On characterization of the range and inversion formulas for the geodesic X-ray transform

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Abstract

We describe a relation between the scattering relation, the Hilbert transform in frequency space and the geodesic ray transform for simple, two dimensional compact Riemannian manifolds with boundary. We use this relation to give a characterization of the range of the geodesic X ray transform acting on scalar functions and vector fields in terms of the scattering relation which involves only measurements at the boundary. We also give inversion formulas for the geodesic X-ray transform acting on scalar functions and vector fields.

1 Introduction

Let (M, g) be a two dimensional compact Riemannian manifold with boundary. In this paper we describe the range of the geodesic X-ray transform for scalar functions and vector fields for simple manifolds in terms of the scattering relation which involves only measurements made at the boundary. We also give inversion formulas for the case of constant curvature and Fredholm type inversion formulas in general. Even in the constant curvature case these formulas seem to be new.

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We first define the geodesic X-ray transform. This is defined for a compact Riemannian manifold (M,g) with boundary of any dimension n. We denote by $\Omega(M)$ the unit sphere bundle of M:

$$\Omega(M)=\bigcup_{x\in M}\Omega_x,\quad \Omega_x=\{\xi\in T_x(M):|\xi|=1\}.$$

It is a (2n-1)-dimensional compact manifold with boundary

$$\partial\Omega(M)=\bigcup_{x\in\partial M}\Omega_x,$$

which can be written as the union $\partial\Omega(M)=\partial_+\Omega(M)\cup\partial_-\Omega(M)$,

$$\partial_\pm\Omega\left(M\right)=\{(x,\xi)\in\partial\Omega\left(M\right),\ \pm\left(\nu\left(x\right),\xi\right)\geq 0\ \},$$

where ν is the unit-inner normal to ∂M. The manifolds of inner vectors $\partial_+\Omega\left(M\right)$ and outer vectors $\partial_-\Omega\left(M\right)$ intersect at the set of tangent vectors

$$\partial_0\Omega\left(M\right)=\{(x,\xi)\in\partial\Omega\left(M\right),\quad \left(\nu\left(x\right),\xi\right)=0\ \}.$$

We embed (M,g) into a compact Riemannian manifold (S,g) of the same dimension and with no boundary. Let φ_t be the geodesic flow on $\Omega(S)$ and $\mathcal{H}=\frac{d}{dt}\varphi_t|_{t=0}$ be the geodesic vector field. Let u^f be the solution of the boundary value problem

$$\mathcal{H}u=-f,\quad u|_{\partial_-\Omega(M)}=0,$$

which can be written as

$$u^f(x,\xi)=\int\limits_0^{\tau(x,\xi)}f(\varphi_t(x,\xi))dt,\quad (x,\xi)\in\Omega(M),\tag{1.1}$$

where $\tau(x,\xi)$ is the length of the geodesic $\gamma(x,\xi,t),t\geq 0$, starting at the point x in the direction $\xi\in\Omega_x$. In particular

$$\mathcal{H}\tau=-1.$$

The trace

$$If=u^f|_{\partial_+\Omega(M)}$$

is called **the geodesic X-ray transform** of the function f. If the manifold (M,g) is non-trapping (that is every geodesic has finite length) and has a strictly convex boundary the operator $I:C^\infty(\Omega(M))\to C^\infty(\partial_+\Omega(M))$ can be extended as a bounded operator $I:L^2(\Omega(M))\to L^2_\mu(\partial_+\Omega(M))$, where the real Hilbert space $L^2_\mu(\partial_+\Omega(M))$, is defined by the scalar product :

$$(u,v)_{L^2_\mu(\partial_+\Omega(M))}=\int\limits_{\partial_+\Omega(M)}\mu uv d\Sigma,\quad \mu(x,\xi)=(\nu(x),\xi).$$

Here $d\Sigma$ is the volume form of $\partial\Omega(M)$.

Clearly a function f is not determined by its geodesic X-ray transform alone, since it depends on more variables than does If. We consider the geodesic X-ray transform acting on symmetric tensor fields.

We denote by $f_m(x,\xi)$ an homogeneous polynomial of degree m with respect to ξ, induced by the symmetric tensor field f on M of m degree :

$$f_m(x,\xi) = f_{i_1 \dots i_m}(x)\,\xi^{i_1} \dots \xi^{i_m}.$$

The operator I_m, defined by

$$I_m f = I f_m$$

is called **the geodesic X-ray transform** of the symmetric tensor field. If the manifold M is non-trapping and the boundary ∂M is strictly convex $I_m : C^\infty(M, S_m(M)) \to C^\infty(\partial_+\Omega(M))$, where $S_m(M)$ denotes the bundle of symmetric tensors over M. It is known that any symmetric smooth enough tensor field f may be decomposed in a potential and solenoidal parts [Sh]:

$$f = dp + h, \quad p|_{\partial M} = 0, \quad \delta h = 0,$$

where δ denotes the divergence and $d = \sigma\nabla$ is the symmetric part of the covariant derivative. It is easy to see that the geodesic X-ray transform of the potential part dp is zero. We denote by $C^\infty_{sol}(M, S_m(M))$ the space of smooth solenoidal symmetric tensor fields.

The adjoint of the operator I_m is the bounded operator $I^*_m : L^2_\mu(\partial_+\Omega(M)) \to L^2(M, S_m(M))$. We give its representation in section 2. The field $I^*_m w$ is solenoidal in the sense of the theory of distributions.

We will consider in this paper the case of the geodesic X-ray transform acting on functions independent of ξ and the geodesic X-ray transform acting on vector fields which, following the notation above, are denoted by I_0 and I_1 respectively.

It is known that I_0 is injective on simple manifolds [Mu] and that I_1 is injective on solenoidal vector fields on simple manifolds [A], [AR]. In [SU] sharp stability estimates are proven for I_0 and I_1. We recall that a manifold is **simple** if all geodesics have no conjugate points and the boundary is strictly convex.

A brief summary of this paper is as follows. In section 2 we define the scattering relation. In section 3 describe a commutator between the Hilbert transform and geodesic flow for two dimensional manifolds (see Theorem 3.1). In section 4 we give a characterization of I_0 and I_1 in terms of the scattering relation (see Theorem 4.4). In section 5 we give Fredholm type inversion formulas for these transforms in the two dimensional case for simple manifolds which are exact for the case of constant curvature. For the case of constant curvature we do not need the assumption of no-conjugate points (see Theorem 5.1).

2 Scattering relation

Let (M,g) be a compact n-dimensional Riemannian manifold with boundary ∂M. Let ν be the unit inner normal to ∂M.

Let (M,g) be non-trapping and the boundary ∂M is strictly convex. Then the function of geodesic length $\tau(x,\xi)$ is smooth on $\Omega(M)\setminus\partial_0\Omega(M)$. The function $\tau^0=\tau|_{\partial\Omega(M)}$ is equal zero on $\partial_-\Omega(M)$ and is smooth on $\partial_+\Omega(M)$. Its odd part with respect to ξ

$$\tau_-^0(x,\xi)=\frac{1}{2}\left(\tau^0(x,\xi)-\tau^0\left(x,-\xi\right)\right)$$

is a smooth function.

Definition 2.1. *Let (M,g) be non-trapping with strictly convex boundary. The scattering relation $\alpha:\partial\Omega(M)\to\partial\Omega(M)$ is defined by $\alpha(x,\xi)=(\gamma(x,\xi,2\tau_-^0(x,\xi)),\dot\gamma(x,\xi,2\tau_-^0(x,\xi)))$ or*

$$\alpha(x,\xi)=\left\{\begin{array}{ll}(\gamma(x,\xi,\tau(x,\xi)),\dot\gamma(x,\xi,\tau(x,\xi))) & (x,\xi)\in\partial_+\Omega\left(M\right)\\ (\gamma(x,\xi,-\tau(x,-\xi)),\dot\gamma(x,\xi,-\tau(x,-\xi))) & (x,\xi)\in\partial_-\Omega(M).\end{array}\right.$$

We remark that the scattering relation appears naturally as the wave front set of the scattering operator [G]. The scattering relation is a diffeomorphism $\partial\Omega\left(M\right)\to\partial\Omega\left(M\right)$. Notice that $\alpha|_{\partial_+\Omega(M)}:\partial_+\Omega\left(M\right)\to\partial_-\Omega\left(M\right)$, $\alpha|_{\partial_-\Omega(M)}:\partial_-\Omega\left(M\right)\to\partial_+\Omega\left(M\right)$ are diffeomorphisms as well. Obviously, α is an involution, $\alpha^2=id$ and $\partial_0\Omega\left(M\right)$ is the hypersurface of its fixed points, $\alpha(x,\xi)=(x,\xi)$, $(x,\xi)\in\partial_0\Omega\left(M\right)$. The Jacobian $|\det\alpha'|$ of scattering relation is given by

$$|\det\alpha'|=-\frac{\mu}{\alpha^*\mu},$$

where $\mu=(\xi,\nu)$.

Define the map $\psi:\Omega(M)\to\partial_-\Omega(M)$ by

$$\psi(x,\xi)=\varphi_{\tau(x,\xi)}(x,\xi),\quad (x,\xi)\in\Omega(M).$$

The solution of the boundary value problem for the transport equation

$$\mathcal{H}u=0,\quad u|_{\partial_+\Omega(M)}=w$$

can be written in the form

$$u=w_\psi=w\circ\alpha\circ\psi.$$

The operator I_m^* is given by

$$\left(I_m^*w\right)^{i_1\dots i_m}(x)=\int\limits_{\Omega_x}w_\psi(x,\xi)\xi^{i_1}\dots\xi^{i_m}d\Omega_x.$$

We also remark that by the fundamental theorem of calculus we have that

$$I\mathcal{H}f = (f \circ \alpha - f)|_{\partial_+\Omega(M)}.$$

Denote by A_+, A_- the even and odd continuations with respect to the involution α:

$$A_\pm w(x,\xi) = \begin{cases} w(x,\xi) & (x,\xi) \in \partial_+\Omega\,(M) \\ \pm\,(\alpha^* w)\,(x,\xi) & (x,\xi) \in \partial_-\Omega\,(M)\,. \end{cases}$$

Introduce the real Hilbert space $L^2_{|\mu|}(\partial\Omega(M))$, defined by scalar product

$$(u,v)_{L^2_{|\mu|}(\partial\Omega(M))} = \int\limits_{\partial\Omega(M)} |\mu|\, uv d\Sigma,$$

Then $A_\pm$ are bounded operators from $L^2_\mu(\partial_+\Omega(M))$ into $L^2_{|\mu|}\,(\partial\Omega\,(M))$. The adjoint of $A_\pm$ is a bounded operator $A^*_\pm : L^2_{|\mu|}\,(\partial\Omega\,(M)) \to L^2_\mu(\partial_+\Omega(M))$ given by

$$A^*_\pm u = (u \pm u \circ \alpha)|_{\partial_+\Omega(M)}.$$

The space $C^\infty_\alpha\,(\partial_+\Omega\,(M))$ is defined by

$$C^\infty_\alpha\,(\partial_+\Omega\,(M)) = \{w \in C^\infty\,(\partial_+\Omega\,(M)) : w_\psi \in C^\infty\,(\Omega\,(M))\}.$$

We have the following characterization of the space of smooth solutions of the transport equation (see [PU])

Lemma 2.1.

$$C^\infty_\alpha(\partial_+\Omega(M)) = \{w \in C^\infty(\partial_+\Omega(M)) : A_+ w \in C^\infty(\partial\Omega(M))\}.$$

3 Hilbert transform and geodesic flow

Now we define the Hilbert transform:

$$Hu(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{1+(\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta)d\Omega_x(\eta), \quad \xi \in \Omega_x,$$

where the integral is understood as a principle value integral. Here $\perp$ means a 90 degree rotation. In coordinates $(\xi_\perp)_i = \varepsilon_{ij}\xi^j$, where

$$\varepsilon = \sqrt{\det g}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Hilbert transform H transforms even (respectively odd) functions with respect to ξ to even (respectively odd) ones. If H_+ (respectively H_-) is the even (respectively odd) part of the operator H:

$$H_+u(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{(\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta)d\Omega_x(\eta),$$
$$H_-u(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{1}{(\xi_\perp,\eta)} u(x,\eta)d\Omega_x(\eta)$$

and u_+, u_- are the even and odd parts of the function u, then $H_+u = Hu_+, H_-u = Hu_-$.

We introduce the notation $\mathcal{H}_\perp = (\xi_\perp, \nabla) = -(\xi, \nabla_\perp)$, where $\nabla_\perp = \varepsilon\nabla$ and ∇ is the so-called horizontal derivative (see for details [Sh]). In local coordinates on the manifold $\Omega(M)$ it is given by

$$\nabla_i u(x,\xi) = \frac{\partial u}{\partial x^i} - \Gamma^k_{ij}\xi^j\partial_k u,$$

where Γ^k_{ij} are the Christoffel symbols of the metric g and

$$\partial_k u(x,\xi) = \frac{\partial u(x,\xi/|\xi|)}{\partial \xi^k}\Big|_{|\xi|=1}.$$

The following commutator formula for the geodesic vector field and the Hilbert transform, which is crucial to our approach, was proven in [PU].

Theorem 3.1. *Let (M,g) be a compact two dimensional Riemannian manifold with boundary. For any smooth function u on $\Omega(M)$ the following identity holds:*

$$[H,\mathcal{H}]u = \mathcal{H}_\perp u_0 + (\mathcal{H}_\perp u)_0 \tag{3.1}$$

where

$$u_0(x) = \frac{1}{2\pi}\int_{\Omega_x} u(x,\xi)d\Omega_x$$

is the average value.

4 The range of the geodesic X-ray transform

In was shown in [PU] that in the scalar case I_0^* is surjective between appropriate spaces. More precisely we have.

Theorem 4.1. *Let (M,g) be a simple, compact Riemannian manifold with boundary. Then the operator $I_0^* : C^\infty_\alpha(\partial_+\Omega(M)) \to C^\infty(M)$ is onto.*

We need a weaker version of this result for I_1^*. The result was proven in [Pe] and follows the proof of [PU]. We include it for completeness.

Theorem 4.2. *Let (M,g) be a simple, compact Riemannian manifold with boundary. Then for any field $v \in C^\infty_{sol}(M,T(M))$ there exists a function $w \in C^\infty_\alpha(\partial_+\Omega(M))$ and harmonic function $h \in C^\infty(M)$, such that*

$$v = I_1^* w + \nabla h.$$

Recall that (M,g) is embedded in a compact manifold (S,g) without boundary. Let r_M denotes the restriction from S onto M. Theorem 4.2 is a corollary of the following result.

Theorem 4.3. *Let (U,g) be a simple neighborhood of the simple manifold (M,g). Then any solenoidal vector field $v \in H^{s+1}(M,T(M))$, $s \geq 0$ can be represented in the form*

$$v = r_M I_1^* I_1 f + \nabla h,$$

where $f \in H^s(U); h \in C^\infty(M)$ is a harmonic function, I_1 is the geodesic X-ray transform acting on vector fields defined on $\overline{U}$.

Proof. Choose a finite atlas of S, which consists of simple open sets U_k with coordinate maps $\kappa_k : U_k \to \mathbf{R}^n$. Let $\{\varphi_k\}$ be a subordinated partition of unity: $\varphi_k \geq 0$, $supp\varphi_k \subset U_k$, $\sum \varphi_k = 1$. We assume without loss of generality that $M \subset U_1$ and $\varphi_1|_M = 1$. We consider the operators $I_{1(k)}$, $I^*_{1(k)}$ for the domain U_k, and the pseudodifferential operator on (S,g)

$$P_1(f) = \sum_k \varphi_k I^*_{1(k)} I_{1(k)}(f|_{U_k}), \;\; f \in D'(S,T(S)).$$

We have that $I^*_{1(k)} I_{1(k)} : C^\infty_0(U_k,T(U_k)) \to C^\infty_{sol}(U_k,T(U_k))$ is a pseudodifferential operator of order -1 with principal symbol

$$c_n(\frac{\delta_i^j}{|\xi|} - \frac{\xi^j\xi_i}{|\xi|^3}),$$

where c_n is a constant. The operator P_1 has the same principal symbol and therefore is a bounded operator from $H^s(S,T(S))$ into $H^{s+1}(S,T(S))$ for all $s \in \mathbf{R}$. We will construct an operator Λ so that the sum $P_1 + \Lambda$ is an elliptic pseudodifferential operator of order -1.

Consider the operators $\nabla : C^\infty(S) \to C^\infty(S,T(S))$, $\delta : C^\infty(S,T(S)) \to C^\infty(S)$. In local coordinates

$$\nabla_j u = \frac{\partial u}{\partial x^j}, \quad \delta u = \frac{\partial u^j}{\partial x^j} + \Gamma^k_{kj} u^j.$$

Their principal symbols $\sigma(\nabla)$, $\sigma(\delta)$ are respectively given by

$$\sigma\left(\nabla\right)(x,\xi)u = i\xi u, \;\; \sigma\left(\delta\right)(x,\xi)u = i\left(\xi,u\right).$$

The Laplace-Beltrami operator $\Delta = \delta\nabla : H^s(S) \to H^{s-2}(S)$ is an elliptic differential operator of order 2 with principal symbol

$$\sigma(\Delta) = -\left|\xi\right|^2.$$

Its kernel and cokernel consist of constant functions. The operator Δ is invertible on the subspace $H_0^{s-2}(S) \subset H^{s-2}(S)$, which consists of the functions orthogonal to 1,

$$H_0^s(S) = \{u \in H^s(S) : \langle u, 1 \rangle = 0\}.$$

We have that the divergence $\delta(H^s(S)) \subset H_0^{s-1}(S)$. We define then the pseudodifferential operator $\Lambda : H^s(S, T(S)) \to H^{s+1}(S, T(S))$ by

$$\Lambda = -c_n \nabla (-\Delta)^{-3/2} \delta.$$

Its principal symbol is given by

$$\sigma(\Lambda)_i^j(x, \xi) = c_n \frac{\xi^j \xi_i}{|\xi|^3}.$$

Then principal symbol of the operator $P_1 + \Lambda$ is given by

$$\sigma\left(P_1 + \Lambda\right) = c_n \frac{g}{|\xi|}$$

and therefore $P_1 + \Lambda$ is an elliptic pseudodifferential operator of order -1. Its range is closed and has finite codimension.

The same arguments as in the scalar case (see [PU]) lead to the conclusion, that the range of the operator $r_M(P_1 + \Lambda) : H^s(U_1, T(U_1)) \to H^{s+1}(M, T(M))$ also has finite codimension and is closed. Now to prove the solvability of

$$r_M(P_1 + \Lambda) f = v \in H^{s+1}\left(M, T\left(M\right)\right) \tag{4.1}$$

is sufficient to show that the adjoint $(r_M(P_1 + \Lambda))^* : H^{-s-1}\left(M, T\left(M\right)\right) \to H^{-s}\left(U_1, T\left(U_1\right)\right)$ is injective.

Using elliptic regularity we have

$$Ker((r_M(P_1 + \Lambda))^*) \subset C_0^\infty(U_1).$$

Furthermore it is not difficult to see that the following equality holds

$$(r_M(P_1 + \Lambda))^* = I_{1(1)}^* I_{1(1)} + r_{U_1} \Lambda.$$

Then using that $(r_M(P_1 + \Lambda))^* u = 0,\ u \in C_0^\infty\left(U_1\right)$ we obtain

$$(u, (I_{1(1)}^* I_{1(1)} + \Lambda) u) = \left\| I_{1(1)} u \right\|_{L_\mu^2(\partial_+ \Omega(\overline{U}_1))}^2 + c_n \left\| (-\Delta)^{-\frac{3}{4}} \delta u \right\|_{L^2(S)}^2 = 0.$$

The fact that $(-\Delta)^{-\frac{3}{4}} \delta u = 0$ for smooth compactly supported vector fields u implies $\delta u = 0$, and then $I_{1(1)} u = 0$ implies $u = 0$, since solenoidal smooth vector fields on a simple manifold are determined uniquely by its geodesic X-ray transform [An].

Thus equation (4.1) is solvable for any right hand side. Then any vector field $v \in H^{s+1}_{sol}(M, S(M))$ may be represented in the form

$$v = r_M I^*_{1(1)} I_{1(1)} f + \nabla h,$$

where $h = c_n(-\Delta)^{-3/2}\delta f|_M$. Since the vector field $r_M I^*_{1(1)} I_{1(1)} f$ is solenoidal, we have that $\delta\nabla h = 0$, i.e. the function h is harmonic.

□

Let $T(M)$ be the tangent bundle of M. We define the operator $\delta_\perp : C^\infty(M, T(M)) \to C^\infty(M)$ by

$$\delta_\perp u = -\delta u_\perp.$$

Proposition 4.1. *The maps* $\delta_\perp : C^\infty_{sol}(M, T(M)) \to C^\infty(M)$ *and* $\nabla_\perp : C^\infty(M) \to C^\infty_{sol}(M, T(M))$ *are surjective.*

Proof. We first prove the first statement. Let $f \in C^\infty(M)$. Then the solution u of the Dirichlet problem

$$\Delta u = \delta\nabla u = f, \quad u|_{\partial M} = 0$$

belongs to $C^\infty(M)$ and defines the smooth solenoidal field $\nabla_\perp u$, so that $\delta_\perp\nabla_\perp u = \delta\nabla u = f$.

Now we prove the second statement. Any smooth solenoidal vector field u has a potential, $u = \nabla_\perp f$, $f \in C^\infty(M)$. This follows since $u_\perp$ satisfies the equation $\delta_\perp u_\perp = 0$. This means that the 1-form $(u_\perp)_i\, dx^i$ is closed. By Poincaré's Lemma this form is exact, i.e. $u_\perp = -\nabla f$, $f \in C^\infty(M)$. Therefore $u = \nabla_\perp f$ concluding the proof that $\nabla_\perp$ is onto.

□

We now give the characterization of the range of I_0 and I_1 in terms of the scattering relation only. Notice again that the scattering relation can be determined by performing measurements at the boundary. We introduce the notations

$$P_- = A^*_- H_- A_+, \quad P_+ = A^*_- H_+ A_+$$

The Theorem states that P_-, P_+ are respectively the projections of I_0, I_1 in appropriate spaces.

Theorem 4.4. *Let* (M, g) *be simple two dimensional compact Riemannian manifold with boundary. Then*
i) The maps

$$\begin{aligned} \delta_\perp I^*_1 &: C^\infty_\alpha(\partial_+\Omega(M)) \to C^\infty(M), \\ \nabla_\perp I^*_0 &: C^\infty_\alpha(\partial_+\Omega(M)) \to C^\infty_{sol}(M, T(M)) \end{aligned}$$

are onto.
ii) A function $u \in C^\infty(\partial_+\Omega(M))$ *belongs to Range* I_0 *iff* $u = P_- w$, $w \in C^\infty_\alpha(\partial_+\Omega(M))$.
iii) A function $u \in C^\infty(\partial_+\Omega(M))$ *belongs to Range* I_1 *iff* $u = P_+ w$, $w \in C^\infty_\alpha(\partial_+\Omega(M))$.

Proof. We have that

$$\delta_\perp \nabla_\perp f = \delta \nabla f = \Delta f, \quad \delta_\perp \nabla f = -\delta \nabla_\perp f = 0.$$

The identity (3.1) applied to a function $w_\psi, w \in C_\alpha^\infty(\partial_+\Omega(M))$ gives

$$\mathcal{H} H w_\psi = \frac{1}{2\pi}\delta_\perp I_1^* w + \frac{1}{2\pi}(\xi, \nabla_\perp I_0^* w). \tag{4.2}$$

Notice, that the operator $\mathcal{H}$ transforms even (with respect to ξ) functions to odd ones and the other way around. Therefore (4.2) gives the following two identities

$$\begin{aligned} \mathcal{H} H_- w_\psi &= \frac{1}{2\pi}\delta_\perp I_1^* w, \\ \mathcal{H} H_+ w_\psi &= \frac{1}{2\pi}(\xi, \nabla_\perp I_0^* w). \end{aligned}$$

After integration, using that $I\mathcal{H} = -A_-^* f^0$ where $f^0 = f|_{\partial\Omega(M)}$, and the fact that $w_\psi|_{\partial\Omega(M)} = A_+ w$, we obtain the following identities

$$\begin{aligned} A_-^* H_- A_+ w &= -\frac{1}{2\pi} I_0 \delta_\perp I_1^* w, \\ A_-^* H_+ A_+ w &= -\frac{1}{2\pi} I_1 \nabla_\perp I_0^* w. \end{aligned}$$

Therefore, the following factorizations holds :

$$P_- = -\frac{1}{2\pi} I_0 \delta_\perp I_1^*, \quad P_+ = -\frac{1}{2\pi} I_1 \nabla_\perp I_0^*. \tag{4.3}$$

We would like to emphasize that the operators $P_\pm$ are defined in terms of the scattering relation only. Clearly $P_\pm : C_\alpha^\infty(\partial_+\Omega(M)) \to C^\infty(\partial_+\Omega(M))$.

By Theorem 4.3, any smooth vector field v can be written in the form

$$v = I_1^* w + \nabla h, \ w \in C_\alpha^\infty(\partial_+\Omega(M)), \ h \in C^\infty(M).$$

Since $\delta_\perp \nabla h \equiv 0$, the surjectivity of the map $\delta_\perp I_1^* : C_\alpha^\infty(\partial_+\Omega(M)) \to C^\infty(M)$ follows from the surjectivity of the operator $\delta_\perp$ proven in Proposition 4.1.

Now since $I_0^* : C_\alpha^\infty(\partial_+\Omega(M)) \to C^\infty(M)$ is onto we get that the map $\nabla_\perp I_0^* : C_\alpha^\infty(\partial_+\Omega(M)) \to C_{sol}^\infty(M, T(M))$ is also surjective since the map $\nabla_\perp : C^\infty(M) \to C_{sol}^\infty(M, T(M))$ is onto as stated in Proposition 4.1.

The other statements in the Theorem are obvious corollaries of the first statement and the identities (4.3). $\square$

5 Inversion formulas

In this section we use the identity (3.1) to give an inversion formula for the geodesic X-ray transforms I_0, I_1 for two dimensional manifolds of constant curvature, which we believe is new since we are considering the case of manifolds with boundary. For the case of I_0 for compactly supported functions see [H] for inversion formulas on the plane (Radon transform), two sphere (Funk transform) and hyperbolic plane. For the general case we give an inversion formula of Fredholm type. We remark that for the constant curvature case we don't need the assumption of no-conjugate points along geodesics.

If we know a solution of the equation $P_- w = -I_0 f$ the function f is determined by the formula

$$f = \frac{1}{2\pi}\delta_\perp I_1^* w.$$

In a similar way if we know a solution of the equation $P_+ w = -I_1 f$ the solenoidal vector field f is determined by the formula

$$f = \frac{1}{2\pi}\nabla_\perp I_0^* w.$$

In this section we describe a procedure that leads to Fredholm type inversion formulas.

We assume that (M, g) is simple and $f \in C_0^\infty(M)$. Then $u^f \in C^\infty(\Omega(M))$ as in (1.1). Its odd part with respect to ξ, u_-^f, satisfies $\mathcal{H} u_-^f = -f$. Applying the Hilbert transform we obtain

$$\mathcal{H} H u_-^f = -(\mathcal{H}_\perp u^f)_0.$$

Proposition 5.1. *The operator $W : C_0^\infty(M) \to C^\infty(M)$, defined by*

$$Wf = (\mathcal{H}_\perp u^f)_0$$

can be extended to a smoothing operator $W : L^2(M) \to C^\infty(M)$.

Proof. We define two Jacobi vector fields along the geodesic $\gamma(x, \xi, t)$:

$$X(x,\xi,t) = \mathcal{H}_\perp \gamma(x,\xi,t), \;\; Y(x,\xi,t) = \partial_\theta \gamma(x,\xi,t),$$

where

$$\partial_\theta = (\xi_\perp, \partial)$$

and ∂ is defined in section 3.

These Jacobi fields have initial data

$$\begin{aligned} X(x,\xi,0) &= \xi_\perp, \;\; D_t X(x,\xi,0) = 0, \\ Y(x,\xi,0) &= 0, \;\; D_t Y(x,\xi,0) = \xi_\perp, \end{aligned}$$

where D_t is covariant derivative along the geodesic and is orthogonal to the geodesic :

$$\langle X, \dot{\gamma} \rangle = \langle Y, \dot{\gamma} \rangle = 0.$$

We denote by $\dot{\gamma}_\perp(x, \xi, t)$ the unit vector field on the geodesic $\gamma(x, \xi, t)$, parallel translated along the geodesic, i.e. $D_t\dot{\gamma}_\perp = 0$ with initial data $\dot{\gamma}_\perp(x, \xi, 0) = \xi_\perp$. This field is orthogonal to the geodesic. The fields X and Y are proportional to $\dot{\gamma}_\perp$,

$$X = a\dot{\gamma}_\perp, \ Y = b\dot{\gamma}_\perp$$

and it follows from the Jacobi equation that the coefficients a and b satisfy the scalar Jacobi equations

$$\ddot{a} + Ka = 0, \ \ \ddot{b} + Kb = 0,$$

where K is the Gaussian curvature. The initial data is given by

$$\begin{aligned} a(x, \xi, 0) &= 1, \ \dot{a}(x, \xi, 0) = 0, \\ b(x, \xi, 0) &= 0, \ \dot{b}(x, \xi, 0) = 1. \end{aligned}$$

Note that if $K = const$ then the functions a and b do not depend on (x, ξ).

Denote by $G \subset \Omega(S) \times R$ the submanifold

$$G = \{(x, \xi, t) : (x, \xi) \in \Omega\left(M\right), \ t \in \left[-\tau\left(x, -\xi\right), \tau\left(x, \xi\right)\right]\}.$$

The functions a and b are smooth on G. Consider on $G\backslash\{t = 0\}$ the function

$$q = \partial_\theta \frac{a}{b}.$$

The simplicity of (M, g) implies that $b \neq 0$ if $t \neq 0$. Considering the initial data we have that $(b\partial_\theta a - a\partial_\theta b)|_{t=0} = 0$. Since $t = 0$ is a simple zero of b then q can be extended smoothly to the manifold G. More precisely it is easy to see that

$$q\left(x, \xi, t\right) = tQ(x, t\xi), \quad Q \in C^\infty\left(T\left(M\right)\right).$$

We have

$$2\pi\left(\mathcal{H}_{\perp}u^{f}\right)_{0}(x)=\int\limits_{\Omega_x}d\Omega_x\mathcal{H}_{\perp}\int\limits_{0}^{\tau(x,\xi)}f\left(\gamma(x,\xi t)\right)dt=$$

$$=\int\limits_{\Omega_x}d\Omega_x\int\limits_{0}^{\tau(x,\xi)}\frac{a\left(x,\xi,t\right)}{b\left(x,\xi,t\right)}\left\langle\nabla f,Y\right\rangle(x,\xi,t)dt=$$

$$=\int\limits_{\Omega_x}d\Omega_x\int\limits_{0}^{\tau(x,\xi)}\frac{a\left(x,\xi,t\right)}{b\left(x,\xi,t\right)}\partial_{\theta}[f\left(\gamma\left(x,\xi t\right)\right)]dt=$$

$$=-\int\limits_{\Omega_x}d\Omega_x\int\limits_{0}^{\tau(x,\xi)}q\left(x,\xi,t\right)f\left(\gamma\left(x,\xi t\right)\right)dt=$$

$$=-\int\limits_{D_x}Q\left(x,\eta\right)f\left(\gamma\left(x,\eta\right)\right)dT_x,$$

where $D_x=Exp_x^{-1}\left(M\right)$ and dT_x denotes the volume form on $T_x(M)$. After the change of variables $\eta\rightarrow y=\gamma\left(x,\eta\right)$ we obtain integral operator with smooth kernel

$$\left(\mathcal{H}_{\perp}u^{f}\right)_{0}(x)=(Wf)(x)=\int\limits_{M}f\left(y\right)W\left(x,y\right)dM_y,$$

where

$$W\left(x,y\right)=-Q\left(x,Exp_x^{-1}y\right)\frac{\det\left(Exp_x^{-1}\right)'\left(x,y\right)\sqrt{\det g\left(x\right)}}{\sqrt{\det g\left(y\right)}}.$$

□

Notice, that in the case of constant Gaussian curvature $W=0$, since $q=0$ (and this does not depend on whether there are conjugate points).

We need the following Lemma in order to compute the adjoint of the operator W. We mean evenness and oddness with respect to the ξ variable.

Lemma 5.1. *For any even function $f\in L^2\left(\Omega\left(M\right)\right)$ and odd function $h\in L^2\left(\Omega\left(M\right)\right)$*

$$\left(If,Ih\right)_{L^2_{\mu}(\partial_{+}\Omega(M))}=0.$$

Proof. It is enough to prove this for smooth f,h. We have

$$\left(If,Ih\right)_{L_{\mu}(\partial_{+}\Omega(M))}=\int\limits_{\partial_{+}\Omega(M)}\mu IfIhd\Sigma.$$

The integral of a function over $\partial\Omega(M)$ is equal to the integral of its even part:

$$(If, Ih)_{L^2_\mu(\partial_+\Omega(M))} = 2 \int\limits_{\partial\Omega(M)} \mu u^f_+ u^h_- d\Sigma.$$

Clearly

$$\mathcal{H}u^f_+ = 0, \quad \mathcal{H}u^h_- = 0.$$

Integrating the equality $0 = \mathcal{H}(u^f_+ u^h_-)$, we obtain

$$0 = \int\limits_{\partial\Omega(M)} \mu u^f_+ u^h_- d\Sigma = \frac{1}{2}(If, Ih)_{L^2_\mu(\partial_+\Omega(M))}.$$

□

Proposition 5.2.

$$W^* h := (u^{\mathcal{H}_\perp h})_0.$$

is the adjoint of W *in* $L^2(M)$.

Proof. For $f, h \in C_0^\infty(M)$ we have

$$(f, W^* h) = \frac{1}{2\pi}\left(f, u^{\mathcal{H}_\perp h}\right) = -\frac{1}{2\pi}\left(\mathcal{H}u^f, u^{\mathcal{H}_\perp h}\right) = -\frac{1}{2\pi}(If, I\mathcal{H}_\perp h)_{L^2_\mu(\partial_+\Omega(M))} + \frac{1}{2\pi}\left(u^f, \mathcal{H}u^{\mathcal{H}_\perp h}\right).$$

By Lemma 5.1 the first term of the last equation is zero. We obtain

$$(f, W^* h) = -\frac{1}{2\pi}\left(u^f, \mathcal{H}_\perp h\right) = \frac{1}{2\pi}\left(\mathcal{H}_\perp u^f, h\right) = \left((\mathcal{H}_\perp u^f)_0, h\right) = (Wf, h).$$

□

Clearly also $W^* : L^2(M) \to C^\infty(M)$ is smoothing.

We denote below by $(I_m f)^+, (I_m f)^-, m = 0, 1$ the even and odd continuations of $I_m f$ with respect to ξ.

Now we are in position to state our inversion results.

Theorem 5.1. *Let* (M, g) *be a two-dimensional simple manifold. Then*

$$\begin{aligned} f + W^2 f &= \frac{1}{2\pi}\delta_\perp I_1^* w, \quad w = \frac{1}{2}\alpha^* H(I_0 f)^-|_{\partial_+\Omega(M)}, \quad f \in L^2(M), \\ h + (W^*)^2 h &= \frac{1}{2\pi} I_0^* w, \quad w = \frac{1}{2}\alpha^* H(I_1 \mathcal{H}_\perp h)^+|_{\partial_+\Omega(M)}, \; h \in H_0^1(M), \end{aligned}$$

where $W, W^* : L^2(M) \to C^\infty(M)$. *In the case of a manifold of constant curvature* $W = 0, \quad W^* = 0$.

Proof. Applying the Hilbert transform to the equality $\mathcal{H}u_-^f = -f$ we conclude

$$\mathcal{H}H_-u^f = -Wf. \tag{5.1}$$

The solution of the boundary value problem for the transport equation

$$\mathcal{H}u = -f, \;\; u|_{\partial_+\Omega(M)} = w$$

is written in the form

$$u = u^f + w_\psi.$$

Then

$$H_-u^f = u^{Wf} + w_\psi, \tag{5.2}$$

where $w = \frac{1}{2}H(I_0 f)^-|_{\partial_+\Omega(M)}$.

Applying the Hilbert transform to (5.1) we get

$$\mathcal{H}H^2u_-^f = -\left(\mathcal{H}_\perp H u_-^f\right)_0.$$

Since $(u_-^f)_0 = 0$, then $H^2u_-^f = -u_-^f$ and we obtain

$$(\mathcal{H}_\perp H u_-^f)_0 = f.$$

Applying the operator $(\mathcal{H}_\perp)_0$ to (5.2) gives

$$f + W^2 f = (\mathcal{H}_\perp w_\psi)_0 = \frac{1}{2\pi}\delta_\perp I_1^* w, \quad w = \frac{1}{2}H(I_0 f)^-|_{\partial_+\Omega(M)}.$$

Since $C_0^\infty(M)$ is dense in $L^2(M)$ then the equality above holds for all $f \in L^2(M)$. proving the first statement of Theorem 5.1.

Now we obtain an analogous identity for the case of vector fields. Assume for simplicity $f = \nabla_\perp h, \; h \in C_0^\infty(M)$. We have

$$\mathcal{H}u_+^{f_1} = -f_1, \;\; \mathcal{H}Hu_+^{f_1} = -\mathcal{H}h - \mathcal{H}_\perp u_0^{f_1}. \tag{5.3}$$

From (5.3) it follows that

$$h + Hu_+^{\mathcal{H}_\perp h} = -u^{\mathcal{H}_\perp W^* h} + w_\psi,$$

where $w = HI_1^+(\mathcal{H}_\perp h)|_{\partial_+\Omega(M)}$. Since $(Hu)_0 \equiv 0$, integration over Ω_x gives that

$$h + (W^*)^2 h = \frac{1}{2\pi}I_0^* w, \quad w = \frac{1}{2}H(I_1\mathcal{H}_\perp h)^+|_{\partial_+\Omega(M)},$$

Since $C_0^\infty(M)$ is dense in $H_0^1(M) = \{f \in H^1(M) : f|_{\partial M} = 0\}$ then this holds for all $h \in H_0^1(M)$ concluding the proof of the theorem. $\square$

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