Visibility and Invisibility

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Abstract. The first part of this article concerns visibility, that is the question of determining the internal properties of a medium by making electromagnetic measurements at the boundary of the medium. We concentrate on the problem of Electrical Impedance Tomography (EIT) which consists in determining the electrical conductivity of a medium by making voltage and current measurements at the boundary. We describe the use of complex geometrical optics solutions in EIT.

In the second part of this article we will review recent theoretical and experimental progress on making objects invisible to electromagnetic waves. This is joint work with A. Greenleaf, Y. Kurylev and M. Lassas. Maxwell's equations have transformation laws that allow for design of electromagnetic parameters that would steer light around a hidden region, returning it to its original path on the far side. Not only would observers be unaware of the contents of the hidden region, they would not even be aware that something was hidden. The object would have no shadow. New advances in metamaterials have given some experimental evidence that this indeed can be made possible at certain frequencies.

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1. Visibility

1.1. Introduction. In this section we consider the subject of *visibility* that is whether we can determine electromagnetic parameters of a medium by probing the medium with time harmonic waves and measuring the response at the boundary. We concentrate in the case of a single frequency and emphasis in particular Calderón's inverse problem, which is the question of whether an unknown conductivity distribution inside a domain in $\mathbb{R}^n$, modelling for example the human thorax, can be determined from voltage and current measurements made on the boundary. This is also known as Electrical Impedance Tomography (EIT). See [8, 14] for reviews of EIT.

The problem was originally proposed by Calderón [11] in 1980 motivated by oil prospecting. In the 40's he worked as an engineer for Yacimientos Petrolíferos Fiscales (YPF), the state oil company of Argentina. Cancerous breast tissue is

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known to differ significantly from healthy breast tissue in terms of conductivity [32]. Given local EIT data measured using electrodes placed on the breast, the problem is to find out whether there is a cancerous region (tumor) inside the breast, and if so, what is the approximate location of the tumor. See [76] and the references therein. Another relevant application of EIT is geophysical sensing for underground objects, see for example [31]. See also [25] for a recent issue of Physiological Measurement dedicated to EIT.

For isotropic conductivities this problem can be mathematically formulated as follows: Let Ω be the measurement domain, and denote by γ a bounded and strictly positive function describing the conductivity in Ω. In Ω the voltage potential u satisfies the equation

$$\nabla \cdot \gamma \nabla u = 0. \tag{1.1}$$

To uniquely fix the solution u it is enough to give its value, f, on the boundary. In the idealized case, one measures for all voltage distributions $u|_{\partial\Omega} = f$ on the boundary the corresponding current flux, $\nu \cdot \gamma \nabla u$, through the boundary, where ν is the exterior unit normal to $\partial\Omega$. Mathematically this amounts to the knowledge of the Dirichlet–Neumann map Λ_γ corresponding to γ, i.e., the map taking the Dirichlet boundary values of the solution to (1.1) to the corresponding Neumann boundary values,

$$\Lambda_\gamma : \quad u|_{\partial\Omega} \mapsto \nu \cdot \gamma \nabla u|_{\partial\Omega}.$$

Calderón's inverse problem is then to reconstruct γ from Λ_γ.

Unique determination of an isotropic conductivity from the DN map was shown in dimension $n > 2$ for C^2 conductivities in [66]. At the writing of this paper this result has been extended to conductivities having $\frac{3}{2}$ derivatives in [8] and [56]. In two dimensions the first unique identifiability result was proven in [50] for C^2 conductivities. This was improved to Lipschitz conductivities in [10] and to merely L^∞ conductivities in [3]. All of these results use *complex geometrical optics* (CGO) solutions with a linear phase that we review in section 1.2.

It is often possible to measure the DN map only on part of the boundary. This is the case in medical and geophysical EIT since it is not practical to cover a patient or the Earth completely by electrodes. This is the partial data problem that it is considered in section 1.3. In dimension three or higher, it is shown in [33], that if one measures the voltage on an open subset of the boundary and measures the current flux in, roughly, the complement, one can determine uniquely the conductivity in the whole domain. This result uses a new set of CGO solutions with a non-linear phase. We review these solutions in section 1.3 as well as the limiting Carleman weight estimates used in their construction. In section 1.4 we use the solutions constructed in section 1.3 for the problem of detecting anomalies in particular inclusions from electrical measurements made on part of the boundary.

In section 1.5 we discuss the case of anisotropic conductivities, that is conductivities that depend also on direction. In this case the problem is invariant under changes of variables that are the identity at the boundary. We review in this section what is known about this problem. The fact that the anisotropic conductivity equation is invariant under transformations played a crucial role on the constructions of electromagnetic parameters that make objects invisible which is the subject of section 2.

1.2. Complex geometrical optics solutions with a linear phase.

If u is a solution of (1.1) with boundary data f, the divergence theorem gives that

$$Q_\gamma(f) := \int_\Omega \gamma |\nabla u|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) f \, dS \tag{1.2}$$

where dS denotes surface measure. In other words $Q_\gamma(f)$ is the quadratic form associated to the linear map $\Lambda_\gamma(f)$, i.e., to know $\Lambda_\gamma(f)$ or $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ is equivalent. The form $Q_\gamma(f)$ measures the energy needed to maintain the potential f at the boundary. Calderón's point of view was to find enough solutions $u \in H^1(\Omega)$ of the conductivity equation $\operatorname{div}(\gamma\nabla u) = 0$ so that $|\nabla u|^2$ is dense in an appropriate topology in order to find γ in Ω. Notice that the DN map (or Q_γ) depends non-linearly on γ. Calderón considered the linearized problem at a constant conductivity. A crucial ingredient in his approach is the use of the harmonic complex exponential solutions:

$$u = e^{x\cdot\rho}, \text{ where } \rho \in \mathbb{C}^n \text{ with } \rho\cdot\rho = 0. \tag{1.3}$$

Sylvester and Uhlmann [66, 67] constructed in dimension $n \geq 2$ complex geometrical optics (CGO) solutions of the conductivity equation for C^2 conductivities similar to Calderón's. This can be reduced to constructing solutions in the whole space (by extending $\gamma = 1$ outside a large ball containing Ω) for the Schrödinger equation with potential. We describe this more precisely below.

Let $\gamma \in C^2(\mathbb{R}^n)$, γ strictly positive in $\mathbb{R}^n$ and $\gamma = 1$ for $|x| \geq R$, $R > 0$. Let $L_\gamma u = \nabla \cdot \gamma \nabla u$. Then we have

$$\gamma^{-\frac{1}{2}} L_\gamma (\gamma^{-\frac{1}{2}}) = \Delta - q \tag{1.4}$$

where

$$q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{1.5}$$

Therefore, to construct solutions of $L_\gamma u = 0$ in $\mathbb{R}^n$ it is enough to construct solutions of the Schrödinger equation $(\Delta - q)u = 0$ with q of the form (1.5). The next result proven in [66, 67] states the existence of complex geometrical optics solutions for the Schrödinger equation associated to any bounded and compactly supported potential.

Theorem 1.1. *Let $q \in L^\infty(\mathbb{R}^n)$, $n \geq 2$, with $q(x) = 0$ for $|x| \geq R > 0$. Let $-1 < \delta < 0$. There exists $\epsilon(\delta)$ and such that for every $\rho \in \mathbb{C}^n$ satisfying*

$$\rho \cdot \rho = 0$$

and

$$\frac{\|(1+|x|^2)^{1/2} q\|_{L^\infty(\mathbb{R}^n)} + 1}{|\rho|} \leq \epsilon$$

there exists a unique solution to

$$(\Delta - q)u = 0$$

of the form

$$u = e^{x\cdot\rho}(1 + \psi_q(x,\rho)) \tag{1.6}$$

with $\psi_q(\cdot,\rho) \in L^2_\delta(\mathbb{R}^n)$. *Moreover* $\psi_q(\cdot,\rho) \in H^2_\delta(\mathbb{R}^n)$ *and for* $0 \le s \le 2$ *there exists* $C = C(n,s,\delta) > 0$ *such that*

$$\|\psi_q(\cdot,\rho)\|_{H^s_\delta} \le \frac{C}{|\rho|^{1-s}} \tag{1.7}$$

Here

$$L^2_\delta(\mathbb{R}^n) = \{f; \int (1+|x|^2)^\delta |f(x)|^2 dx < \infty\}$$

with the norm given by $\|f\|^2_{L^2_\delta} = \int (1+|x|^2)|f(x)|^2 dx$ and $H^m_\delta(\mathbb{R}^n)$ denotes the corresponding Sobolev space. Note that for large $|\rho|$ these solutions behave like Calderón's exponential solutions $e^{x\cdot\rho}$. The equation for ψ_q is given by

$$(\Delta + 2\rho\cdot\nabla)\psi_q = q(1+\psi_q). \tag{1.8}$$

The equation (1.8) is solved by constructing an inverse for $(\Delta + 2\rho\cdot\nabla)$ and solving the integral equation

$$\psi_q = (\Delta + 2\rho\cdot\nabla)^{-1}(q(1+\psi_q)). \tag{1.9}$$

Lemma 1.2. *Let* $-1 < \delta < 0$, $0 \le s \le 1$. *Let* $\rho \in \mathbb{C}^n - 0$, $\rho\cdot\rho = 0$. *Let* $f \in L^2_{\delta+1}(\mathbb{R}^n)$. *Then there exists a unique solution* $u_\rho \in L^2_\delta(\mathbb{R}^n)$ *of the equation*

$$\Delta_\rho u_\rho := (\Delta + 2\rho\cdot\nabla)u_\rho = f. \tag{1.10}$$

Moreover $u_\rho \in H^2_\delta(\mathbb{R}^n)$ *and*

$$\|u_\rho\|_{H^s_\delta(\mathbb{R}^n)} \le \frac{C_{s,\delta}\|f\|_{L^2_{\delta+1}}}{|\rho|^{s-1}}$$

for $0 \le s \le 1$ *and for some constant* $C_{s,\delta} > 0$.

The integral equation (1.8) can then be solved in $L^2_\delta(\mathbb{R}^n)$ for large $|\rho|$ since

$$(I - (\Delta + 2\rho\cdot\nabla)^{-1}q)\psi_q = (\Delta + 2\rho\cdot\nabla)^{-1}q$$

and $\|(\Delta + 2\rho\cdot\nabla)^{-1}q\|_{L^2_\delta\to L^2_\delta} \le \frac{C}{|\rho|}$ for some $C > 0$ where $\|\quad\|_{L^2_\delta\to L^2_\delta}$ denotes the operator norm between $L^2_\delta(\mathbb{R}^n)$ and $L^2_\delta(\mathbb{R}^n)$. We will not give details of the proof of Lemma 1.2 here. We refer to the papers [66, 67] .

If 0 is not a Dirichlet eigenvalue for the Schrödinger equation we can also define the DN map

$$\Lambda_q(f) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$$

where u solves

$$(\Delta - q)u = 0; \quad u|_{\partial\Omega} = f.$$

The DN map associated to the Schrödinger equation $\Delta - q$ determines in dimension $n > 2$ uniquely a bounded potential (see [66]). The case of more singular conormal potentials than $L^{\frac{n}{2}}$ was considered in [19].

The two dimensional results of [50],[10], [3] use similar CGO solutions and the $\overline{\partial}$ method in the complex frequency introduced by Beals and Coifman in [6] and generalized to higher dimensions in several articles [7],[1], [52].

We also remark that it is an open problem in two dimensions whether one can determine uniquely a potential q from the DN map for the Schrödinger equation. It is shown that this is the case in [64] for a generic class of potentials.

Other applications to inverse problems using the CGO solutions described above with a linear phase are:

- Quantum Scattering. It is shown in [51] and [53] that in dimension $n > 2$ the scattering amplitude at a fixed energy determines uniquely a two body compactly supported potential. This result also follows from [66] (see for instance [70], [71]). Applications of CGO solutions to the 3-body problem were given in [72].

- Optics. The DN map associated to the Helmholtz equation $-\Delta + k^2 n(x)$ with an isotropic index of refraction n determines uniquely a bounded index of refraction in dimension $n > 2$.

- Optical tomography in the diffusion approximation. In this case we have $\nabla \cdot \mathcal{D}(x)\nabla u - \sigma_a(x)u - i\omega u = 0$ in Ω where u represents the density of photons, $\mathcal{D}$ the diffusion coefficient, and σ_a the optical absorption. Using the result of [66] one can show in dimension three or higher that if $\omega \neq 0$ one can recover both $\mathcal{D}$ and σ_a from the corresponding DN map. If $\omega = 0$ then one can recover one of the two parameters.

- Electromagnetics. The DN map for isotropic Maxwell's equations determines uniquely the isotropic electric permittivity, magnetic permeability and conductivity [54]. This system can in fact be reduced to the Schrödinger equation $\Delta - Q$ with Q an 8×8 system and Δ the Laplacian times the identity matrix [55].

For further discussion and other applications of CGO solutions with a linear phase, including inverse problems for the magnetic Schrödinger operator, see [70].

1.3. The partial data problem. In several applications one can only measure data on part of the boundary. Substantial progress has been made recently on the problem of whether one can determine the conductivity in the interior by measuring the DN map on part of the boundary.

The paper [10] used the method of Carleman estimates with a linear weight to prove that, roughly speaking, knowledge of the DN map in "half" of the boundary is enough to determine uniquely a C^2 conductivity. The regularity assumption on the conductivity was relaxed to $C^{1+\epsilon}, \epsilon > 0$ in [37]. Stability estimates for the uniqueness result of [10] were given in [24]. Stability estimates for the magnetic Schrödinger operator with partial data in the setting of [10] can be found in [68].

The [10] result was substantially improved in [33]. The latter paper contains a global identifiability result where it is assumed that the DN map is measured on any open subset of the boundary for all functions supported, roughly, on the complement. We state the theorem more precisely below.

Let $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$, where $\mathrm{ch}\,(\Omega)$ denotes the convex hull of Ω. Define the front and the back faces of $\partial\Omega$ by

$$F(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu \le 0\},\ B(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu > 0\}.$$

The main result of [33] is the following:

Theorem 1.3. *Let $n > 2$. With Ω, x_0, $F(x_0)$, $B(x_0)$ defined as above, let $q_1, q_2 \in L^\infty(\Omega)$ be two potentials and assume that there exist open neighborhoods $\widetilde{F}$, $\widetilde{B} \subset \partial\Omega$ of $F(x_0)$ and $B(x_0) \cup \{x \in \partial\Omega; (x - x_0) \cdot \nu = 0\}$ respectively, such that*

$$\Lambda_{q_1} u = \Lambda_{q_2} u \ \text{in } \widetilde{F},\ \text{for all } u \in H^{\frac{1}{2}}(\partial\Omega) \cap \mathscr{E}'(\widetilde{B}). \tag{1.11}$$

Then $q_1 = q_2$.

Here $\mathscr{E}'(\widetilde{B})$ denotes the space of compactly supported distributions in $\widetilde{B}$.

We remark that this theorem has not been proven in two dimensions.

The proof of this result uses Carleman estimates for the Laplacian with limiting Carleman weights (LCW). The Carleman estimates allow one to construct, for large τ, a larger class of CGO solutions for the Schrödinger equation than previously used. These have the form

$$u = e^{\tau(\phi + i\psi)}(a + r), \tag{1.12}$$

where $\nabla\phi \cdot \nabla\psi = 0, |\nabla\phi|^2 = |\nabla\psi|^2$ and ϕ is the LCW. Moreover a is smooth and non-vanishing and $\|r\|_{L^2(\Omega)} = O(\frac{1}{\tau})$, $\|r\|_{H^1(\Omega)} = O(1)$. Examples of LCW are the linear phase $\phi(x) = x \cdot \omega, \omega \in S^{n-1}$, used previously, and the non-linear phase $\phi(x) = \ln|x - x_0|$, where $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$ which was used in [33]. Any conformal transformation of these would also be a LCW. For a characterization of all the LCW in $\mathbb{R}^n$, $n > 2$, see [13]. In two dimensions any harmonic function is a LCW [74].

1.3.1. Limiting Carleman weights. We here only recall the main ideas in the construction of the solutions. We will denote $\tau = \frac{1}{h}$ in order to use the standard semiclassical notation. Let $P_0 = -h^2\Delta$, where $h > 0$ is a small semi-classical parameter. The weighted L^2-estimate

$$\|e^{\phi/h} u\| \le C\|e^{\phi/h} P_0 u\|$$

is of course equivalent to the unweighted estimate for a conjugated operator:

$$\|v\| \le C\|e^{\phi/h} P_0 e^{-\phi/h} v\|.$$

The semi-classical principal symbol of P_0 is $p(x,\xi) = \xi^2$, and that of the conjugated operator $e^{\phi/h}P_0e^{-\phi/h}$ is

$$p(x,\xi + i\phi'(x)) = a(x,\xi) + ib(x,\xi),$$

where

$$a(x,\xi) = \xi^2 - \phi'(x)^2,\ b(x,\xi) = 2\xi\cdot\phi'(x).$$

Here we denote by ϕ' the gradient of ϕ.

Write the conjugated operator as $A + iB$, with A and B formally selfadjoint and with a and b as their associated principal symbols. Then

$$\|(A+iB)u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u|u).$$

The principal symbol of $i[A,B]$ is $h\{a,b\}$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket. In order to get enough positivity we require that

$$a(x,\xi) = b(x,\xi) = 0 \Rightarrow \{a,b\} \geq 0.$$

It is then indeed possible to get an apriori estimate for the conjugated operator. Since we need these estimates to fit nicely with the construction of WKB-solutions, we are led to consider especially the limiting case, as it appears in the following.

Definition 1.4. ϕ is a limiting Carleman weight (LCW) on some open set Ω if $\nabla\phi(x)$ is non-vanishing there and we have

$$a(x,\xi) = b(x,\xi) = 0 \Rightarrow \{a,b\} = 0.$$

Proposition 1.5. *Let $\phi \in C^\infty(\mathrm{neigh}\,(\overline{\Omega}))$ be an LCW, $P = -h^2\Delta + h^2q$, $q \in L^\infty(\Omega)$. Then, for $u \in C^\infty(\overline{\Omega})$, with $u_{|\partial\Omega} = 0$, we have*

$$\begin{aligned} -\tfrac{h^3}{C}((\phi'_x\cdot\nu)e^{\phi/h}\partial_\nu u|e^{\phi/h}\partial_\nu u)_{\partial\Omega_-} + \tfrac{h^2}{C}(\|e^{\phi/h}u\|^2 + \|e^{\phi/h}\nabla u\|^2) \\ \leq Ch^3((\phi'_x\cdot\nu)e^{\phi/h}\partial_\nu u|e^{\phi/h}\partial_\nu u)_{\partial\Omega_+} + \|e^{\phi/h}Pu\|^2, \end{aligned} \tag{1.13}$$

where norms and scalar products are in $L^2(\Omega)$ unless a subscript A (like for instance $A = \partial\Omega_-$) indicates that they should be taken in $L^2(A)$. Here

$$\partial\Omega_\pm = \{x \in \partial\Omega;\ \pm\nu(x)\cdot\phi'(x) \geq 0\}.$$

We remark that if ϕ is a LCW so is $-\phi$. The proof of existence of solutions of the form (1.15) follows by using the Hahn-Banach theorem for the adjoint equation $e^{-\phi/h}Pe^{\phi/h}u = v$.

1.3.2. Special null solutions. Let ϕ be a LCW and write $p(x,\phi'(x)+\xi) = a(x,\xi) + ib(x,\xi)$. Then we know that a and b are in involution on their common zero set, and in this case it is well-known and exploited in [15] that we can find plenty of local solutions to the Hamilton-Jacobi system

$$\left\{\begin{array}{l} a(x,\psi'(x)) = 0 \\ b(x,\psi'(x)) = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \psi'^2 = \phi'^2 \\ \psi'\cdot\phi' = 0 \end{array}\right. \tag{1.14}$$

We need the following more global statement:

Proposition 1.6. *Let $\phi \in C^\infty(\text{neigh}\,(\overline{\Omega}))$ be an LCW, where Ω is a domain in $\mathbf{R}^n$ and define the hypersurface $G = p^{-1}(C_0)$ for some fixed value of C_0. Assume that each integral curve of $\phi' \cdot \partial_x$ through a point in Ω also intersects G and that the corresponding projection map $\Omega \to G$ is proper. Then we get a solution of (1.14) in $C^\infty(\Omega)$ by solving first $g'(x)^2 = \phi'(x)^2$ on G and then defining ψ by $\psi_{|G} = g$, $\phi'(x) \cdot \partial_x \psi = 0$. The vector fields $\phi' \cdot \partial_x$ and $\psi' \cdot \partial_x$ commute.*

This result will be applied with a new domain Ω that contains the original one. Next consider the WKB-problem

$$P_0(e^{\frac{1}{h}(-\phi+i\psi)}a(x)) = e^{\frac{1}{h}(-\phi+i\psi)}\mathcal{O}(h^2). \tag{1.15}$$

The transport equation for a is of Cauchy-Riemann type along the two-dimensional integral leaves of $\{\phi' \cdot \partial_x, \psi' \cdot \partial_x\}$. We have solutions that are smooth and everywhere $\neq 0$. (See [15]).

The existence result for $e^{\phi/h}Pe^{-\phi/h}$ mentioned in one of the remarks after Proposition 1.5 permits us to to replace the right hand side of (1.15) by zero, more precisely, we can find $r = \mathcal{O}(h)$ in the semi-classical Sobolev space H^1 equipped with the norm $\|r\| = \|\langle hD\rangle r\|$, such that

$$P(e^{\frac{1}{h}(-\phi+i\psi)}(a+r)) = 0. \tag{1.16}$$

1.3.3. The uniqueness proof. We sketch the proof for the case that $\widetilde{B} = \partial\Omega$. All the arguments in this section are in dimension $n > 2$. Here we repeat the argument of [10] with richer spaces of null-solutions. Let ϕ be an LCW for which the constructions of Section 1.3.2 are available. Let $q_1, q_2 \in L^\infty(\Omega)$ be as in Theorem 1.3 with

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f) \text{ in } \partial\Omega_{-,\epsilon_0},\ \forall f \in H^{\frac{1}{2}}(\partial\Omega), \tag{1.17}$$

where

$$\begin{aligned} \partial\Omega_{-,\epsilon_0} &= \{x \in \partial\Omega;\ \nu(x)\cdot\phi'(x) < \epsilon_0\} \\ \partial\Omega_{+,\epsilon_0} &= \{x \in \partial\Omega;\ \nu(x)\cdot\phi'(x) \geq \epsilon_0\}. \end{aligned}$$

Let

$$u_2 = e^{\frac{1}{h}(\phi+i\psi_2)}(a_2+r_2)$$

solve

$$(\Delta - q_2)u_2 = 0 \text{ in } \Omega,$$

with $\|r_2\|_{H^1} = \mathcal{O}(h)$. Let u_1 solve

$$(\Delta - q_1)u_1 = 0 \text{ in } \Omega,\ u_{1|_{\partial\Omega}} = u_{2|_{\partial\Omega}}.$$

Then according to the assumptions in the theorem, we have $\partial_\nu u_1 = \partial_\nu u_2$ in $\partial\Omega_{-,\epsilon_0}$ if $\epsilon_0 > 0$ has been fixed sufficiently small and we choose $\phi(x) = \ln|x - x_0|$.

Put $u = u_1 - u_2$, $q = q_2 - q_1$, so that

$$(\Delta - q_1)u = qu_2,\ u_{|_{\partial\Omega}} = 0,\ \text{supp}\,(\partial_\nu u_{|_{\partial\Omega}}) \subset \partial\Omega_{+,\epsilon_0}. \tag{1.18}$$

For $v \in H^1(\Omega)$ with $\Delta v \in L^2$, we get from Green's formula

$$\begin{aligned} \int_\Omega q u_2 \overline{v} dx &= \int_\Omega (\Delta - q_1) u \overline{v} dx \\ &= \int_\Omega u \,\overline{(\Delta - \overline{q}_1) v} dx + \int_{\partial\Omega_{+,\epsilon_0}} \partial_\nu u \overline{v} S(dx). \end{aligned} \tag{1.19}$$

Similarly to u_2, we choose

$$v = e^{-\frac{1}{h}(\phi + i\psi_1)}(a_1 + r_1),$$

with

$$(\Delta - \overline{q}_1)v = 0.$$

Then

$$\int_\Omega q e^{\frac{i}{h}(\psi_1 + \psi_2)}(a_2 + r_2)(\overline{a_1 + r_1}) dx = \int_{\partial\Omega_{+,\epsilon_0}} \partial_\nu u \, e^{-\frac{1}{h}(\phi - i\psi_1)}(\overline{a_1 + r_1}) S(dx). \tag{1.20}$$

Assume that ψ_1, ψ_2 are slightly h-dependent with

$$\frac{1}{h}(\psi_1 + \psi_2) \to f, \ \ h \to 0.$$

The left hand side of (1.20) tends to

$$\int_\Omega q e^{if} a_2 \overline{a}_1 dx,$$

when $h \to 0$. The modulus of the right hand side is

$$\leq \| a_1 + r_1 \|_{\partial\Omega_{+,\epsilon_0}} \big(\int_{\partial\Omega_{+,\epsilon_0}} e^{-2\phi/h} |\partial_\nu u|^2 S(dx) \big)^{\frac{1}{2}}.$$

Here the first factor is bounded when $h \to 0$. In the Carleman estimate (1.13) we can replace ϕ by $-\phi$ and make the corresponding permutation of $\partial\Omega_-$ and $\partial\Omega_+$. Applying this variant to the equation (1.18), we see that the second factor tends to 0, when $h \to 0$. Thus,

$$\int_\Omega e^{if(x)} a_2(x) \overline{a_1(x)} q(x) dx = 0.$$

Here we can arrange so that f, a_2, a_1 are real-analytic and so that a_1, a_2 are non-vanishing. Moreover if f can be attained as a limit of $(\psi_1 + \psi_2)/h$ when $h \to 0$, so can λf for any $\lambda > 0$. Thus we get the conclusion

$$\int_\Omega e^{i\lambda f(x)} a_2(x) \overline{a_1(x)} q(x) dx = 0. \tag{1.21}$$

To show that $q = 0$ one uses arguments of analytic microlocal analysis [33].

Isakov [29] proved a uniqueness result in dimension three or higher when the DN map is given on an arbitrary part of the boundary assuming that the remaining part is an open subset of a plane or a sphere. In [46] there are several results for inverse problems with partial data on a slab.

The DN map with partial data for the magnetic Schrödinger operator was studied in [13], [38], [68]. We also mention that in [22] (resp. [30]) CGO approximate solutions are concentrated near planes (resp. spheres) and provided some local results related to the local DN map. For further application of these solutions see the next section.

1.4. Determination of cavities and inclusions. The CGO solutions with a linear phase have the property that they grow exponentially in a direction where the inner product of the real part of the complex phase with the direction is strictly positive, they are exponentially decaying if this inner product is negative and oscillatory if the inner product is zero. This was exploited by Ikehata in [27] to give a reconstruction procedure from the DN map of a cavity D with strongly convex C^2 boundary ∂D inside a conductive medium Ω with conductivity 1 such that $\Omega \setminus \overline{D}$ is connected. We sketch some of the details here. We define the DN map Λ_D by

$$\Lambda_D(f) := \frac{\partial u(f)}{\partial \nu}|_{\partial\Omega}, \tag{1.22}$$

where $u(f) \in H^2(\Omega)$ is the solution to

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \frac{\partial u}{\partial \nu}|_{\partial D} = 0, \\ u|_{\partial\Omega} = f \in H^{3/2}(\partial\Omega) \end{cases} \tag{1.23}$$

and ν is the unit normal of ∂D. If $D = \emptyset$, we denote Λ_D by Λ_0. Let $\omega, \omega^\perp$ be unit real vectors perpendicular to each other. For $\tau > 0$, consider the Calderón harmonic functions

$$v(x, \tau, \omega, \omega^\perp) = e^{-t\tau} e^{\tau x \cdot (\omega + i\omega^\perp)}. \tag{1.24}$$

Note that this function grows exponentially in the half space $x \cdot \omega > t$ and decays exponentially in the half space $x \cdot \omega < t$. For $t \in \mathbb{R}$, define the indicator function by

$$I_{\omega,\omega^\perp}(\tau, t) := \int_{\partial\Omega} ((\Lambda_D - \Lambda_0) v|_{\partial\Omega}) \overline{v|_{\partial\Omega}}\, dS. \tag{1.25}$$

We also define the support function $h_D(\omega)$ of D by

$$h_D(\omega) := \sup_{x \in D} x \cdot \omega. \tag{1.26}$$

Ikehata characterizes the support function in terms of the indicator function. More precisely we have

$$h_D(\omega) - t = \lim_{\tau \to \infty} \frac{I_{\omega,\omega^\perp}(\tau, t)}{2\tau}. \tag{1.27}$$

Hence, by taking many ω's, we can recover the shape of D. See [27], [28] for more details and references, including numerical implementation of this method.

Using methods of hyperbolic geometry similar to [30] it is shown in [26] that one can reconstruct inclusions from the *local* DN map using CGO solutions that decay exponentially inside a ball and grow exponentially outside. These are called *complex spherical waves*. A numerical implementation of this method has been done in [26]. The construction of complex spherical waves can also be done using the CGO solutions constructed in [33]. This was done in [73] in order to detect elastic inclusions, and in [74] to detect inclusions in the two dimensional case for a large class of systems with inhomogeneous background.

1.5. Anisotropic conductivities. Anisotropic conductivities depend on direction. Muscle tissue in the human body is an important example of an anisotropic conductor. For instance cardiac muscle has a conductivity of 2.3 mho in the transverse direction and 6.3 in the longitudinal direction. The conductivity in this case is represented by a positive definite, smooth, symmetric matrix $\gamma = (\gamma^{ij}(x))$ on Ω.

Under the assumption of no sources or sinks of current in Ω, the potential u in Ω, given a voltage potential f on $\partial\Omega$, solves the Dirichlet problem

$$\begin{cases} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) &= 0 \text{ on } \Omega \\ u|_{\partial\Omega} &= f. \end{cases} \tag{1.28}$$

The DN map is defined by

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega} \tag{1.29}$$

where $\nu = (\nu^1, \ldots, \nu^n)$ denotes the unit outer normal to $\partial\Omega$ and u is the solution of (1.28). The inverse problem is whether one can determine γ by knowing Λ_γ. Unfortunately, Λ_γ doesn't determine γ uniquely. This observation is due to L. Tartar (see [40] for an account).

Let $\psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism with $\psi|_{\partial\Omega} = Id$ where Id denotes the identity map. We have

$$\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma \tag{1.30}$$

where

$$\widetilde{\gamma} = \left(\frac{(D\psi)^T \circ \gamma \circ (D\psi)}{|\det D\psi|}\right) \circ \psi^{-1}. \tag{1.31}$$

Here $D\psi$ denotes the (matrix) differential of ψ, $(D\psi)^T$ its transpose and the composition in (1.31) is to be interpreted as multiplication of matrices.

We have then a large number of conductivities with the same DN map: any change of variables of Ω that leaves the boundary fixed gives rise to a new conductivity with the same electrostatic boundary measurements. The question is then whether this is the only obstruction to unique identifiability of the conductivity. This was done in two dimensions for C^3 conductivities by reducing the anisotropic problem to the isotropic one by using isothermal coordinates [65] and using Nachman's result [50]. The regularity was improved in [63] to Lipschitz conductivities using the techniques of [10] and to L^∞ conductivities in [4] using the results of [3].

In the case of dimension $n \geq 3$, as was pointed out in [43], this is a problem of geometrical nature and makes sense for general compact Riemannian manifolds with boundary.

Let (M, g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric g is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det g} g^{ij} \frac{\partial u}{\partial x_j} \right) \tag{1.32}$$

where (g^{ij}) is the matrix inverse of the matrix (g_{ij}). Let us consider the Dirichlet problem associated to (1.32)

$$\Delta_g u = 0 \text{ on } \Omega, \quad u|_{\partial\Omega} = f \tag{1.33}$$

We define the DN map in this case by

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^i g^{ij} \frac{\partial u}{\partial x_j} \sqrt{\det g}|_{\partial\Omega} \tag{1.34}$$

The inverse problem is to recover g from Λ_g.

We have that

$$\Lambda_{\psi^* g} = \Lambda_g \tag{1.35}$$

where ψ is a C^∞ diffeomorphism of $\overline{M}$ which is the identity on the boundary. As usual $\psi^* g$ denotes the pull back of the metric g by the diffeomorphism ψ.

In the case that M is an open, bounded subset of $\mathbb{R}^n$ with smooth boundary, it is easy to see ([43]) that for $n \geq 3$

$$\Lambda_g = \Lambda_\gamma \tag{1.36}$$

where

$$(g_{ij}) = (\det \gamma^{kl})^{\frac{1}{n-2}} (\gamma^{ij})^{-1}, \quad (\gamma_{ij}) = (\det g_{kl})^{\frac{1}{2}} (g_{ij})^{-1}. \tag{1.37}$$

In the two dimensional case there is an additional obstruction since the Laplace-Beltrami operator is conformally invariant. More precisely we have

$$\Delta_{\alpha g} = \frac{1}{\alpha} \Delta_g$$

for any function α, $\alpha \neq 0$. Therefore we have that for $n = 2$

$$\Lambda_{\alpha(\psi^* g)} = \Lambda_g \tag{1.38}$$

for any smooth function $\alpha \neq 0$ so that $\alpha|_{\partial M} = 1$.

Lassas and Uhlmann ([41]) proved that (1.35) is the only obstruction to unique identifiability of the conductivity for real-analytic manifolds in dimension $n \geq 3$. In the two dimensional case they showed that (1.38) is the only obstruction to unique identifiability for smooth Riemannian surfaces. Moreover these results assume that Λ measured only on an open subset of the boundary. We state the two basic results.

Let Γ be an open su ∂M. We define for f, $\operatorname{supp} f \subseteq \Gamma$

$$\Lambda_{g,\Gamma}(f) = \Lambda_g(f)|_{\Gamma}.$$

Theorem 1.7 ($n \geq 3$). *Let (M, g) be a real-analytic compact, connected Riemannian manifold with boundary. Let $\Gamma \subseteq \partial M$ be real-analytic and assume that g is real-analytic up to Γ. Then $(\Lambda_{g,\Gamma}, \partial M)$ determines uniquely (M, g) up to an isometry.*

Theorem 1.8 ($n = 2$). *Let (M, g) be a compact Riemannian surface with boundary. Let $\Gamma \subseteq \partial M$ be an open subset. Then $(\Lambda_{g,\Gamma}, \partial M)$ determines uniquely the conformal class of (M, g) up to an isometry.*

Notice that these two results don't assume any condition on the topology of the manifold except for connectedness. An earlier result of [43] assumed that (M, g) was strongly convex and simply connected and $\Gamma = \partial M$ in both results. Theorem 1.7 was extended in [42] to non-compact, connected real-analytic manifolds with boundary.

In two dimensions the invariant form of the conductivity equation is given by

$$\operatorname{div}_g(\beta \nabla_g) u = 0 \tag{1.39}$$

where β is the conductivity and div_g (resp. ∇_g) denotes divergence (resp. gradient) with respect to the Riemannian metric g. This includes the case considered by Calderón with g the Euclidian metric, and the anisotropic case by taking ($g^{ij} = \gamma^{ij}$ and $\beta = \sqrt{\det g}$). It was shown in [63] for bounded domains of Euclidian space that the isometric class of (β, g) is determined uniquely by the corresponding DN map.

2. Invisibility

This section is joint work with A. Greenleaf, Y. Kurylev and M. Lassas and is based on the papers [19], [20], [16], [17], [18].

There have recently been many studies [2, 16, 49, 44, 57, 48, 75] on the possibility, both theoretical and practical, of a region or object being shielded, or cloaked from detection via electromagnetic waves. The interest in cloaking was raised in particular in 2006 when it was realized that practical cloaking constructions are possible using so-called metamaterials which allow fairly arbitrary specification of electromagnetic material parameters. At the present moment such materials have been implemented at microwave frequencies [61]. On the practical limitations of cloaking, we note that, with current technology, above microwave frequencies the required metamaterials are difficult to fabricate and assemble, although research is presently progressing on metamaterial engineering at optical frequencies [62].

Furthermore, metamaterials are inherently prone to dispersion, so that realistic cloaking must currently be considered as occurring at a single wavelength, or very narrow range of wavelengths.

Theoretical considerations related to cloaking were introduced already in 2003, before the appearance of practical possibilities for cloaking. Indeed, the cloaking constructions in the zero frequency case, i.e., for electrostatics, were introduced as counterexamples in the study of inverse problems. In [20, 21] it was shown that passive objects can be coated with a layer of material with a degenerate conductivity which makes the object undetectable by electrical impedance tomography (EIT), that is, in the electrostatic measurements. This gave counterexamples for uniqueness in the Calderón inverse problem for the conductivity equation. The counterexamples were motivated by consideration of certain degenerating families of Riemannian metrics, which in the limit correspond to singular conductivities, i.e., that are not bounded below or above. A related example of a complete but noncompact two-dimensional Riemannian manifold with boundary having the same Dirichlet-to-Neumann map as a compact one was given in [42].

We emphasize that for the positive results for inverse problems described in section 1 it is assumed that the eigenvalues of the conductivity are bounded below and above by positive constants. Thus, a key point in the current works on invisibility that allows one to avoid the known uniqueness theorems is the lack of positive lower and/or upper bounds on the eigenvalues of these symmetric tensor fields.

In 2006, several cloaking constructions were proposed. The constructions in [44] are based on conformal mapping in two dimensions and are justified via a change of variables on the exterior of the cloaked region. At the same time, [57] proposed a cloaking construction for Maxwell's equations based on a singular transformation of the original space, again observing that, outside the cloaked region, the solutions of the homogeneous Maxwell equations in the original space become solutions of the transformed equations. The transformations used there are the same as used in [20, 21] in the context of Calderón's inverse conductivity problem. The paper [58] contained analysis of cloaking on the level of ray-tracing, full wave numerical simulations were discussed in [12], and the cloaking experiment at 8.5Ghz is in [61].

The electromagnetic material parameters used in cloaking constructions are degenerate and, due to the degeneracy of the equations at the surface of the cloaked region, it is important to consider rigorously (weak) solutions to Maxwell's equations on *all* of the domain, not just the exterior of the cloaked region. This analysis was carried out in [16]. There, various constructions for cloaking from observation are analyzed on the level of physically meaningful electromagnetic waves, i.e., finite energy distributional solutions of the equations. In the analysis of the problem, it turns out that the cloaking structure imposes hidden boundary conditions on such waves at the surface of the cloak. When these conditions are overdetermined, finite energy solutions typically do not exist.

The time-domain physical interpretation of this was at first not entirely clear, but it now seems to be intimately related with blow-up of the fields, which may compromise the desired cloaking effect [18]. We review the results here and give the possible remedies to restore invisibility.

We note that [20, 21] gave, in dimensions $n \geq 3$, counterexamples to uniqueness for the inverse conductivity problem. Such counterexamples have now also been given and studied further in two dimensional case [39, 5].

2.1. Basic constructions.

The material parameters of electromagnetism, the electrical permittivity, $\varepsilon(x)$; magnetic permeability, $\mu(x)$; and the conductivity $\gamma(x)$ can be considered as coordinate invariant objects. As already mention in section 1.5 if $F:\Omega_1 \longrightarrow \Omega_2$, $y=F(x)$, is a diffeomorphism between domains in $\mathbb{R}^n$, then $\gamma(x)=[\gamma^{jk}(x)]_{j,k=1}^n$ on Ω_1 pushes forward to $(F_*\gamma)(y)$ on Ω_2, given by

$$(F_*\gamma)^{jk}(y) = \frac{1}{\det\left[\frac{\partial F}{\partial x}(x)\right]} \sum_{p,q=1}^{n} \frac{\partial F^j}{\partial x^p}(x)\, \frac{\partial F^k}{\partial x^q}(x)\gamma^{pq}(x)\Bigg|_{x=F^{-1}(y)}. \tag{2.1}$$

The same transformation rule is valid for permittivity ε and permeability μ.

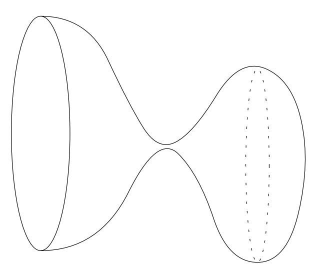

Figure 1. A family of manifolds that develops a singularity when the width of the neck connecting two parts goes to zero.

On the other hand, a Riemannian metric $g=[g_{jk}(x)]_{j,k=1}^n$ is a covariant symmetric two-tensor. As already observed in Section 1.5 in dimension three or higher, a material parameter tensor and a Riemannian metric can be associated with each other. Using this correspondence, examples of singular anisotropic conductivities in $\mathbb{R}^n, n\geq 3$, that are indistinguishable from a constant isotropic conductivity, in that they have the same Dirichlet-to-Neumann map, are given in [21]. This construction is based on degenerations of Riemannian metrics, whose singular limits can be considered as coming from singular changes of variables. If one considers Figure 1, where the "neck" of the surface (or a manifold in the higher dimensional cases) is pinched, the manifold contains in the limit a pocket about which the boundary measurements do not give any information. If the collapsing of the manifold is done in an appropriate way, in the limit we have a Riemannian manifold which is indistinguishable from a flat surface. This can be considered as a singular conductivity that appears the same as a constant conductivity to all boundary measurements.

To consider the above precisely, let $B(0,R)\subset\mathbb{R}^3$ be an open ball with center 0 and radius R. We use in the sequel the set $N=B(0,2)$, decomposed to two parts, $N_1=B(0,2)\setminus\overline{B}(0,1)$ and $N_2=B(0,1)$. Let $\Sigma=\partial N_2$ be the interface (or "cloaking surface") between N_1 and N_2.

We use also a "copy" of the ball $B(0,2)$, with the notation $M_1=B(0,2)$. Let $g_{jk}=\delta_{jk}$ be the Euclidian metric in M_1 and let $\gamma=1$ be the corresponding homogeneous conductivity. Define a singular transformation

$$F: M_1\setminus\{0\}\to N_1,\qquad F(x)=(\frac{|x|}{2}+1)\frac{x}{|x|},\quad 0<|x|\le 2. \tag{2.2}$$

The pushforward $\widetilde{g}=F_*g$ of the metric g in F is the metric in N_1 given by

$$(F_*g)_{jk}(y)=\left.\sum_{p,q=1}^{n}\frac{\partial F^p}{\partial x^j}(x)\,\frac{\partial F^q}{\partial x^k}(x)g_{pq}(x)\right|_{x=F^{-1}(y)}. \tag{2.3}$$

We use it to define a singular conductivity

$$\widetilde{\gamma}=\begin{cases}|\widetilde{g}|^{1/2}\widetilde{g}^{jk} & \text{for } x\in N_1,\\ \delta^{jk} & \text{for } x\in N_2\end{cases} \tag{2.4}$$

in N. Then, denoting by $(r,\phi,\theta)\mapsto(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ the spherical coordinates, we have

$$\widetilde{\gamma}=\begin{pmatrix}2(r-1)^2\sin\theta & 0 & 0\\ 0 & 2\sin\theta & 0\\ 0 & 0 & 2(\sin\theta)^{-1}\end{pmatrix},\quad 1<|x|\le 2.$$

This means that in the Cartesian coordinates the conductivity $\widetilde{\gamma}$ is given by

$$\widetilde{\gamma}(x)=2(I-P(x))+2(|x|-1)^2P(x),\quad 1<|x|<2,$$

where I is the identity matrix and $P(x)=|x|^{-2}xx^t$ is the projection to the radial direction. We note that the anisotropic conductivity $\widetilde{\gamma}$ is singular on Σ in the sense that it is not bounded from below by any positive multiple of I. (See [39] for a similar calculation for $n=2$.)

Consider now the *Cauchy data* of all $H^1(N)$-solutions of the conductivity equation corresponding to $\widetilde{\gamma}$, that is,

$$C_1(\widetilde{\gamma})=\{(u|_{\partial N},\nu\cdot\widetilde{\gamma}\nabla u|_{\partial N})\ :\ u\in H^1(N),\ \nabla\cdot\widetilde{\gamma}\nabla u=0\},$$

where ν is the Euclidian unit normal vector of ∂N.

Theorem 2.1. *([21]) The Cauchy data of H^1-solutions for the conductivities $\widetilde{\gamma}$ and γ on N coincide, that is, $C_1(\widetilde{\gamma})=C_1(\gamma)$.*

This means that all boundary measurements for the homogeneous conductivity $\gamma=1$ and the degenerated conductivity $\widetilde{\gamma}$ are the same. In the figure below there are analytically obtained solutions on a disc with metric $\widetilde{\gamma}$

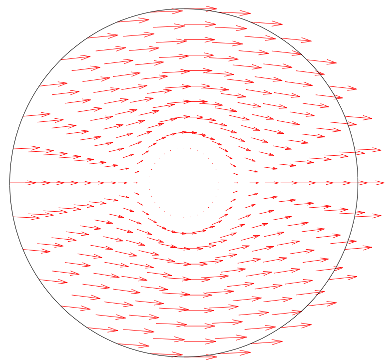

Figure 2. Analytic solutions for the currents

As seen in the figure, no currents appear near the center of the disc, so that if the conductivity is changed near the center, the measurements on the boundary ∂N do not change.

We note that a similar type of theorem is valid also for a more general class of solutions. Consider an unbounded quadratic form, A in $L^2(N)$,

$$A_{\widetilde{\gamma}}[u,v] = \int_N \widetilde{\gamma}\nabla u\cdot \nabla v\, dx$$

defined for $u, v \in \mathcal{D}(A_{\widetilde{\gamma}}) = C_0^\infty(N)$. Let $\overline{A}_{\widetilde{\gamma}}$ be the closure of this quadratic form and say that

$$\nabla\cdot \widetilde{\gamma}\nabla u = 0 \quad \text{in } N$$

is satisfied in the finite energy sense if there is $u_0 \in H^1(N)$ supported in N_1 such that $u - u_0 \in \mathcal{D}(A_{\widetilde{\gamma}})$ and

$$\overline{A}_{\widetilde{\gamma}}[u-u_0, v] = -\int_N \widetilde{\gamma}\nabla u_0\cdot \nabla v\, dx, \quad \text{for all } v \in \mathcal{D}(\overline{A}_{\widetilde{\gamma}}).$$

Then Cauchy data set of the finite energy solutions, denoted

$$C_f(\widetilde{\gamma}) = \{(u|_{\partial N}, \nu\cdot \widetilde{\gamma}\nabla u|_{\partial N}) \ :\ u \text{ is finite energy solution of } \nabla\cdot \widetilde{\gamma}\nabla u = 0\}$$

coincides with $C_f(\gamma)$. Using the above more general class of solutions, one can consider the non-zero frequency case,

$$\nabla \cdot \widetilde{\gamma}\nabla u = \lambda u,$$

and show that the Cauchy data set of the finite energy solutions to the above equation coincides with the corresponding Cauchy data set for γ, cf. [16].

2.2. Maxwell's equations. In what follows, we treat Maxwell's equations in non-conducting media, that is, for which $\gamma = 0$. We consider the electric and magnetic fields, E and H, as differential 1-forms, given in some local coordinates by

$$E = E_j(x)dx^j, \quad H = H_j(x)dx^j.$$

For a 1-form $E(x) = E_1(x)dx^1 + E_2(x)dx^2 + E_3(x)dx^3$ we define the push-forward of E in F, denoted $\widetilde{E} = F_*E$, by

$$\begin{aligned} \widetilde{E}(\widetilde{x}) &= \widetilde{E}_1(\widetilde{x})d\widetilde{x}^1 + \widetilde{E}_2(\widetilde{x})d\widetilde{x}^2 + \widetilde{E}_3(\widetilde{x})d\widetilde{x}^3 \\ &= \sum_{j=1}^{3}\left(\sum_{k=1}^{3}(DF^{-1})_j^k(\widetilde{x})\, E_k(F^{-1}(\widetilde{x}))\right)d\widetilde{x}^j, \quad \widetilde{x} = F(x). \end{aligned}$$

A similar kind of transformation law is valid for 2-forms. We interpret the curl operator for 1-forms in $\mathbb{R}^3$ as being the exterior derivative, d. Maxwell's equations then have the form

$$\operatorname{curl} H = -ikD + J, \quad \operatorname{curl} E = ikB$$

where we consider the D and B fields and the external current J (if present) as 2-forms. The constitutive relations are

$$D = \varepsilon E, \quad B = \mu H,$$

where the material parameters ε and μ are linear maps mapping 1-forms to 2-forms.

Let g be a Riemannian metric in $\Omega \subset \mathbb{R}^3$. Using the metric g, we define a specific permittivity and permeability by setting

$$\varepsilon^{jk} = \mu^{jk} = |g|^{1/2}g^{jk}.$$

To introduce the material parameters $\widetilde{\varepsilon}(x)$ and $\widetilde{\mu}(x)$ that make cloaking possible, we consider the map F given by (2.2), the Euclidian metric g in M_1 and $\widetilde{g} = F_*g$ in N_1 as before, and define the singular permittivity and permeability by the formula analogous to (2.4),

$$\varepsilon^{jk} = \mu^{jk} = \begin{cases} |\widetilde{g}|^{1/2}\widetilde{g}^{jk} & \text{for } x \in N_1, \\ \delta^{jk} & \text{for } x \in N_2. \end{cases} \tag{2.5}$$

These material parameters are singular on Σ, and thus we must take care in defining what it means for the fields $(\widetilde{E}, \widetilde{H})$ to solve Maxwell's equations.

2.2.1. Definition of solutions of Maxwell equations. Since the material parameters $\widetilde{\varepsilon}$ and $\widetilde{\mu}$ are again singular, we need to define solutions carefully.

Definition 2.2. We say that $(\widetilde{E}, \widetilde{H})$ is a finite energy solution to Maxwell's equations on N,

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{on } N, \tag{2.6}$$

if $\widetilde{E}$, $\widetilde{H}$ are one-forms and $\widetilde{D} := \widetilde{\varepsilon}\,\widetilde{E}$ and $\widetilde{B} := \widetilde{\mu}\,\widetilde{H}$ two-forms in N with $L^1(N, dx)$-coefficients satisfying

$$\|\widetilde{E}\|^2_{L^2(N,|\widetilde{g}|^{1/2}dV_0(x))} = \int_N \widetilde{\varepsilon}^{jk}\,\widetilde{E}_j\,\overline{\widetilde{E}_k}\,dV_0(x) < \infty, \tag{2.7}$$

$$\|\widetilde{H}\|^2_{L^2(N,|\widetilde{g}|^{1/2}dV_0(x))} = \int_N \widetilde{\mu}^{jk}\,\widetilde{H}_j\,\overline{\widetilde{H}_k}\,dV_0(x) < \infty; \tag{2.8}$$

where dV_0 is the standard Euclidian volume, $(\widetilde{E}, \widetilde{H})$ is a classical solution of Maxwell's equations on a neighborhood $U \subset \overline{N}$ of ∂N:

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\varepsilon(x)\widetilde{E} + \widetilde{J} \quad \text{in } U,$$

and finally,

$$\int_N ((\nabla \times \widetilde{h}) \cdot \widetilde{E} - ik\widetilde{h} \cdot \widetilde{\mu}(x)\widetilde{H})\,dV_0(x) = 0,$$
$$\int_N ((\nabla \times \widetilde{e}) \cdot \widetilde{H} + \widetilde{e} \cdot (ik\widetilde{\varepsilon}(x)\widetilde{E} - \widetilde{J}))\,dV_0(x) = 0$$

for all $\widetilde{e}, \widetilde{h} \in C_0^\infty(\Omega^1 N)$.

Here, $C_0^\infty(\Omega^1 N)$ denotes smooth 1-forms on N whose supports do not intersect ∂N, and the inner product "$\cdot$" denotes the Euclidian inner product.

Surprisingly, the finite energy solutions do not exists for generic currents. To consider this, let M be the disjoint union of a ball $M_1 = B(0, 2)$ and a ball $M_2 = B(0, 1)$. These will correspond to sets N, N_1, N_2 after an appropriate changes of coordinates. We thus consider a map $F : M \setminus \{0\} = (M_1 \setminus \{0\}) \cup M_2 \to N \setminus \Sigma$, where F mapping $M_1 \setminus \{0\}$ to N_1 is the map defined by formula (2.2) and F mapping M_2 to N_2 is the identity map.

Theorem 2.3. *([16]) Let E and H be 1-forms with measurable coefficients on $M \setminus \{0\}$ and $\widetilde{E}$ and $\widetilde{H}$ be 1-forms with measurable coefficients on $N \setminus \Sigma$ such that $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$. Let J and $\widetilde{J}$ be 2-forms with smooth coefficients on $M \setminus \{0\}$ and $N \setminus \Sigma$, that are supported away from $\{0\}$ and Σ.*

Then the following are equivalent:

1. *The 1-forms $\widetilde{E}$ and $\widetilde{H}$ on N satisfy Maxwell's equations*

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{on } N, \tag{2.9}$$
$$\nu \times \widetilde{E}|_{\partial N} = f$$

in the sense of Definition 2.2.

2. *The forms E and H satisfy Maxwell's equations on M,*

$$\nabla \times E = ik\mu(x)H, \quad \nabla \times H = -ik\varepsilon(x)E + J \quad \text{on } M_1, \tag{2.10}$$
$$\nu \times E|_{\partial M_1} = f$$

and

$$\nabla \times E = ik\mu(x)H, \quad \nabla \times H = -ik\varepsilon(x)E + J \quad \text{on } M_2 \tag{2.11}$$

with Cauchy data

$$\nu \times E|_{\partial M_2} = b^e, \quad \nu \times H|_{\partial M_2} = b^h \tag{2.12}$$

that satisfies $b^e = b^h = 0$.

Moreover, if E *and* H *solve (2.10), (2.11), and (2.12) with non-zero* b^e *or* b^h*, then the fields* $\widetilde{E}$ *and* $\widetilde{H}$ *are not solutions of Maxwell equations on* N *in the sense of Definition 2.2.*

The above theorem can be interpreted by saying that the cloaking of active objects is difficult, as the idealized model with non-zero currents present within the region to be cloaked, leads to non-existence of finite energy distributional solutions. We find two ways of dealing with this difficulty. One is to simply augment the above coating construction around a ball by adding a perfect electrical conductor (PEC) lining at Σ, so that $\nu \times \widetilde{E} = 0$ at the inner surface of Σ, i.e., when approaching Σ from N_2. Physically, this corresponds to a surface current J along Σ which shields the interior of N_2 and make the object inside the coating material appear like a passive object. Other boundary conditions making the problem solvable in some sense, using a different definition based on self-adjoint extensions of the operators, have been recently characterized in [75]. Alternatively to considering a boundary condition on Σ, one can introduce a more elaborate construction, which we refer to as the *double coating.* Mathematically, this corresponds to a singular Riemannian metric which degenerates in the same way as one approaches Σ from both sides; physically it would correspond to surrounding both the inner and outer surfaces of Σ with appropriately matched metamaterials.

2.3. Cloaking an infinite cylindrical domain. In the following we consider a different geometric situation, and redefine the meaning of the notation.

We consider next an infinite cylindrical domain. Below, $B_2(0,r) \subset \mathbb{R}^2$ is Euclidian disc with center 0 and radius r. Let us use in the following the notations $N = B_2(0,2) \times \mathbb{R}$, $N_1 = (B_2(0,2) \setminus B_2(0,1)) \times \mathbb{R}$, and $N_2 = B_2(0,1) \times \mathbb{R}$. Moreover, let M be the disjoint union of $M_1 = B_2(0,2) \times \mathbb{R}$ and $M_2 = B_2(0,1) \times \mathbb{R}$. Finally, let us denote in this section $\Sigma = \partial B_2(0,1) \times \mathbb{R}$, $L = \{(0,0)\} \times \mathbb{R} \subset M_1$. We define the map $F : M \setminus L \to N \setminus \Sigma$ in cylindrical coordinates by

$$F(r,\theta,z) = (1 + \frac{r}{2}, \theta, z), \quad \text{on } M_1 \setminus L,$$
$$F(r,\theta,z) = (r,\theta,z), \quad \text{on } M_2.$$

Again, let g be the Euclidian metric on M, and $\varepsilon = 1$ and $\mu = 1$ be homogeneous material parameters in M. Using map F we define $\widetilde{g} = F_* g$ in $N \setminus \Sigma$ and define $\widetilde{\varepsilon}$ and $\widetilde{\mu}$ as in formula (2.5). By finite energy solutions of Maxwell's equations on N we will mean one-forms $\widetilde{E}$ and $\widetilde{H}$ satisfying the conditions analogous to Definition 2.2.

Theorem 2.4. *([16]) Let E and H be 1-forms with measurable coefficients on $M \setminus L$ and $\widetilde{E}$ and $\widetilde{H}$ be 1-forms with measurable coefficients on $N \setminus \Sigma$ such that $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$. Let J and $\widetilde{J}$ be 2-forms with smooth coefficients on $M \setminus L$ and $N \setminus \Sigma$, that are supported away from L and Σ, respectively, $J = F^*\widetilde{J}$.*

Then the following are equivalent:

1. *On N, the 1-forms $\widetilde{E}$ and $\widetilde{H}$ satisfy Maxwell's equations*

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \textit{in } N, \qquad (2.13)$$
$$\nu \times \widetilde{E}|_{\partial N} = f$$

   and $\widetilde{E}$ and $\widetilde{H}$ are finite energy solutions.

2. *On M, the forms E and H are classical solutions to Maxwell's equations on M, with data*

$$b_1^e = \zeta\cdot E|_L, \quad b_2^e = \zeta\cdot E|_{\Sigma_-}, \quad b_1^h = \zeta\cdot H|_L, \quad b_2^h = \zeta\cdot H|_{\Sigma_-}, \qquad (2.14)$$

   that satisfy

$$b_1^e(z) = b_2^e(r,\theta,z,)|_{r=1}, \quad b_1^h(z) = b_2^h(r,\theta,z)|_{r=1}, \quad and dt_2^e = t_h^e = 0. \quad z \in \mathbb{R}(2.15)$$

   Here, $\zeta = \partial_\theta$ is the angular vector field tangential to Σ.

Moreover, if E and H solve Maxwell's equations on M with the boundary values (2.14) that do not satisfy (2.15), then then the fields $\widetilde{E}$ and $\widetilde{H}$ are not finite energy solutions of Maxwell equations on N.

Further analysis and numerical simulations, exploring the consequences of this non-existence result for cloaking, can be found in [18].

2.4. Cloaking a cylinder with the Soft-and-Hard boundary condition. Next, we consider N_2 as an obstacle, while the domain N_1 is equipped with a metric corresponding to the above coating in the cylindrical geometry. Motivated by the conditions at Σ in the previous section, we impose the soft-and-hard surface (SHS) boundary condition on the boundary of the obstacle. In classical terms, the SHS condition on a surface Σ [23, 34] is

$$\zeta \cdot E|_\Sigma = 0 \quad \text{and} \quad \zeta \cdot H|_\Sigma = 0,$$

where $\zeta = \zeta(x)$ is a tangential vector field on Σ, that is, $\zeta \times \nu = 0$. In other words, the part of the tangential component of the electric field E that is parallel to ζ vanishes, and the same is true for the magnetic field H. This was originally introduced in antenna design and can be physically realized by having a surface with thin parallel gratings filled with dielectric material [34, 35, 47, 23]. Here, we consider this boundary condition when ζ is the vector field $\eta = \partial_\theta$, that is, the angular vector field that is tangential to Σ.

To this end, let us give still one more definition of weak solutions, appropriate for this construction. We consider only solutions on the set N_1; nevertheless, we continue to denote $\partial N = \partial N_1 \setminus \Sigma$.

Definition 2.5. We say that the 1-forms $\widetilde{E}$ and $\widetilde{H}$ are *finite energy solutions* of Maxwell's equations on N_1 with the soft-and-hard (SH) boundary conditions on Σ,

$$\nabla\times\widetilde{E}=ik\widetilde{\mu}(x)\widetilde{H},\quad \nabla\times\widetilde{H}=-ik\widetilde{\varepsilon}(x)\widetilde{E}+\widetilde{J}\quad \text{on } N_1, \tag{2.16}$$
$$\eta\cdot\widetilde{E}|_{\Sigma}=0,\quad \eta\cdot\widetilde{H}|_{\Sigma}=0, \tag{2.17}$$
$$\nu\times\widetilde{E}|_{\partial N}=f,$$

if $\widetilde{E}$ and $\widetilde{H}$ are 1-forms on N_1 and $\widetilde{\varepsilon}\widetilde{E}$ and $\widetilde{\mu}\widetilde{H}$ are 2-forms with measurable coefficients satisfying

$$\|\widetilde{E}\|^2_{L^2(N_1,|\widetilde{g}|^{1/2}dV_0)}=\int_{N_1}\widetilde{\varepsilon}^{jk}\,\widetilde{E}_j\,\overline{\widetilde{E}_k}\,dV_0(x)<\infty, \tag{2.18}$$
$$\|\widetilde{H}\|^2_{L^2(N_1,|\widetilde{g}|^{1/2}dV_0)}=\int_{N_1}\widetilde{\mu}^{jk}\,\widetilde{H}_j\,\overline{\widetilde{H}_k}\,dV_0(x)<\infty; \tag{2.19}$$

Maxwell's equation are valid in the classical sense in a neighborhood U of ∂N:

$$\nabla\times\widetilde{E}=ik\widetilde{\mu}(x)\widetilde{H},\quad \nabla\times\widetilde{H}=-ik\varepsilon(x)\widetilde{E}+\widetilde{J}\quad \text{in } U,$$
$$\nu\times\widetilde{E}|_{\partial N}=f;$$

and finally,

$$\int_{N_1}((\nabla\times\widetilde{h})\cdot\widetilde{E}-ik\widetilde{h}\cdot\widetilde{\mu}(x)\widetilde{H})\,dV_0(x)=0,$$
$$\int_{N}((\nabla\times\widetilde{e})\cdot\widetilde{H}+\widetilde{e}\cdot(ik\widetilde{\varepsilon}(x)\widetilde{E}-\widetilde{J}))\,dV_0(x)=0,$$

for all $\widetilde{e},\widetilde{h}\in C_0^\infty(\Omega^1 N_1)$ satisfying

$$\eta\cdot\widetilde{e}|_{\Sigma}=0,\quad \eta\cdot\widetilde{h}|_{\Sigma}=0. \tag{2.20}$$

We then have the following invisibility result.

Theorem 2.6. *([16]) Let E and H be 1-forms with measurable coefficients on $M_1\setminus L$ and $\widetilde{E}$ and $\widetilde{H}$ be 1-forms with measurable coefficients on N_1 such that $E=F^*\widetilde{E}$, $H=F^*\widetilde{H}$. Let J and $\widetilde{J}$ be 2-forms with smooth coefficients on $M_1\setminus L$ and $N_1\setminus\Sigma$, that are supported away from L and Σ respectively, $J=F^*\widetilde{J}$..*

Then the following are equivalent:

1. *On N_1, the 1-forms $\widetilde{E}$ and $\widetilde{H}$ satisfy Maxwell's equations with SH boundary conditions in the sense of Definition 2.5.*

2. *On M_1, the forms E and H are classical solutions of Maxwell's equations,*

$$\nabla\times E=ik\mu(x)H,\quad \text{in } M_1 \tag{2.21}$$
$$\nabla\times H=-ik\varepsilon(x)E+J,\quad \text{in } M_1,$$
$$\nu\times E|_{\partial M_1}=f.$$

This result implies that when the surface Σ is lined with a material implementing the SHS boundary condition, the finite energy distributional solutions exist for all incoming waves.

2.5. Electromagnetic wormholes. Cloaking a ball or cylinder are particularly extreme examples of what has come to be known as *transformation optics* in the physics literature, and other interesting effects are possible. We sketch the construction of artificial electromagnetic wormholes, introduced in [17]. Consider first as in Fig. 3 a 3-dimensional wormhole manifold (or handlebody) $M = M_1 \# M_2$ where the components

$$M_1 = \mathbb{R}^3 \setminus (B(O,1) \cup B(P,1)),$$
$$M_2 = \mathbb{S}^2 \times [0,1]$$

are glued together smoothly.

An optical device that acts as a wormhole for electromagnetic waves at a given frequency k can be constructed by starting with a two-dimensional finite cylinder

$$T = \mathbb{S}^1 \times [0,L] \subset \mathbb{R}^3$$

and taking its neighborhood $K = \{x \in \mathbb{R}^3 : \ \mathrm{dist}(x,T) < \rho\}$, where $\rho > 0$ is small enough and $N = \mathbb{R}^3 \setminus K$. Let us put on ∂K the SHS boundary condition

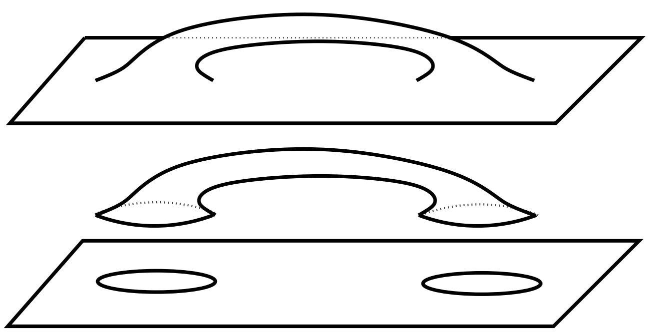

Figure 3. A schematic figure of two dimensional wormhole construction by gluing surfaces. Note that in the artificial wormhole construction components are three dimensional.

and cover K with "invisibility cloaking material", that in the boundary normal coordinates around K has the same representation as $\widetilde{\varepsilon}$ and $\widetilde{\mu}$ when cloaking an infinite cylinder. Finally, let

$$U = \{x : \ \mathrm{dist}(x,K) > 1\} \subset \mathbb{R}^3 .$$

The set U can be considered both as a subset of $N \subset \mathbb{R}^3$ and the wormhole manifold M, $U \subset M_1$. Then all measurements of fields E and H in $U \subset M$ and $U \subset N$

coincide with currents that are supported in U, that is, thus $(N, \widetilde{\varepsilon}, \widetilde{\mu})$ behaves as the wormhole M in all external measurements.

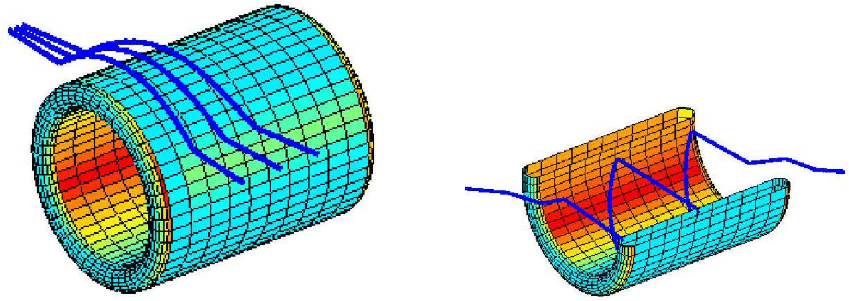

Fig. 4 (a) Rays travelling outside. (b) A ray travelling inside.

In Fig. 4, we give ray-tracing simulations in and near the wormhole. The obstacle in the figures is K, and the metamaterial corresponding to $\widetilde{\varepsilon}$ and $\widetilde{\mu}$ is not shown.

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