Increasing stability in an inverse problem for the acoustic equation

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Abstract

In this work we study the inverse boundary value problem of determining the refractive index in the acoustic equation. It is known that this inverse problem is ill-posed. Nonetheless, we show that the ill-posedness decreases when we increase the frequency and the stability estimate changes from logarithmic type for low frequencies to a Lipschitz estimate for large frequencies.

1 Introduction

In this paper we study the issue of stability for determining the refractive index in the acoustic equation by boundary measurements. It is well known that this inverse problem is ill-posed. However, one anticipates that the stability will increase if one increases the frequency. This phenomenon was observed numerically in the inverse obstacle scattering problem [5]. Several rigorous justifications of the increasing stability phenomena in different settings were obtained by Isakov *et al* [6, 7, 8, 10, 11]. Especially, in [8], Isakov considered the Helmholtz equation with a potential

$$-\Delta u - k^2 u + qu = 0 \quad \text{in} \quad \Omega. \tag{1.1}$$

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He obtained stability estimates of determining q by the Dirichlet-to-Neumann map for different ranges of k's. All of these results demonstrate the increasing stability phenomena in k. For the case of the inverse source problem for Helmholtz equation and an homogeneous background it was shown in [3] that the ill-posedness of the inverse problem decreases as the frequency increases.

In this paper, we study the acoustic wave equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, where $n \geq 3$. Let $\partial\Omega$ be smooth. We consider the equation

$$\big(\Delta + k^2 q(x)\big)u(x) = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

where the real-valued $q(x)$ is the refractive index. Assume that the kernel of the operator $\Delta + k^2 q(x)$ on $H_0^1(\Omega)$ is trivial. Associated with (1.2), we define the Dirichlet-to-Neumann map (DN map) $\Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda f = \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega},$$

where u is the solution to (1.2) with the Dirichlet condition $u = f$ on $\partial\Omega$, and ν is the unit outer normal vector of $\partial\Omega$. The uniqueness of this inverse problem is well known [13]. This inverse problem is notoriously ill-posed. For this aspect, Alessandrini proved that the stability estimate for this problem is of log type [1] and Mandache showed that the log type stability is optimal [9]. In this paper, we would like to focus on how the stability behaves when the frequency k increases. Now we state the main result.

Theorem 1.1. *Assume that $q_1(x)$ and $q_2(x)$ are two sound speeds with associated DN maps Λ_1 and Λ_2, respectively. Let $s > (n/2) + 1$, $M > 0$. Suppose $\|q_l\|_{H^s(\Omega)} \leq M$ $(l = 1, 2)$ and $\operatorname{supp}(q_1 - q_2) \subset \Omega$. Denote $\widetilde{q}$ a zero extension of $q_1 - q_2$. Then there exists a constant C_1, depending only on n, s, and Ω, such that if $k^2 \geq 1/(C_1 M)$ and $\|\Lambda_1 - \Lambda_2\|_* \leq 1/e$ then*

$$\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)} \leq \frac{C}{k^2}\exp(Ck^2)\|\Lambda_1 - \Lambda_2\|_* + C\left(k^2 + \log\frac{1}{\|\Lambda_1 - \Lambda_2\|_*}\right)^{-(2s-n)} \tag{1.3}$$

holds, where $C > 0$ depends only on n, s, Ω, M and $\operatorname{supp}(q_1 - q_2)$. Here $\|\cdot\|_*$ is the operator norm from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$.

Remark 1.2. 1. *The estimate* (1.3) *consists two parts – Lipschitz and logarithmic estimates. As k increases, the logarithmic part decreases and the Lipschitz part* *becomes dominated. In other words, the ill-posedness is alleviated when k is large.*

2. *We would like to remark on the constant $C\exp(Ck^2)/k^2$ appearing in the Lipschitz part of* (1.3). *$1/k^2$ comes from k^2q in the equation, which appears naturally, while, $\exp(Ck^2)$ is due to the fact that we use the complex geometrical optics solutions in the proof. Even so, we expect that the there is an exponential growth of the constant with frequency since we do not assume any geometrical restriction on $q(x)$ other than regularity. For the wave equation it has been shown by Burq for the obstacle problem* [4] *that the local energy decay is log-slow and this is due to the presence of trapped rays. Notice that in our case we can have trapped rays. For the case of simple sound speeds we expect that there is no exponential increase in the constant. In* [12] *a Hölder stability estimate was obtained for the hyperbolic DN map for generic simple metrics. For very general metrics there is not known modulus of continuity for the hyperbolic DN map, see* [2] *for convergence results.*

However, in practice, k is fixed and so is the constant. Therefore, one should expect to obtain a better resolution of q from boundary measurements when the chosen k is large.

3. *Unlike the result in* [8, Theorem 2.1] (for equation (1.1)) *where the stability estimates were derived in different ranges of k, estimate* (1.3) *is valid for all range of k provided $k^2 \geq 1/(C_1M)$.*

The proof of Theorem 1.1 makes use of Alessandrini's arguments [1] and the CGO solutions constructed in [13]. The main task is to keep track of how k appears in the proof of the stability estimates.

2 Complex geometrical optics solutions

In this section, we construct CGO solutions to the equation (1.2) by using the idea in [13]. The main point is to express the dependence of constants on k explicitly. We first state two easy consequences from the results in [13].

Lemma 2.1 (see [13, Proposition 2.1 and Corollary 2.2])**.** *Let $s \geq 0$ be an integer. Let $\varepsilon_0 > 0$. Let $\xi \in \mathbb{C}^n$ satisfy $\xi \cdot \xi = 0$ and $|\xi| \geq \varepsilon_0$. Then for any $f \in H^s(\Omega)$ there exists $w \in H^s(\Omega)$ such that w is a solution to*

$$\Delta w + \xi \cdot \nabla w = f \textit{ in } \Omega$$

and satisfies the estimate

$$\|w\|_{H^s(\Omega)} \leq \frac{C_0}{|\xi|}\|f\|_{H^s(\Omega)},$$

where a positive constant C_0 depends only on n, s, ε_0 and Ω.

By using this lemma, we can obtain a solution to the equation

$$\Delta\psi + \xi \cdot \nabla\psi + g\psi = f \tag{2.1}$$

satisfying some decaying property as in the following lemma.

Lemma 2.2 ([13, Theorem 2.3 and Corollary 2.4])**.** *Let $s > n/2$ be an integer. Let $\varepsilon_0 > 0$. Let $\xi \in \mathbb{C}^n$ satisfy $\xi \cdot \xi = 0$ and $|\xi| \geq \varepsilon_0$. Let $f, g \in H^s(\Omega)$. Then there exists $C_1 > 0$ depending only on n, s, ε_0 and Ω such that if*

$$|\xi| \geq C_1\|g\|_{H^s(\Omega)}$$

then there exists a solution $\psi \in H^s(\Omega)$ to the equation (2.1) *satisfying the estimate*

$$\|\psi\|_{H^s(\Omega)} \leq \frac{2C_0}{|\xi|}\|f\|_{H^s(\Omega)},$$

where C_0 is the positive constant in Lemma 2.1.

The needed CGO solutions are constructed as follows.

Proposition 2.3. *Let $s > n/2$ be an integer. Let $\varepsilon_0 > 0$. Let $\xi \in \mathbb{C}^n$ satisfy $\xi \cdot \xi = 0$ and $|\xi| \geq \varepsilon_0$. Define the constants C_0 and C_1 as in Lemma 2.2. Then if*

$$|\xi| \geq C_1 k^2\|q\|_{H^s(\Omega)}$$

then there exists a solution u to the equation (1.2) *with the form of*

$$u(x) = \exp\left(\frac{\xi}{2} \cdot x\right)\big(1 + \psi(x)\big), \tag{2.2}$$

where ψ has the estimate

$$\|\psi\|_{H^s(\Omega)} \leq \frac{2C_0 k^2}{|\xi|}\|q\|_{H^s(\Omega)}.$$

Proof. Substituting (2.2) into (1.2), we have

$$\Delta\psi + \xi \cdot \nabla\psi + k^2 q\psi = -k^2 q.$$

Then by Lemma 2.2, we obtain this proposition. □

3 Proof of stability estimate

This section is devoted to the proof of Theorem 1.1. We begin with Alessandrini's identity.

Proposition 3.1. *Let u_l be a solution to* (1.2) *with $q = q_l$, then we have*

$$k^2 \int_\Omega (q_2 - q_1) u_1 u_2 \, dx = \big\langle (\Lambda_1 - \Lambda_2) u_1|_{\partial\Omega},\, u_2|_{\partial\Omega} \big\rangle.$$

Now we would like to estimate the Fourier transform of the difference of two q's. We denote $\mathcal{F}(f)$ the Fourier transformation of a function f.

Lemma 3.2. *Let $s > (n/2) + 1$ be an integer and $M > 0$. Assume $\|q_l\|_{H^s(\Omega)} \le M$, $\operatorname{supp}(q_1 - q_2) \subset \Omega$ and $k^2 \ge 1/C_1 M$, where C_1 is the constant defined in Lemma 2.2 corresponding to $\varepsilon_0 = 1$. Let $\widetilde{q}$ be a zero extension of $q_1 - q_2$ and $a_0 \ge C_1$. Suppose that $\chi \in C_0^\infty(\Omega)$ satisfies $\chi \equiv 1$ near $\operatorname{supp}(q_1 - q_2)$. Then for $r \ge 0$ and $\eta \in \mathbb{R}^n$ with $|\eta| = 1$ the following statements hold: if $0 \le r \le a_0 k^2 M$ then*

$$|\mathcal{F}\widetilde{q}(r\eta)| \le \frac{C\|\chi\|_{H^s(\Omega)}}{a_0} \|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)} + \frac{C}{k^2} \exp(C a_0 k^2 M) \|\Lambda_1 - \Lambda_2\|_* \tag{3.1}$$

holds; if $r \ge C_1 k^2 M$ then

$$|\mathcal{F}\widetilde{q}(r\eta)| \le \frac{CMk^2\|\chi\|_{H^s(\Omega)}}{r} \|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)} + \frac{C}{k^2} \exp(Cr) \|\Lambda_1 - \Lambda_2\|_* \tag{3.2}$$

holds, where $C > 0$ depends only on n, s and Ω.

Proof. In the following proof, the letter C stands for a general constant depending only on n, s and Ω. By Proposition 2.3, we can construct CGO solutions $u_l(x)$ to the equation (1.2) with $q = q_l$ having the form of

$$u_l(x) = \exp\left(\frac{\xi_l}{2} \cdot x\right) \big(1 + \psi_l(x)\big)$$

for $l = 1, 2$, and we have

$$\begin{aligned} &\int_\Omega (q_2 - q_1) \exp\left(\frac{1}{2}(\xi_1 + \xi_2) \cdot x\right) (1 + \psi_1 + \psi_2 + \psi_1\psi_2) \, dx \\ &= \frac{1}{k^2} \big\langle (\Lambda_1 - \Lambda_2) u_1|_{\partial\Omega},\, u_2|_{\partial\Omega} \big\rangle \end{aligned} \tag{3.3}$$

from Proposition 3.1, where ψ_l satisfies

$$\|\psi_l\|_{H^s(\Omega)} \leq \frac{Ck^2}{|\xi_l|}\|q_l\|_{H^s(\Omega)}$$

if $\xi_l \in \mathbb{C}^n$ satisfies $\xi_l \cdot \xi_l = 0$, $|\xi_l| \geq 1$ and

$$|\xi_l| \geq C_1 k^2 \|q_l\|_{H^s(\Omega)}. \tag{3.4}$$

We remark that $\|\psi_l\|_{H^s(\Omega)} \leq C$ also holds. Indeed, we have

$$\|\psi_l\|_{H^s(\Omega)} \leq \frac{Ck^2\|q_l\|_{H^s(\Omega)}}{|\xi_l|} \leq \frac{Ck^2\|q_l\|_{H^s(\Omega)}}{C_1 k^2\|q_l\|_{H^s(\Omega)}} = \frac{C}{C_1} = C.$$

Now, let $r \geq 0$, and $\eta \in \mathbb{R}^n$ satisfy $|\eta| = 1$. We assume that $\alpha, \zeta \in \mathbb{R}^n$ satisfy

$$\alpha \cdot \eta = \alpha \cdot \zeta = \eta \cdot \zeta = 0 \text{ and } |\zeta|^2 = |\alpha|^2 + r^2. \tag{3.5}$$

Define ξ_1 and ξ_2 as

$$\xi_1 = \zeta + i\alpha - ir\eta \quad \text{and} \quad \xi_2 = -\zeta - i\alpha - ir\eta.$$

Then we have

$$\xi_l \cdot \xi_l = 0,\ |\xi_l|^2 = |\zeta|^2 + |\alpha|^2 + r^2 = 2|\zeta|^2\ (l = 1, 2) \text{ and } \frac{1}{2}(\xi_1 + \xi_2) = -ir\eta.$$

Hence by (3.3), we immediately obtain that

$$\begin{aligned} \mathcal{F}\widetilde{q}(r\eta) = &- \int_\Omega (q_2 - q_1)\exp(-ir\eta \cdot x)(\psi_1 + \psi_2 + \psi_1\psi_2)\,dx \\ &+ \frac{1}{k^2}\big\langle (\Lambda_1 - \Lambda_2)u_1|_{\partial\Omega},\, u_2|_{\partial\Omega} \big\rangle \end{aligned} \tag{3.6}$$

provided $|\xi_l| \geq 1$ and (3.4) are satisfied. We first estimate the first term on the right hand side of (3.6) by

$$\begin{aligned}
&\left|\int_\Omega (q_2 - q_1)\exp(-ir\eta\cdot x)(\psi_1 + \psi_2 + \psi_1\psi_2)\,dx\right| \\
&= \left|\int_\Omega (q_2 - q_1)\exp(-ir\eta\cdot x)\chi(\psi_1 + \psi_2 + \psi_1\psi_2)\,dx\right| \\
&\le \|q_2 - q_1\|_{H^{-s}(\Omega)}\big\|\chi(\psi_1 + \psi_2 + \psi_1\psi_2)\big\|_{H^s(\Omega)} \\
&\le \|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)}\|\chi\|_{H^s(\Omega)}\big(\|\psi_1\|_{H^s(\Omega)} + \|\psi_2\|_{H^s(\Omega)} + \|\psi_1\|_{H^s(\Omega)}\|\psi_2\|_{H^s(\Omega)}\big) \\
&\le \|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)}\|\chi\|_{H^s(\Omega)}\left(\frac{Ck^2}{\sqrt{2}|\zeta|} + \frac{Ck^2}{\sqrt{2}|\zeta|} + C\frac{Ck^2}{\sqrt{2}|\zeta|}\right)\sum_{l=1}^{2}\|q_l\|_{H^s(\Omega)} \\
&= \frac{Ck^2\|\chi\|_{H^s(\Omega)}}{|\zeta|}\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)}\sum_{l=1}^{2}\|q_l\|_{H^s(\Omega)}.
\end{aligned}$$

since $\chi(\psi_1 + \psi_2 + \psi_1\psi_2) \in H_0^s(\Omega)$ and $s > n/2$.

On the other hand, by taking R large enough such that $\Omega \subset B_R(0)$, we have

$$\begin{aligned}
\big\|u_l|_{\partial\Omega}\big\|_{L^2(\partial\Omega)} &\le |\partial\Omega|^{1/2}\|u_l\|_{C^0(\Omega)} \le |\partial\Omega|^{1/2}\exp\left(\frac{|\mathrm{Re}\,\xi_l|}{2}R\right)\big(1 + \|\psi_l\|_{L^\infty(\Omega)}\big) \\
&\le C\exp\left(\frac{|\mathrm{Re}\,\xi_l|}{2}R\right)\big(1 + \|\psi_l\|_{H^s(\Omega)}\big) \\
&\le C\exp\left(\frac{|\mathrm{Re}\,\xi_l|}{2}R\right)(1 + C) = C\exp\left(\frac{|\zeta|}{2}R\right).
\end{aligned}$$

Likewise, we can get that

$$\begin{aligned}
\big\|\nabla u_l|_{\partial\Omega}\big\|_{L^2(\partial\Omega)} &= \left\|\frac{\xi_l}{2}u_l + \exp\left(\frac{\xi_l}{2}\cdot\bullet\right)(\nabla\psi_l)\right\|_{L^2(\partial\Omega)} \\
&\le \frac{\sqrt{2}|\zeta|}{2}C\exp\left(\frac{|\zeta|}{2}R\right) + |\partial\Omega|^{1/2}\exp\left(\frac{|\zeta|}{2}R\right)\|\nabla\psi_l\|_{C^0(\Omega)} \\
&\le C|\zeta|\exp\left(\frac{|\zeta|}{2}R\right) + C\exp\left(\frac{|\zeta|}{2}R\right)\|\nabla\psi_l\|_{H^{s-1}(\Omega)} \\
&\le C|\zeta|\exp\left(\frac{|\zeta|}{2}R\right) + C\exp\left(\frac{|\zeta|}{2}R\right)\|\psi_l\|_{H^s(\Omega)} \\
&\le C\exp(C|\zeta|)
\end{aligned}$$

since $s-1>n/2$. Consequently, we have

$$\big\|u_l|_{\partial\Omega}\big\|_{H^{1/2}(\partial\Omega)} \le C\exp(C|\zeta|).$$

Therefore, we can estimate the second term of the right-hand side of (3.6) by

$$\begin{aligned}\big|\big\langle(\Lambda_1-\Lambda_2)u_1|_{\partial\Omega},\, u_2|_{\partial\Omega}\big\rangle\big| &\le \|\Lambda_1-\Lambda_2\|_*\big\|u_1|_{\partial\Omega}\big\|_{H^{1/2}(\partial\Omega)}\big\|u_2|_{\partial\Omega}\big\|_{H^{1/2}(\partial\Omega)}\\ &\le C\exp(C|\zeta|)\|\Lambda_1-\Lambda_2\|_*.\end{aligned}$$

Summing up, we have shown that for $r>0$ and for $\eta\in\mathbb{R}^n$ with $|\eta|=1$ if we take α and ζ satisfying the conditions (3.5), $|\zeta|\ge 2^{-1/2}$ and

$$|\zeta| \ge 2^{-1/2}C_1k^2\|q_l\|_{H^s(\Omega)} \tag{3.7}$$

then

$$\begin{aligned}|\mathcal{F}\widetilde{q}(r\eta)| \le{}& \frac{Ck^2\|\chi\|_{H^s(\Omega)}}{|\zeta|}\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)}\sum_{l=1}^{2}\|q_l\|_{H^s(\Omega)}\\ &+\frac{C}{k^2}\exp(C|\zeta|)\|\Lambda_1-\Lambda_2\|_*\end{aligned} \tag{3.8}$$

holds.

Now assume that $\|q_l\|_{H^s(\Omega)}\le M$ and $k^2\ge 1/C_1M$. Thus if

$$|\zeta| \ge C_1k^2M \tag{3.9}$$

holds, then (3.7) and $|\zeta|\ge 2^{-1/2}$ are satisfied. Pick $a_0\ge C_1$. We first consider the case where $0\le r\le a_0k^2M$. By choosing α and ζ satisfying

$$\alpha\cdot\eta=\alpha\cdot\zeta=\eta\cdot\zeta=0,\ |\zeta|=a_0k^2M(\ge r) \text{ and } |\alpha|=\sqrt{(a_0k^2M)^2-r^2}$$

both (3.5) and (3.9) are then satisfied since $a_0\ge C_1$. Hence we obtain (3.8), that is (3.1). On the other hand, when $r\ge C_1k^2M$, we can choose $\alpha=0$, $\eta\cdot\zeta=0$ and $|\zeta|=r$. Then (3.5), (3.9) are satisfied and thus (3.8) holds and consequently (3.2) is valid. □

Now we prove our main result.

Proof. As above, C denotes a general constant depending only on n, s and Ω. Written in polar coordinates, we have

$$\begin{aligned}
\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} &= C\int_0^\infty \int_{|\eta|=1} |\mathcal{F}\widetilde{q}(r\eta)|^2 (1+r^2)^{-s} r^{n-1}\, d\eta\, dr \\
&= C\Big(\int_0^{a_0k^2M} \int_{|\eta|=1} |\mathcal{F}\widetilde{q}(r\eta)|^2 (1+r^2)^{-s} r^{n-1}\, d\eta\, dr \\
&\qquad + \int_{a_0k^2M}^{T} \int_{|\eta|=1} |\mathcal{F}\widetilde{q}(r\eta)|^2 (1+r^2)^{-s} r^{n-1}\, d\eta\, dr \\
&\qquad + \int_{T}^{\infty} \int_{|\eta|=1} |\mathcal{F}\widetilde{q}(r\eta)|^2 (1+r^2)^{-s} r^{n-1}\, d\eta\, dr\Big) \\
&=: C(I_1+I_2+I_3), \qquad (3.10)
\end{aligned}$$

where $a_0 \geq C_1$ and $T \geq a_0k^2M$ are parameters which will be chosen later. Here C_1 is the constant given in Lemma 3.2. From now on, we take $k^2 \geq 1/(C_1M)$.

Our task now is to estimate each integral separately. We begin with I_3. Since $|\mathcal{F}\widetilde{q}(r\eta)| \leq C\|q_1-q_2\|_{L^2(\Omega)}$, $q_1-q_2 \in H_0^s(\Omega)$, and $s > n/2$, we have that

$$\begin{aligned}
I_3 &\leq C\int_T^\infty \|q_1-q_2\|^2_{L^2(\Omega)} (1+r^2)^{-s} r^{n-1}\, dr \leq CT^{-m}\|q_1-q_2\|^2_{L^2(\Omega)} \\
&\leq CT^{-m}\left(\varepsilon\|q_1-q_2\|^2_{H^{-s}(\Omega)} + \frac{C}{\varepsilon}\|q_1-q_2\|^2_{H^s(\Omega)}\right) \\
&\leq CT^{-m}\left(\varepsilon\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{M^2}{\varepsilon}\right) \qquad (3.11)
\end{aligned}$$

for $\varepsilon > 0$, where $m := 2s-n$.

On the other hand, by Lemma 3.2, we can estimate

$$\begin{aligned}
I_1 &\leq C\int_0^{a_0k^2M} (1+r^2)^{-s} r^{n-1}\, dr \\
&\quad \times \left[\frac{\|\chi\|^2_{H^s(\Omega)}}{a_0^2}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{\exp(2Ca_0k^2M)}{k^4}\|\Lambda_1-\Lambda_2\|_*^2\right] \\
&\leq C\int_0^\infty (1+r^2)^{-s} r^{n-1}\, dr \left[\frac{C_\chi^2}{a_0^2}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{\exp(Ca_0k^2M)}{k^4}\|\Lambda_1-\Lambda_2\|_*^2\right] \\
&= \frac{CC_\chi^2}{a_0^2}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{C\exp(Ca_0k^2M)}{k^4}\|\Lambda_1-\Lambda_2\|_*^2, \qquad (3.12)
\end{aligned}$$

where $\chi \in C_0^\infty(\Omega)$ satisfies $\chi \equiv 1$ near $\operatorname{supp}(q_2 - q_1)$ and $C_\chi := \|\chi\|_{H^s(\Omega)}$. In view of

$$\int_{a_0 k^2 M}^{T} (1+r^2)^{-s} r^{n-3}\, dr \le \int_{a_0 k^2 M}^{T} r^{-2s+n-3}\, dr \le C(a_0 k^2 M)^{-2s+n-2}$$
$$\le C(a_0 k^2 M)^{-2}(C_1 k^2 M)^{-m} \le \frac{C}{a_0^2 k^4 M^2}$$

and

$$\int_{a_0 k^2 M}^{T} \exp(Cr)(1+r^2)^{-s} r^{n-1}\, dr \le \exp(CT) \int_{a_0 k^2 M}^{T} (1+r^2)^{-s} r^{n-1}\, dr$$
$$\le \exp(CT) \int_{0}^{\infty} (1+r^2)^{-s} r^{n-1}\, dr$$
$$\le C\exp(CT),$$

we have that

$$I_2 \le CM^2 k^4 \|\chi\|^2_{H^s(\Omega)} \|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} \int_{a_0 k^2 M}^{T} (1+r^2)^{-s} r^{n-3}\, dr$$
$$+ \frac{C}{k^4} \|\Lambda_1 - \Lambda_2\|_*^2 \int_{a_0 k^2 M}^{T} \exp(Cr)(1+r^2)^{-s} r^{n-1}\, dr$$
$$\le \frac{CC_\chi^2}{a_0^2} \|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{C}{k^4}\exp(CT)\|\Lambda_1 - \Lambda_2\|_*^2. \tag{3.13}$$

Combining (3.10)–(3.13) gives

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} \le C(I_1 + I_2 + I_3)$$
$$\le \frac{CC_\chi^2}{a_0^2} \|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{C\exp(Ca_0 k^2 M)}{k^4} \|\Lambda_1 - \Lambda_2\|_*^2$$
$$+ \frac{CC_\chi^2}{a_0^2} \|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{C}{k^4}\exp(CT)\|\Lambda_1 - \Lambda_2\|_*^2$$
$$+ CT^{-m}\left(\varepsilon \|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{M^2}{\varepsilon}\right)$$
$$= \left(\frac{C_2^2 C_\chi^2}{a_0^2} + C_3 T^{-m}\varepsilon\right) \|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}$$
$$+ \frac{C}{k^4}\big(\exp(Ca_0 k^2 M) + \exp(CT)\big)\|\Lambda_1 - \Lambda_2\|_*^2 + \frac{CM^2}{\varepsilon} T^{-m},$$

where positive constants C_2 and C_3 depend only on n, s and Ω.

Now we pick a_0 and ε as

$$a_0 = 2C_2C_\chi \geq C_1 \text{ and } \varepsilon = \frac{T^m}{4C_3}$$

(if needed, we take C_2 large enough). We then obtain that

$$\begin{aligned}\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} &\leq \frac{C}{k^4}\left[\exp(2C_2CC_\chi k^2 M) + \exp(CT)\right]\|\Lambda_1 - \Lambda_2\|^2_* + CT^{-2m}M^2 \\ &= \frac{C}{k^4}\exp(Cak^2)A + C\Phi(T) \qquad (3.14)\end{aligned}$$

for $T \geq a_0 k^2 M = 2C_2C_\chi k^2 M = ak^2$, where

$$\Phi(T) := \frac{1}{k^4}\exp(C_4 T)A + M^2 T^{-2m},$$

$A := \|\Lambda_1 - \Lambda_2\|^2_*$, $a := 2C_2C_\chi M^2$ and $C_4 > 0$ depends only on n, s and Ω.

To continue, we consider two cases:

$$ak^2 \leq p\log\frac{1}{A} \qquad (3.15)$$

and

$$ak^2 \geq p\log\frac{1}{A}, \qquad (3.16)$$

where p will be determined later (see (3.24)).

For the first case (3.15), our aim is to show that there exists $T \geq ak^2$ such that

$$\Phi(T) \leq 2C_5\left(k^2 + \log\frac{1}{A}\right)^{-2m}. \qquad (3.17)$$

Substituting (3.17) into (3.14) clearly implies (1.3). Now to derive (3.17), it is enough to prove that

$$\frac{1}{k^4}\exp(C_4T)A \leq C_5\left(k^2 + \log\frac{1}{A}\right)^{-2m} \qquad (3.18)$$

and

$$M^2T^{-2m} \leq C_5\left(k^2 + \log\frac{1}{A}\right)^{-2m}. \qquad (3.19)$$

Remark that (3.19) in equivalent to

$$T \geq C_5^{-1/2m} M^{1/m} \left(k^2 + \log \frac{1}{A} \right),$$

which holds if

$$T \geq C_5^{-1/2m} M^{1/m} \left(1 + \frac{p}{a} \right) \log \frac{1}{A} \tag{3.20}$$

because of (3.15). Setting $T = p \log(1/A)$ ($\geq ak^2$ by (3.15)), then (3.20) holds provided

$$p \geq C_5^{-1/2m} M^{1/m} \left(1 + \frac{p}{a} \right). \tag{3.21}$$

Now we turn to (3.18). It is clear that (3.18) is equivalent to

$$C_4 p \log \frac{1}{A} \leq \log C_5 + 2 \log k^2 + \log \frac{1}{A} - 2m \log \left(k^2 + \log \frac{1}{A} \right) \tag{3.22}$$

since $T = p \log(1/A)$. It follows from (3.15) that

$$\log \left(k^2 + \log \frac{1}{A} \right) \leq \log \left(\frac{p}{a} \log \frac{1}{A} + \log \frac{1}{A} \right) = \log \left(\frac{p}{a} + 1 \right) + \log \log \frac{1}{A}.$$

Hence (3.22) is verified if we can show that

$$C_4 p \log \frac{1}{A} \leq \log C_5 - 2 \log(MC_1) + \log \frac{1}{A} - 2m \left(\log \left(\frac{p}{a} + 1 \right) + \log \log \frac{1}{A} \right),$$

i.e.

$$(1 - C_4 p) \log \frac{1}{A} - 2m \log \log \frac{1}{A} + \log C_5 - 2 \log(MC_1) - 2m \log \left(\frac{p}{a} + 1 \right) \geq 0 \tag{3.23}$$

for $\log(1/A) \geq 1$. Now we choose

$$p = \frac{1}{2C_4}. \tag{3.24}$$

Then (3.23) becomes

$$\log \frac{1}{A} - 4m \log \log \frac{1}{A} + 2 \log C_5 - 4 \log(MC_1) - 4m \log \left(\frac{p}{a} + 1 \right) \geq 0. \tag{3.25}$$

Notice that

$$\inf_{0<A\le 1/e}\left(\log\frac{1}{A}-4m\log\log\frac{1}{A}\right)=\inf_{z\ge 1}(z-4m\log z)$$
$$\ge \inf_{z>0}(z-4m\log z)=4m\log\frac{e}{4m}.$$

Hence if we choose C_5 such that

$$C_5\ge (MC_1)^2\left(\frac{p}{a}+1\right)^{2m}\left(\frac{4m}{e}\right)^{2m} \tag{3.26}$$

then (3.25) follows. Finally, we take

$$C_5:=\max\left\{C_1^2\left(\frac{4m}{e}\right)^{2m},\ p^{-2m}\right\}M^2\left(1+\frac{p}{a}\right)^{2m},$$

which depends only on n,Ω,s,M and χ. With such choice of C_5, the conditions (3.26) and (3.21) hold, and thus estimate (3.17) is satisfied.

Next we consider the second case (3.16). By (3.14) with $T=ak^2$, we get that

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}\le \frac{C}{k^4}\exp(Cak^2)A+\frac{C}{k^4}\exp(C_4ak^2)A+CM^2(ak^2)^{-2m}$$
$$\le \frac{C}{k^4}\exp(Cak^2)A+CM^2a^{-2m}k^{-4m}.$$

Hence it remains to show that

$$k^{-4m}\le C_6\left(k^2+\log\frac{1}{A}\right)^{-2m},$$

i.e.

$$k^2\ge C_6^{-1/2m}\left(k^2+\log\frac{1}{A}\right). \tag{3.27}$$

Since

$$k^2+\log\frac{1}{A}\le\left(1+\frac{a}{p}\right)k^2$$

by (3.16), we have (3.27) if we take C_6 large enough so that

$$C_6\ge\left(1+\frac{a}{p}\right)^{2m}.$$

The proof is completed. □

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