\alpha^d)$ is independent of the set of points $\{z_j\}_{j=1}^m$. We want to make suitable choices for the test functions v in $\tilde{N}(\Omega)$ and the boundary condition $f \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$ in order to get simple equations for the unknown parameters, namely, for the points $\{z_j\}_{j=1}^m$ and the values $\{q_j(z_j)\}_{j=1}^m$.

Similar idea was used and the associated numerical experiments have been successfully conducted in the case of the (piecewise constant) conductivity problem [3] with boundary measurements on all of $\partial\Omega$.

Similarly to the reconstruction procedure we can take complex geometrical optics solutions so that

$$u(z_j) = e^{z_j\cdot\rho_1}(1+\theta_q(z_j,\rho_1))$$
$$v(z_j) = e^{z_j\cdot\rho_2}(1+\theta_q(z_j,\rho_2)).$$

We obtain that

$$\begin{aligned}\int_\Gamma\Big(\frac{\partial u_\alpha}{\partial\nu}v - \frac{\partial v}{\partial\nu}u_\alpha\Big)\ ds &= \int_\Gamma\Big(u_\alpha(\Lambda_{q_\alpha}-\Lambda_q)v\Big)\\ &= \alpha^d\sum_{j=1}^m (q(z_j)-q_j(z_j))|B_j|e^{ik\cdot z_j}+o(\alpha^d).\end{aligned}\tag{20}$$

Now by Theorem 2 we have

$$\begin{aligned}&\int_\Gamma\Big(u_\alpha(\Lambda_{q_\alpha}-\Lambda_q)v\Big) =\\ &\int_\Gamma(\Lambda_q+N_{\rho_1})^{-1}(\rho_1\cdot\nu(x)e^{x\cdot\rho_1}|_\Gamma)(x)(\Lambda_{q_\alpha}-\Lambda_q)(\Lambda_q+N_{\rho_2})^{-1}(\rho_2\cdot\nu(x)e^{x\cdot\rho_2}|_\Gamma)(x)\ ds(x)\\ &+o(\alpha^d),\end{aligned}\tag{21}$$

after neglecting the remainders $o(\alpha^d)$ in (20) and (21), that the locations $\{z_j\}_{j=1}^m$ are obtained as supports of the inverse Fourier transform of

$$\int_\Gamma(\Lambda_q+N_{\rho_1})^{-1}(\rho_1\cdot\nu(x)e^{x\cdot\rho_1}|_\Gamma)(x)(\Lambda_{q_\alpha}-\Lambda_q)(\Lambda_q+N_{\rho_2})^{-1}(\rho_2\cdot\nu(x)e^{x\cdot\rho_2}|_\Gamma)(x)\ ds(x).$$

Once we get the points $\{z_j\}_{j=1}^m$, the values $\{q_j(z_j)\}_{j=1}^m$ could be obtained by solving a linear system arising from (20).

Similarly we can reconstruct the locations and values of small volume perturbations of the refractive index $n(x)$ at fixed frequency ω by taking the corresponding potential $q=\omega^2 n$.

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