COMPLEX GEOMETRICAL OPTICS SOLUTIONS SOLUTIONS FOR LIPSCHITZ CONDUCTIVITIES

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ABSTRACT.. We prove the existence of complex geometrical optics solutions for Lipschitz conductivities. Moreover we show that, in dimensions $n \geq 3$ that one can uniquely recover a $W^{3/2,\infty}$ conductivity from its associated Dirichlet-to-Neumann map or voltage to current map.

1. INTRODUCTION.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $\gamma \in L^\infty(\Omega)$ be the electrical conductivity of Ω. We assume throughout the paper that the conductivity is strictly positive on $\overline{\Omega}$. Given a voltage potential f on $\partial\Omega$, under the assumption of no sources or sinks of current on Ω, the induced potential u on Ω satisfies the Dirichlet problem

$$\begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0 \quad \text{on} \quad \Omega \\ u_{|\partial\Omega} &= f. \end{aligned} \tag{1.1}$$

The Dirichlet-to-Neumann map (DN), or voltage to current map, is defined by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega}, \tag{1.2}$$

where u is a solution to (1.1) and ν denotes the unit outer normal to $\partial\Omega$.

The inverse problem of determining γ from Λ_γ has been extensively studied since Calderón's pioneer paper [4]. The essential tool for the

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solution is the construction of complex geometrical optics solutions for the underlying conductivity equation. See [16] for a recent survey.

Kohn and Vogelius proved that one can uniquely determine piecewise real-analytic conductivities from the DN map [6]. Sylvester and Uhlmann proved in dimension $n \geq 3$ a global identifiability result for smooth conductivities [12]. Brown extended this result to conductivities in $C^{3/2+\varepsilon}(\overline{\Omega})$ for any $\varepsilon > 0$, [2]. In two dimensions Nachman proved a global identifiability result for conductivities having two derivatives [8]. This was improved to Lipschitz conductivities in [3]. We point out that the above inverse conductivity problem makes sense for conductivities that are only in L^∞. There are neither proofs nor counter-examples for this case in any dimension.

As mentioned above a crucial ingredient in the proof of the results in [2], [3], [8], and [12] is the construction of complex geometrical optics solutions for the equation (1.1), [12, 13].

Let $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$. Moreover, let $\gamma \in C^2(\mathbb{R}^n)$, $n \geq 2$, with $\gamma = 1$ for $|x| \geq R$ and R sufficiently large. Then for $|\rho|$ sufficiently large there exist solutions of $\operatorname{div}(\gamma \nabla u) = 0$ in $\mathbb{R}^n$ of the form

$$u = e^{x \cdot \rho} \gamma^{-1/2} \left(1 + \psi_\gamma(x, \rho)\right). \tag{1.3}$$

In two dimensions these solutions are constructed for all $\rho \in \mathbb{C}^2 \setminus \{0\}$ with $\rho \cdot \rho = 0$, [8].

The function ψ_γ satisfies

$$\left(\Delta + 2\rho \cdot \nabla\right) \psi_\gamma = q(1 + \psi_\gamma), \tag{1.4}$$

where

$$q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}. \tag{1.5}$$

An important property of ψ_γ is that $\psi_\gamma \overset{|\rho| \to \infty}{\longrightarrow} 0$ in an appropriate sense so that the solutions of (1.4) behave like $e^{x \cdot \rho} \gamma^{-1/2}$ for large $|\rho|$.

In this paper we construct complex geometrical optics solutions for conductivities $\gamma \in W^{1,\infty}(\mathbb{R}^n)$. We sketch some of the steps in the construction. We first rewrite (1.1) in the form

$$\Delta u + A \cdot \nabla u = 0, \tag{1.6}$$

where $A = \nabla \log \gamma \in L^\infty(\mathbb{R}^n)$ has compact support. Let $A_\varepsilon = A * \Phi_\varepsilon$ where $\Phi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\varepsilon > 0$, is an approximation to the δ-function.

We consider instead of (1.6) the Schrödinger equation with a smooth first order term

$$\Delta u_\varepsilon + A_\varepsilon \cdot \nabla u_\varepsilon = 0. \tag{1.7}$$

In [9] complex geometrical optics solutions to (1.7) are constructed of the form

$$u_\varepsilon = e^{x\cdot\rho} e^{-\frac{\varphi_\varepsilon(x)}{2}} r_\varepsilon(x,\rho),$$

with r_ε satisfying appropriate decay conditions as $|\rho| \to \infty$. The function φ_ε are defined by

$$\varphi_\varepsilon = \varphi * \Phi_\varepsilon. \tag{1.8}$$

We remark that in the case that $A_\varepsilon = \nabla \log \alpha$ with α smooth, then $e^{-\frac{\varphi_\varepsilon(x)}{2}} = \alpha^{-1/2}$. Our complex geometrical optics solutions are of the form

$$u = e^{x\cdot\rho}(\omega_0(x,\varepsilon) + \omega_1(x,\varepsilon,\rho)), \tag{1.9}$$

with

$$\omega_0(x,\varepsilon) = e^{-\frac{\varphi_\varepsilon(x)}{2}}. \tag{1.10}$$

In Section 2 we see that $\omega_0 \to \gamma^{-1/2}$ as $\varepsilon \to 0$, $|\rho| \to \infty$, in appropriate sense. Since ω_0 is smooth we have avoided the problem of taking two derivatives of γ.

Now ω_1 solves the equation

$$((\Delta + 2\rho\cdot\nabla) + A\cdot(\nabla+\rho))\omega_1 = -((\Delta + 2\rho\cdot\nabla) + A\cdot(\nabla+\rho))\omega_0. \tag{1.11}$$

In this paper we show the existence of the complex geometrical solutions for all Lipschitz conductivities and give a global identifiability result for $W^{3/2,\infty}$ conductivities in dimension three and higher. This latter result improves on the result of [2]. More precisely, we will prove

Theorem 1.1. *Let $\gamma \in W^{1,\infty}(\mathbb{R}^n)$, γ strictly positive and $\gamma = 1$ outside a large ball. Let $-1 < \delta < 0$. Then for $|\rho|$ sufficiently large there is a unique solution of*

$$\mathrm{div}(\gamma\nabla \mathrm{u}) = 0 \quad in \quad \mathbb{R}^{\mathrm{n}}$$

of the form

$$u = e^{x\cdot\rho}(\gamma^{-1/2} + \psi_\gamma(x,\rho)) \tag{1.12}$$

with $\psi_\gamma \in L^2_\delta(\mathbb{R}^n)$.

Moreover, ψ_γ has the form

$$\psi_\gamma(x,\rho) = (\omega_0(x,\rho) - \gamma^{-1/2}) + \omega_1(x,\rho) \tag{1.13}$$

where ω_0 and ω_1 satisfy

$$\lim_{|\rho|\to\infty} \|\omega_0(x,\rho) - \gamma^{-1/2}\|_{H^1_\delta} = 0, \tag{1.14}$$

and

$$\lim_{|\rho|\to\infty} \|\omega_1(x,\rho)\|_{L^2_\delta} = 0. \tag{1.15}$$

Theorem 1.2. *Let $n \geq 3$. Let $\gamma_i \in W^{3/2,\infty}(\Omega)$, be strictly positive on $\overline{\Omega}$, $i = 1,2$. Assume $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then $\gamma_1 = \gamma_2$ on Ω.*

Note that if one could replace L^2_δ–norm in (1.15) with H^1_δ–norm then Theorem 1.2 would easily follow for any $W^{1,\infty}(\Omega)$ conductivity.

In Section 2 we construct solutions of Theorem 1.1. The key point of the proof is the observation that the operator on the left-hand side of (1.11) has an explicit approximative inverse that involves only first order derivatives of γ (c.f. Theorem 2.1 below).

In Section 3 we prove Theorem 1.2 , using a new identity that is satisfied if the DN maps of two conductivities are the same and plugging the solutions (1.9) into this identity. The standard identity used in previous result doesn't seem to give the desired result.

As for the case of $W^{2,\infty}(\Omega)$ conductivities, we expect that the methods of this paper will lead to stability estimates and a reconstruction method. Several of the results of this paper generalize to $L^\infty(\Omega)$ first order perturbations of the Laplacian. The case of $C^{2/3+\varepsilon}(\overline{\Omega})$, $\varepsilon > 0$, first order perturbations of the Laplacian was considered in [15].

2. Construction of the complex geometrical optics solutions.

In this section we construct ω_0 and ω_1 as in Theorem 1.2 and we derive their asymptotic properties as $|\rho| \to \infty$.

We denote by $\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$, an approximation to the δ-function. More specifically we assume

$$\Phi(x) = \prod_{j=1}^{n} \Phi_0(x_j)$$

with $\Phi_0 \in C_0^\infty(\mathbb{R})$, $\Phi_0 \geq 0$, $\Phi_0 = 1$ near 0 and $\int \Phi_0(x)dx = 1$.

We define the weighted L^2-space, L^2_α, by

$$L^2_\alpha(\mathbb{R}^n) = \left\{ f : \int (1+|x|^2)^{\alpha/2} |f(x)|^2 dx < \infty \right\},$$

with norm given by

$$\|f\|^2_{L^2_\alpha} = \int (1+|x|^2)^\alpha |f(x)|^2 dx.$$

We denote by $H^s_\alpha(\mathbb{R}^n)$ the corresponding Sobolev space defined by interpolation for non integer s.

Throughout this section we assume that $\gamma \in W^{1,\infty}(\mathbb{R}^n)$, with $\gamma = 1$ outside a large ball. Let

$$\varphi_\varepsilon = \Phi_\varepsilon * \log \gamma, \tag{2.1}$$

$$A = \nabla \log \gamma \tag{2.2}$$

and

$$A_\varepsilon = \Phi_\varepsilon * A. \tag{2.3}$$

Clearly

$$\rho \cdot \nabla \varphi_\varepsilon = \rho \cdot A_\varepsilon. \tag{2.4}$$

Next we define

$$\omega_0(x, \varepsilon, \theta) = e^{-\frac{\varphi_\varepsilon(x,\theta)}{2}}. \tag{2.5}$$

The proof of the following proposition is standard

Proposition 2.1. *If $A \in W^{1/2,\infty}(\Omega)$ we have as $\varepsilon \to 0$*

$$||A_\varepsilon - A||_{L^2} = \varepsilon^{1/2} o(\varepsilon)$$

and

$$||\partial_{x_j} A_\varepsilon||_{L^2} = \varepsilon^{-1/2} o(\varepsilon)$$

From the above proposition we conclude

Lemma 2.1. *Let ω_0 be as in (2.5). Then for any $\delta > 0$*

$$\lim_{\varepsilon \to 0} ||\omega_0 - \gamma^{-1/2}||_{H^1_\delta} = 0$$

.

Let $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$. We define the operators

$$\Delta_\rho u = e^{-x\cdot\rho} \Delta(e^{x\cdot\rho} u) = \Delta u + 2\rho \cdot \nabla u, \tag{2.6}$$

$$\nabla_\rho v = e^{-x\cdot\rho} \nabla(e^{x\cdot\rho} u) = \nabla v + \rho \cdot v. \tag{2.7}$$

Let $f \in C_0^\infty(\mathbb{R}^n)$. We define

$$\Delta_\rho^{-1} f = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \frac{\hat{f}(\xi)}{-|\xi|^2 + 2i\rho \cdot \xi} d\xi \tag{2.8}$$

and recall

Proposition 2.2. *([12, 13]). Let $n \geq 2$. Let $-1 < \delta < 0$ and $s \geq 0$. Then*

$$\Delta_\rho^{-1} \quad \textit{extends to a bounded operator}$$
$$\Delta_\rho^{-1} : H^s_{\delta+1}(\mathbb{R}^n) \to H^s_\delta(\mathbb{R}^n)$$

with

$$\|\Delta_\rho^{-1}\|_{H^s_{\delta+1}\to H^s_\delta} \leq \frac{C(s,\delta,n)}{|\rho|}, \tag{2.9}$$

for some $C > 0$. Moreover

$$\Delta_\rho^{-1} : L^2_{\delta+1}(\mathbb{R}^n) \to H^k_\delta(\mathbb{R}^n), \quad k = 1,2$$

is bounded and

$$\|\Delta_\rho^{-1}\|_{L^2_{\delta+1}\to H^k_\delta} \leq C(\delta,k,n)|\rho|^{k-1}, \quad k = 1,2, \tag{2.10}$$

for some $C > 0$.

Next we construct ω_1. To have a solution of the form (1.9), ω_1 must satisfy

$$(\Delta_\rho + A\cdot\nabla_\rho)\omega_1 = -(\Delta_\rho + A\cdot\nabla_\rho)\omega_0 =: -f_\varepsilon. \tag{2.11}$$

We write

$$\omega_1 = \Delta_\rho^{-1}\tilde{\omega},$$

then $\tilde{\omega}$ must satisfy

$$(I + A\cdot\nabla_\rho\Delta_\rho^{-1})\tilde{\omega} = -f_\varepsilon. \tag{2.12}$$

A straightforward calculation gives that

$$f_\varepsilon = -e^{-\frac{\varphi_\varepsilon}{2}}\left(-\frac{1}{2}\Delta\varphi_\varepsilon + \frac{1}{4}(\nabla\varphi_\varepsilon)^2 - \frac{1}{2}A\cdot\nabla\varphi_\varepsilon - \rho\cdot\nabla\varphi_\varepsilon + \rho\cdot A\right). \tag{2.13}$$

Since $\rho\cdot\nabla\varphi_\varepsilon = \rho\cdot A_\varepsilon$, we get

$$f_\varepsilon = -e^{-\frac{\varphi_\varepsilon}{2}}\left(-\frac{1}{2}\Delta\varphi_\varepsilon + \frac{1}{4}(\nabla\varphi_\varepsilon)^2 - \frac{1}{2}A\cdot\nabla\varphi_\varepsilon + (A - A_\varepsilon)\cdot\rho\right). \tag{2.14}$$

In order to solve (2.12) we define the operators $T_\rho(\gamma)$ and $S_\rho(\gamma)$ by

$$T_\rho(\gamma) := (I + A\cdot\nabla_\rho\Delta_\rho^{-1}) \tag{2.15}$$

and

$$S_\rho(\gamma) := \gamma^{-1/2}(I - A\cdot\nabla_\rho\Delta_\rho^{-1})\gamma^{1/2}. \tag{2.16}$$

Note that

$$S_\rho(\gamma) = \gamma^{-1/2}T_\rho(\gamma^{-1})\gamma^{1/2}. \tag{2.17}$$

The main result of this section is that $T_\rho(\gamma)$ $S_\rho(\gamma)$ are approximate inverses to each other. This result is used to prove that (2.12) has a unique solution in an appropriate space.

Theorem 2.1. *Let $-1 < \delta < 0$ and $\gamma \in W^{1,\infty}(\mathbb{R}^n)$ with γ strictly positive and $\gamma = 1$ outside a large ball. Then the operator*

$$T_\rho(\gamma) : L^2_{\delta+1}(\mathbb{R}^n) \to L^2_{\delta+1}(\mathbb{R}^n), \tag{2.18}$$

is bounded and it has a bounded inverse T_ρ^{-1} for $|\rho|$ large. Moreover

$$\|T_\rho^{-1}(\gamma) - S_\rho(\gamma)\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \to 0 \tag{2.19}$$

as $|\rho| \to \infty$.

We first prove a stronger version of (2.19) for C^2–conductivities. More precisely we show

Lemma 2.2. *Let $\gamma \in C^2(\mathbb{R}^2)$, $\gamma = 1$ outside a large ball and strictly positive. Then for $|\rho|$ large we have the estimate*

$$\|T_\rho^{-1}(\gamma) - S_\rho(\gamma)\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \le \frac{C\|\gamma\|_{C^2}}{|\rho|}, \tag{2.20}$$

for some $C > 0$.

Proof. We have the following identity which was implicitly used in [12, 13] to reduce the construction complex geometrical optics solutions for the conductivity equation to the construction of complex geometrical optics solutions for the Schrödinger equation. The identity was explicitly stated in [5] and used to construct complex geometrical optics solutions for Maxwell's equations.

$$(\Delta_\rho + A\cdot\nabla_\rho)\gamma^{-1/2} = \gamma^{-1/2}(\Delta_\rho - q), \tag{2.21}$$

with $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$. Then

$$(\Delta_\rho + A\cdot\nabla_\rho)\Delta_\rho^{-1} = \gamma^{-1/2}(\Delta_\rho - q)\gamma^{1/2}\Delta_\rho^{-1}$$

and therefore

$$I + (A\cdot\nabla_\rho)\Delta_\rho^{-1} = \gamma^{-1/2}\Delta_\rho\gamma^{1/2}\Delta_\rho^{-1} - q\Delta_\rho^{-1}. \tag{2.22}$$

By Proposition 2.2

$$\|q\Delta_\rho^{-1}\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \le \frac{C\|q\|_\infty}{|\rho|},$$

for some $C > 0$. Therefore

$$\left(I + (A\cdot\nabla_\rho)\Delta_\rho^{-1}\right)^{-1} = \left(\gamma^{-1/2}\Delta_\rho\gamma^{1/2}\Delta_\rho^{-1}\right)^{-1} + B_\rho,$$

for $|\rho|$ large with

$$\|B_\rho\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \leq \frac{C\|q\|_\infty}{|\rho|}.$$

Since

$$\left(\gamma^{-1/2}\Delta_\rho\gamma^{1/2}\Delta_\rho^{-1}\right)^{-1} = \Delta_\rho\gamma^{-1/2}\Delta_\rho^{-1}\gamma^{1/2},$$

we conclude that

(2.23)

$$\|\left(I+(A\cdot\nabla_\rho)\Delta_\rho^{-1}\right)^{-1} - \Delta_\rho\gamma^{-1/2}\Delta_\rho^{-1}\gamma^{1/2}\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \leq \frac{C\|\gamma\|_{C^2}}{|\rho|}.$$

Now we use formula (2.22) again by changing γ to $\frac{1}{\gamma}$, to conclude

$$\begin{aligned}&\Delta_\rho\gamma^{-1/2}\Delta_\rho^{-1}\gamma^{1/2}\\ &= \gamma^{1/2}\left(I-(A\cdot\nabla_\rho)\Delta_\rho^{-1}\right)\gamma^{-1/2} + \gamma^{1/2}\Delta(\gamma^{-1/2})\Delta_\rho^{-1}.\end{aligned} \tag{2.24}$$

From (2.23) and (2.24) we conclude the proof of the Lemma since by Proposition 2.2 again

$$\|\gamma^{1/2}\Delta(\gamma^{-1/2})\Delta_\rho^{-1}\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \leq \frac{C\|\gamma\|_{C^2}}{|\rho|}.$$

$\Diamond$

Lemma 2.3. *Let $\gamma\in W^{1,\infty}$, γ strictly positive and $\gamma=1$ outside a large ball. Let $\gamma_\varepsilon=\gamma*\Phi_\varepsilon$. Then*

$$\|T_\rho(\gamma_\varepsilon)-T_\rho(\gamma)\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \xrightarrow{\varepsilon\to 0} 0. \tag{2.25}$$

Proof. We have that

$$T_\rho(\gamma_\varepsilon)-T_\rho(\gamma) = I + (\nabla\log\gamma_\varepsilon - \nabla\log\gamma)\cdot\nabla_\rho\Delta_\rho^{-1}.$$

Using Proposition 2.2 we conclude that

$$\|\nabla_\rho\Delta_\rho^{-1}\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \leq C,$$

for some $C>0$ independent of $|\rho|$. Therefore

$$\|T_\rho(\gamma_\varepsilon)-T_\rho(\gamma)\|_{L^2_{\delta}\to L^2_{\delta+1}} \leq C\|\nabla\log\gamma_\varepsilon-\nabla\log\gamma\|_{L^\infty(\mathbb{R}^n)}$$

proving the Lemma. $\Diamond$

Theorem 2.1 follows directly from Lemma 2.2 and Lemma 2.3 by standard approximation procedure. Namely, for $\gamma\in W^{1,\infty}(\mathbb{R}^n)$ as before we have

$$\begin{aligned}S_\rho T_\rho - I = \; &(S_\rho(\gamma_\varepsilon)T_\rho(\gamma_\varepsilon)-I)\\ &-(S_\rho(\gamma_\varepsilon)-S_\rho(\gamma))T_\rho(\gamma_\varepsilon) - S_\rho(\gamma)(T_\rho(\gamma_\varepsilon)-T_\rho(\gamma)).\end{aligned} \tag{2.26}$$

Since $||\gamma_\varepsilon||_{C^2} = O(\varepsilon^{-2})$ we obtain from Lemma 2.2

$$\|S_\rho(\gamma_\varepsilon)T_\rho(\gamma_\varepsilon) - I\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \le \frac{C||\gamma_\varepsilon||_{C^2}}{|\rho|} = O\left(\frac{1}{|\rho|\varepsilon^2}\right). \tag{2.27}$$

Now by Proposition 2.2

$$\|S_\rho(\tilde\gamma)\|_{L^2_{\delta+1}\to L^2_{\delta+1}} + \|T_\rho(\tilde\gamma)\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \le C\|\nabla \log\tilde\gamma\|_{L^\infty(\mathbb{R}^n)}$$

holds for any conductivity $\tilde\gamma \in W^{1,\infty}(\mathbb{R}^n)$. Therefore by taking $\varepsilon = |\rho|^{-1/4}$ and $|\rho|$ large we obtain from Lemma 2.3 and (2.27) the desired conclusion (2.19). $\Diamond$

Next we study the behavior of ω_1 as $\varepsilon \to 0$ and$|\rho| \to \infty$. By Theorem 2.1 and(2.12) we can define

$$\omega_1(x,\varepsilon,\rho) = -\Delta_\rho^{-1}S_\rho f_\varepsilon + h_\rho \tag{2.28}$$

where

$$h_\rho = \Delta_\rho^{-1}C_\rho f_\varepsilon \tag{2.29}$$

with $C_\rho = T_\rho^{-1} - S_\rho$.

Lemma 2.4.

$$\lim_{|\rho|\to\infty} \|h_\rho\|_{H^1_\delta(\mathbb{R}^n)} = 0$$

Proof. By Theorem 2.1 we have that

$$\|T_\rho^{-1}\|_{L^2_{\delta+1}\to L^2_{\delta+1}} \le C$$

with C independent of ρ. By definition

$$T_\rho\Delta_\rho\omega_1 = -f_\varepsilon, \tag{2.30}$$

and

$$\Delta_\rho\omega_1 = S_\rho f_\varepsilon + C_\rho f_\varepsilon.$$

Therefore by Theorem 2.1

$$||\Delta_\rho\omega_1 - S_\rho f_\varepsilon||_{L^2_{\delta+1}\to L^2_{\delta+1}} \to 0 \text{ as } \rho\to\infty.$$

This implies that

$$A := \|\nabla\omega_1 - \Delta_\rho^{-1}\nabla S_\rho f_\varepsilon\|_{L^2_\delta\to L^2_\delta} \to 0 \text{ as } |\rho|\to\infty \tag{2.31}$$

since

$$A = ||\nabla\Delta_\rho^{-1}(\Delta_\rho\omega_1 - S_\rho f_\varepsilon)||_{L^2_\delta\to L^2_\delta} \le C||\Delta_\rho\omega_1 - S_\rho f_\varepsilon||_{L^2_{\delta+1}\to L^2_{\delta+1}}$$

The proof of Theorem1.1: The choice $\omega_0(x,\rho) = \omega_0(x,\varepsilon)$ and

$\omega_1(x,\rho) = \omega_1(x,\varepsilon,\rho)$, with say $\varepsilon = |\rho|^{-\alpha}$ for any $0 < \alpha \leq 1$ will lead to the desired result. Indeed, by Proposition 2.2 and by (2.16)

$$||f_\varepsilon|| \leq C(\varepsilon^{-1} + |\rho|)o(\varepsilon).$$

On the other hand by Lemma 2.3

$$||\omega_1||_{L^2_\delta} \leq ||\Delta_\rho^{-1} S_\rho f_\varepsilon||_{L^2_\delta} + ||\Delta_\rho h_\rho||_{L^2_\delta} \leq \frac{C}{|\rho|}||f_\varepsilon||$$

which proves that

$$||\omega_1||_{L^2_\delta} \overset{|\rho|\to\infty}{\longrightarrow} 0.$$

This together with Lemma 2.1 prove Theorem 1.1. $\Diamond$

3. The uniqueness proof.

In this section we prove Theorem 1.2. We shall use the following identity

Lemma 3.1. *Let $\gamma \in W^{1,\infty}(\Omega)$, γ strictly positive on $\overline{\Omega}$ and $a = \sqrt{\gamma}$. Let $u \in H^1(\Omega)$ be a solution of* $\mathrm{div}(\gamma\nabla \mathrm{u}) = 0$. *Let $v \in H^1(\Omega)$. Then*

$$\int_\Omega (\nabla a \cdot \nabla(uv) - \nabla(au)\cdot\nabla v)dx = -\int_{\partial\Omega} av\frac{\partial u}{\partial \nu}dS, \tag{3.1}$$

where dS denotes surface measure on $\partial\Omega$.

Proof. By the divergence theorem we have

$$\int_\Omega a^{-1}\mathrm{div}(a^2\nabla u)v dx = -\int_\Omega a^2\nabla u\cdot\nabla\left(\frac{v}{a}\right)dx - \int_{\partial\Omega} a\frac{\partial u}{\partial\nu}vdS. \tag{3.2}$$

Therefore

$$\int_\Omega a^2\nabla u\cdot\nabla\left(\frac{v}{a}\right)dx = -\int_{\partial\Omega} a\frac{\partial u}{\partial\nu}vdS. \tag{3.3}$$

Now setting $w = au$ and using that $\nabla a^{-1} = -a^{-2}\nabla a$ we obtain

$$\begin{aligned}
&\int_\Omega a^2\nabla(a^{-1}w)\cdot\nabla(a^{-1}v)dx \\
&= \int_\Omega \left(\nabla w\cdot\nabla v - \nabla a\cdot\left(w\nabla(a^{-1}v) + v\nabla(a^{-1}w)\right)\right)dx \\
&= \int_\Omega \nabla w\cdot\nabla v - \nabla a\cdot\nabla(a^{-1}wv)dx \\
&= \int_\Omega \nabla(au)\cdot\nabla v - \nabla a\cdot\nabla(uv)dx,
\end{aligned}$$

proving the Lemma. $\Diamond$

The right-hand side of (3.1) is determined by the DN map if we know $v_{|\partial\Omega}$. This is a consequence of the following two Lemmas.

Lemma 3.2. [7, 14, 1] *Let $\gamma_i \in W^{1,\infty}(\Omega)$ conductivities $i = 1, 2$ with $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then*

$$\gamma_{1|\partial\Omega} = \gamma_{2|\partial\Omega} \quad \text{and} \quad \left(\frac{\partial\gamma_1}{\partial\nu}\right)\Big|_{\partial\Omega} = \left(\frac{\partial\gamma_2}{\partial\nu}\right)\Big|_{\partial\Omega}.$$

Lemma 3.3. [12] *Assume $\gamma_i \in W^{1,\infty}(\Omega)$ conductivities $i = 1, 2$ with $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Extend $\gamma_i \in W^{1,\infty}(\mathbb{R}^n)$ with $\gamma_1 = \gamma_2$ in Ω^c and $\gamma_i = 1$ outside a large ball. Let u_i, $i = 1, 2$ be the complex geometrical solutions as in Theorem 1.1. Then*

$$u_1 = u_2 \quad \text{in} \quad \Omega^c.$$

We include the proof here for the sake of completeness

Proof. Let $z \in H^1(\Omega)$ be the solution of

$$\operatorname{div}(\gamma_1 \nabla z) = 0 \quad \text{on} \quad \Omega$$
$$z_{|\partial\Omega} = u_{2|\partial\Omega}.$$

By the assumption $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, we have

$$\left(\gamma_1 \frac{\partial z}{\partial\nu}\right)\Big|_{\partial\Omega} = \left(\gamma_2 \frac{\partial u_2}{\partial\nu}\right)\Big|_{\partial\Omega}, \tag{3.4}$$

and therefore, since γ_1 equals γ_2 on the boundary,

$$\left(\frac{\partial z}{\partial\nu}\right)\Big|_{\partial\Omega} = \left(\frac{\partial u_2}{\partial\nu}\right)\Big|_{\partial\Omega}.$$

Then z solves $\operatorname{div}(\gamma_1 \nabla z) = 0$ in $\mathbb{R}^n$ if we extend $z = u_2$ on Ω^c. Since z has the same asymptotic behavior as u_1 we have from the uniqueness of the solutions in Theorem 1.1 that $z = u_1$ and therefore $u_1 = u_2$ in Ω^c. $\Diamond$

Combining Lemmas 3.1–3.3 we arrive at the following

Proposition 3.1. *Let $\gamma_i \in W^{1,\infty}(\Omega)$ conductivities with $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ and let $a_i = \gamma_i^{1/2}$. Then*

$$\begin{aligned}\int_\Omega (\nabla a_1 \cdot \nabla(u_1 v) \; - \; \nabla a_2 \cdot \nabla(u_2 v))\, dx \\ - \int_\Omega (\nabla(a_1 u_1) \cdot \nabla v - \nabla(a_2 u_2) \cdot \nabla v)\, dx = 0,\end{aligned} \tag{3.5}$$

for any $v \in H^1(\Omega)$ and $u_i \in H^1(\Omega)$ solution of $div(\gamma_i \nabla u_i) = 0$ on Ω, $i = 1, 2$.

The next lemma is the only place where the assumption $\gamma \in W^{3/2,\infty}(\Omega)$ is needed.

Lemma 3.4. *Let $\gamma \in W^{3/2,\infty}(\mathbb{R}^n)$, γ strictly positive and equal to one outside a large ball. Then for $\omega_1(x,\rho) = \omega_1(x,\varepsilon,\rho)$, as in (2.28), $\varepsilon = |\rho|^{-1}$ we have*

$$\int e^{ix\cdot\xi}\nabla\gamma^{1/2}\cdot\nabla\omega_1 \; dx \stackrel{|\rho|\to\infty}{\longrightarrow} 0. \tag{3.6}$$

Proof. The statement (3.6) is equivalent to

$$\int e^{ix\cdot\xi}\gamma^{1/2}A\cdot\nabla\omega_1 \; dx \stackrel{|\rho|\to\infty}{\longrightarrow} 0. \tag{3.7}$$

Using (2.12), (2.16) and (2.19), (3.7) follows if we show

$$\int e^{ix\cdot\xi}\gamma^{-1/2}A\cdot\nabla\Delta_\rho^{-1}f_\varepsilon \; dx \stackrel{|\rho|\to\infty}{\longrightarrow} 0. \tag{3.8}$$

and

$$\int e^{ix\cdot\xi}\gamma^{-1/2}A\cdot\nabla\Delta_\rho^{-1}\gamma^{-1/2}A\cdot\nabla_\rho\Delta_\rho^{-1}\gamma^{1/2}f_\varepsilon \; dx \stackrel{|\rho|\to\infty}{\longrightarrow} 0. \tag{3.9}$$

We know from Lemma 2.1 that $e^{-\frac{-\varphi_\varepsilon}{2}} \to \gamma^{1/2}$ in H^1_δ so that it can be easily estimated. The first two terms of f_ε involve $\Delta\varphi_\varepsilon$ and $\nabla\varphi_\varepsilon$. Recall from Proposition 2. that $\nabla\varphi_\varepsilon = A_\varepsilon$ and therefore $\Delta\varphi_\varepsilon = \mathrm{divA}_\varepsilon$. Below we will see that the choice $\varepsilon \simeq |\rho|^{-1}$ is the only possibility to obtain (3.6). Accordingly in (2.14) the terms $\nabla\cdot A_\varepsilon = \Delta\varphi_\varepsilon$ and $(A-A_\varepsilon)\cdot\rho$ behave both as $\varepsilon^{-1/2}o(\varepsilon)$. The other two terms are bounded and cause no trouble. We will give detailed estimates for the term involving $(A-A_\varepsilon)\cdot\rho$. The same reasoning works for the other terms, as well. To this end we notice that for (3.9) we need to show

$$C_{\varepsilon,\rho} := \int e^{ix\cdot\xi}\gamma^{-1/2}(A-A_\varepsilon)\cdot\nabla\Delta_\rho^{-1}\gamma^{-1/2}A\cdot\nabla_\rho\Delta_\rho^{-1}\gamma^{1/2}(A-A_\varepsilon)\cdot\rho \, dx \stackrel{|\rho|\to\infty}{\longrightarrow} 0.$$

and

$$D_{\varepsilon,\rho} := \int e^{ix\cdot\xi}\gamma^{-1/2}A_\varepsilon\cdot\nabla\Delta_\rho^{-1}\gamma^{-1/2}A\cdot\nabla_\rho\Delta_\rho^{-1}\gamma^{1/2}(A-A_\varepsilon)\cdot\rho \, dx \stackrel{|\rho|\to\infty}{\longrightarrow} 0.$$

Since $\nabla\Delta_\rho^{-1}$ and $\nabla_\rho\Delta_\rho^{-1}$ are both bounded operators from $L^2_{\delta+1} \longrightarrow L^2_\delta$ we have by choosing $\varepsilon = |\rho|^{-1}$

$$C_{\varepsilon,\rho} = \varepsilon o(\varepsilon)|\rho| \stackrel{|\rho|\to\infty}{\longrightarrow} 0.$$

On the other hand, since

$$||\nabla(e^{ix\cdot\xi}\gamma^{-1/2}A_\varepsilon)||_{L^2} = \varepsilon^{-1/2}o(\varepsilon)$$

the integration by parts gives

$$D_{\varepsilon,\rho} = \varepsilon^{-1/2}\varepsilon^{1/2}o(\varepsilon) \overset{|\rho|\to\infty}{\longrightarrow} 0.$$

proving (3.9) concerning this term. The proof of (3.8) is similar but easier. $\Diamond$

Note that the above proof uses the result of Theorem 2.1. The proof of this theorem uses the choice $\varepsilon = |\rho|^{-1/4}$ but the statement of the theorem is independent of ε.

Proof of Theorem 1.2

We now extend $\gamma_i \in W^{3/2,\infty}(\mathbb{R}^n)$ with $\gamma_1 = \gamma_2$ on Ω^c, $\gamma_i = 1$ outside a large ball $i = 1, 2$ and γ_i strictly positive.

Let $\xi \in \mathbb{R}^n$. We choose $\rho_1 \in \mathbb{C}^n$, $\rho_1 \cdot \rho_1 = 0$, $|\rho_1|$ large enough and $\rho_2 \in \mathbb{C}^n$, $\rho_2 \cdot \rho_2 = 0$ so that $\rho_1 + \rho_2 = i\xi$. Let

$$u_i = e^{x\cdot\rho_1}\left(\omega_0^{(i)} + \omega_1^{(i)}\right), \quad i = 1, 2, \tag{3.10}$$

as in Theorem 1.1 with γ replaced by $\gamma_i, i = 1, 2$. Let

$$v = e^{x\cdot\rho_2}. \tag{3.11}$$

Now we substitute (3.10) and (3.11) in (3.5). We note that by Lemmas 3.2 and 3.3 we can replace the integration over Ω on (3.5) by integration over $\mathbb{R}^n$.

By taking the limit as $|\rho| \to \infty$ and $\varepsilon \to 0$ we can replace by Lemmas 2.1 and 3.3 u_1 by $e^{x\cdot\rho_1}\gamma_1^{-1/2}$ and u_2 by $e^{x\cdot\rho_2}\gamma_2^{-1/2}$ in (3.5). We obtain

$$\int_{\mathbb{R}^n}\left(\nabla\gamma_1^{1/2}\cdot\nabla(e^{ix\cdot\xi}\gamma_1^{-1/2}) - \nabla\gamma_2^{1/2}\cdot\nabla(e^{ix\cdot\xi}\gamma_2^{-1/2})\right)dx = 0. \tag{3.12}$$

This leads to

$$\begin{aligned}&\frac{i}{2}\xi\cdot\int e^{ix\cdot\xi}(\nabla\log\gamma_1 - \nabla\log\gamma_2)dx\\ &-\frac{1}{4}\int e^{ix\cdot\xi}((\nabla\log\gamma_1)^2 - (\nabla\log\gamma_2)^2)dx = 0.\end{aligned} \tag{3.13}$$

Then by using the Fourier inversion formula we obtain in the sense of distributions

$$\Delta(\log\gamma_1 - \log\gamma_2) + \frac{1}{2}(\nabla\log\gamma_1 + \nabla\log\gamma_2)\cdot\nabla(\log\gamma_1 - \log\gamma_2) = 0. \tag{3.14}$$

Since ${\gamma_1}_{|\partial\Omega} = {\gamma_2}_{|\partial\Omega}$, by the uniqueness of the solutions of the Dirichlet problem we conclude that $\gamma_1 = \gamma_2$ ending the proof of Theorem 1.1. $\Diamond$

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