Complex spherical waves for the elasticity system and probing of inclusions

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Abstract

We construct complex geometrical optics solutions for the isotropic elasticity system concentrated near spheres. We then use these special solutions, called complex spherical waves, to identify inclusions embedded in an isotropic, inhomogeneous, elastic background.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with smooth boundary. The domain Ω is modelled as an inhomogeneous, isotropic, elastic medium characterized by the Lamé parameters $\lambda(x)$ and $\mu(x)$. Assume that $\lambda(x) \in C^2(\overline{\Omega})$, $\mu(x) \in C^4(\overline{\Omega})$ and the following inequalities hold

$$\mu(x) > 0 \quad \text{and} \quad \lambda(x) + 2\mu(x) > 0 \quad \forall\, x \in \overline{\Omega} \quad \text{(strong ellipticity)}. \tag{1.1}$$

We consider the static isotropic elasticity system without sources

$$\mathcal{L}u := \nabla \cdot (\lambda(\nabla \cdot u)I + 2\mu \,\mathrm{Sym}(\nabla u)) = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

where $\mathrm{Sym}(A) = (A + A^T)/2$ denotes the symmetric part of the matrix $A \in \mathbb{C}^{3\times 3}$. Equivalently, if we denote $\sigma(u) = \lambda(\nabla \cdot u)I + 2\mu \,\mathrm{Sym}(\nabla u)$ the stress tensor, then (1.2) becomes

$$\mathcal{L}u = \nabla \cdot \sigma = 0 \quad \text{and} \quad \Omega.$$

On the other hand, since the Lamé parameters are differentiable, we can also write (1.2) in the non-divergence form

$$\mathcal{L}u = \mu\Delta u + (\lambda + \mu)\nabla(\nabla \cdot u) + \nabla\lambda\nabla \cdot u + 2\mathrm{Sym}(\nabla u)\nabla\mu = 0 \quad \text{in} \quad \Omega. \tag{1.3}$$

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Special type solutions for elliptic equations or systems have played an important role in inverse problems since the pioneering work of Calderón [2]. In 1987, Sylvester and Uhlmann [20] introduced complex geometrical optics solutions to solve the inverse boundary value problem for the conductivity equation. For the system (1.2), complex geometrical optics solutions were constructed in [5], using [4], and in [16] and [17]. In [16] and [17] the authors introduced an intertwining technique using pseudodifferential operators. In both [5] and [16], [17], the phase functions of the complex geometrical optics solutions are linear. Other type of special solutions, called oscillating-decaying solutions, were constructed for general elliptic systems in [18] and [19]. These oscillating-decaying solutions have been used in solving inverse problems, in particular in detecting inclusions and cavities [18].

In developing the theory for inverse boundary value problems with partial or local measurements, approximate complex geometrical optics solutions concentrated near hyperplanes and near hemispheres for the Schrödinger equation were given in [7] and [14], respectively. In [14], the construction was based on hyperbolic geometry and was applied in [8] to construct complex geometrical optics solutions for the Schrödinger equation where the real part of the phase function is a radial function, i.e., its level surfaces are spheres. They call these solutions *complex spherical waves*. The hyperbolic geometry approach does not work for the Laplacian with first order perturbations such as the Schrödinger equation with magnetic potential and the isotropic elasticity (1.2) (see below). Recently, complex geometrical optics solutions with more general phase functions were constructed in [15] for the Schrödinger equation and in [3] for the Schrödinger equation with magnetic potential. The method used in [15] and [3] relies on Carleman type estimates, which is a more flexible tool in treating lower order perturbations. Hence, we shall apply the method in [15] and [3] to construct complex geometrical optics solutions for (1.2) with the real part of the phase function being a radial function, i.e., complex spherical waves.

With these complex spherical waves at hand, we can study the inverse problem of detecting unknown inclusions inside an elastic body with known isotropic background medium. The investigation of this inverse problem is motivated by [8] in which the same problem was treated for the conductivity equation. There are several results, both theoretical and numerical, concerning the object identification problem by boundary measurements for the conductivity equation. We will not try to give a full account of these developments here. For detailed references, we refer to [8]. For the elasticity system, we will compare our result to some existing ones. In [10], Ikehata generalized his *probe method* to the isotropic elasticity system. Ikehata's probe method is based on singular solutions and Runge's approximation property (which is closely related to the unique continuation property). These ideas are due to Isakov [13]. On the other hand, for the general (anisotropic) elasticity system, a reconstruction method using oscillating-decaying solutions was given by the authors in [18]. The method in [18] shares the same spirit as Ikehata's *enclosure method* (see Ikehata's survey article [9]). Both methods enable us to reconstruct the support function of the inclusion by the Dirichlet-to-Neumann map. It should be noted that Runge's property was used in [18]. Ikehata's results on the enclosure method did not rely on Runge's property because he used the Laplacian as the background and explicit complex geometrical optics solutions are available for this case. Our approach here lies between the method in [18] and Ikehata's enclosure method in the sense that we treat the isotropic elasticity without using Runge's property. Furthermore, since we probe the region by complex spherical waves, it is possible to recover some concave parts of inclusions. Also, as in [8], we can localize the measurements with these complex spherical waves.

This paper is organized as follows. In Section 2, (1.2) or (1.3) is transformed to a system of dimension four and a Carleman estimate is derived for the new system. The construction of complex spherical waves for (1.2) is given in Section 3. The study of the inverse problem is carried out in Section 4.

2 Carleman estimate and its consequence

It suffices to work with the system (1.3) here. Since the leading order of (1.3) is strongly coupled, we want to find a reduced system whose leading part is decoupled (precisely, the Laplacian) and solutions of (1.3) can be constructed more easily. We will use the reduced system derived by Ikehata [11]. This reduction had already been mentioned in [21]. Let $W = \begin{pmatrix} w \\ g \end{pmatrix}$ satisfy

$$PW := \Delta \begin{pmatrix} w \\ g \end{pmatrix} + \tilde{A}_1(x) \begin{pmatrix} \nabla g \\ \nabla \cdot w \end{pmatrix} + \tilde{A}_0(x) \begin{pmatrix} w \\ g \end{pmatrix} = 0, \tag{2.1}$$

where

$$\tilde{A}_1(x) = \begin{pmatrix} 2\mu^{-1/2}(-\nabla^2 + \Delta)\mu^{-1} & -\nabla \log \mu \\ 0 & \frac{\lambda+\mu}{\lambda+2\mu}\mu^{1/2} \end{pmatrix}$$

and

$$\tilde{A}_0(x) = \begin{pmatrix} -\mu^{-1/2}(2\nabla^2 + \Delta)\mu^{1/2} & 2\mu^{-5/2}(\nabla^2 - \Delta)\mu \; \nabla\mu \\ -\frac{\lambda-\mu}{\lambda+2\mu}(\nabla\mu^{1/2})^T & -\mu\Delta\mu^{-1} \end{pmatrix}.$$

Here $\nabla^2 f$ is the Hessian of the scalar function f. Then

$$u := \mu^{-1/2} w + \mu^{-1}\nabla g - g\nabla\mu^{-1}$$

satisfies (1.3). A similar form was also used in [5] for studying the inverse boundary value problem for the isotropic elasticity system.

With (2.1) at hand, we now consider the matrix operator $P_h = -h^2 P$. More precisely, we have

$$P_h = (hD)^2 + ihA_1(x) \begin{pmatrix} hD \\ hD\cdot \end{pmatrix} + h^2 A_0$$

where $D = -i\nabla$, $A_1 = -\tilde{A}_1$, and $A_0 = -\tilde{A}_0$. Later on we shall denote the matrix operator

$$iA_1(x) \begin{pmatrix} hD \\ hD\cdot \end{pmatrix} = A_1(x, hD).$$

To construct complex geometrical optics solutions, we will follow closely the papers [3] and [15]. The construction here is simpler than the one given in [16] where the technique of intertwining operators were first introduced. Furthermore, we do not need to work with C^∞ coefficients here. As in [3] and [15], we will use semiclassical Weyl calculus. Our goal here is to derive a Carleman estimate with semiclassical H^{-2} norm for P_h.

The conjugation of P_h with $e^{\varphi/h}$ is given by

$$e^{\varphi/h}\circ P_h\circ e^{-\varphi/h}=(hD+i\nabla\varphi)^2+hA_1(x,hD+i\nabla\varphi)+h^2A_0(x).$$

We first consider the leading operator $(hD+i\nabla\varphi)^2$ and denote

$$(hD+i\nabla\varphi)^2=A+iB,$$

where $A=(hD)^2-(\nabla\varphi)^2$ and $B=\nabla\varphi\circ hD+hD\circ\nabla\varphi$. The Weyl symbols of A and B are given as

$$a(x,\xi)=\xi^2-(\nabla\varphi)^2\quad\text{and}\quad b(x,\xi)=2\nabla\varphi\cdot\xi,$$

respectively. Let Ω_0 be an open bounded domain such that $\bar{\Omega}\subset\Omega_0$. Accordingly, we extend λ and μ to Ω_0 by preserving their smoothness. We now let φ have nonvanishing gradient in Ω_0 and be a limit Carleman weight in Ω_0:

$$\{a,b\}=0\quad\text{when}\quad a=b=0,$$

i.e.,

$$<\varphi''|\nabla\varphi\otimes\nabla\varphi+\xi\otimes\xi>=0\quad\text{when}\quad\xi^2=(\nabla\varphi)^2\text{ and }\nabla\varphi\cdot\xi=0.$$

In order to get positivity in proving the Carleman estimate, we will modify the weight φ as in [3] and [15]. Let us denote $\varphi_\varepsilon=\varphi+h\varphi^2/(2\varepsilon)$, where $\varepsilon>0$ will be chosen later. Also, we denote a_ε and b_ε the corresponding symbols as φ is replaced by φ_ε. Then one can easily check that

$$\{a_\varepsilon,b_\varepsilon\}=\frac{4h}{\varepsilon}(1+\frac{h}{\varepsilon}\varphi)^2(\nabla\varphi)^4>0\quad\text{when}\quad a_\varepsilon=b_\varepsilon=0.$$

Arguing as in [15], we get

$$\{a_\varepsilon,b_\varepsilon\}=\frac{4h}{\varepsilon}(1+\frac{h}{\varepsilon}\varphi)^2(\nabla\varphi)^4+\alpha(x)a_\varepsilon+\beta(x,\xi)b_\varepsilon,$$

where $\beta(x,\xi)$ is linear in ξ. Therefore, at the operator level, we have

$$i[A_\varepsilon,B_\varepsilon]=\frac{4h^2}{\varepsilon}(1+\frac{h}{\varepsilon}\varphi)^2(\nabla\varphi)^4+\frac{h}{2}(\alpha\circ A_\varepsilon+A_\varepsilon\circ\alpha)+\frac{h}{2}(\beta^w\circ B_\varepsilon+B_\varepsilon\circ\beta^w)+h^3c(x),\tag{2.2}$$

where β^w denotes the Weyl quantization of β.

With the help (2.2), we now can estimate

$$\|(A_\varepsilon+iB_\varepsilon)V\|^2=\|A_\varepsilon V\|^2+\|B_\varepsilon V\|^2+i<B_\varepsilon V|A_\varepsilon V>-i<A_\varepsilon V|B_\varepsilon V>$$

for $V \in C_0^\infty(\Omega)$. Here and below, we define the norm $\|\cdot\|$ and the inner $< \cdot \mid \cdot >$ in term of $L^2(\Omega)$. Integrating by parts, we conclude

$$< B_\varepsilon V | A_\varepsilon V > = < A_\varepsilon B_\varepsilon V | V > \quad \text{and} \quad < A_\varepsilon V | B_\varepsilon V > = < B_\varepsilon A_\varepsilon V | V > . \tag{2.3}$$

On the other hand, we observe that

$$\|h\nabla V\|^2 = < A_\varepsilon V | V > + \|\sqrt{(\nabla\varphi)} V\|^2 \lesssim \|A_\varepsilon V\|^2 + \|V\|^2 \tag{2.4}$$

and the obvious estimate

$$\|(h\nabla)^2 V\|^2 \lesssim \|A_\varepsilon V\|^2 + \|V\|^2. \tag{2.5}$$

Using (2.2), (2.3), (2.4), and (2.5) gives

$$\begin{aligned}
&\|(A_\varepsilon + iB_\varepsilon)V\|^2 \\
&\gtrsim \|A_\varepsilon V\|^2 + \|B_\varepsilon V\|^2 + \tfrac{h^2}{\varepsilon}\|V\|^2 - h(\|A_\varepsilon V\|\|V\| + \|B_\varepsilon V\|\|h\nabla V\|) \\
&\gtrsim \|A_\varepsilon V\|^2 + \|B_\varepsilon V\|^2 + \tfrac{h^2}{\varepsilon}\|V\|^2 - \tfrac{1}{2}\|A_\varepsilon V\|^2 - \tfrac{h^2}{2}\|V\|^2 - \tfrac{1}{2}\|B_\varepsilon V\|^2 \\
&\quad -\tfrac{h^2}{2}(\|A_\varepsilon V\|^2 + \|V\|^2) \\
&\gtrsim (1 - O(\tfrac{h^2}{\varepsilon}))\|A_\varepsilon V\|^2 + \tfrac{h^2}{\varepsilon}(\|A_\varepsilon V\|^2 + \|V\|^2)
\end{aligned}$$

Thus, taking h and ε ($h \ll \varepsilon$) sufficiently small, we arrive at

$$\|(A_\varepsilon + iB_\varepsilon)V\|^2 \gtrsim \frac{h^2}{\varepsilon}(\|V\|^2 + \|h\nabla V\|^2 + \|(h\nabla)^2 V\|^2),$$

namely,

$$\|(A_\varepsilon + iB_\varepsilon)V\|^2 \gtrsim \frac{h^2}{\varepsilon}\|V\|^2_{H_h^2(\Omega)}. \tag{2.6}$$

Here we define the semiclassical Sobelov norms

$$\|v\|^2_{H_h^m(\Omega)} = \sum_{|\alpha| \le m} \|(h\nabla)^\alpha v\|^2 \quad \forall\, m \in \mathbb{N}$$

and

$$\|v\|^2_{H_h^s(\mathbb{R}^3)} = \int (1 + |h\xi|^2)^s |\hat{v}(\xi)|^2 d\xi = \| < hD >^s v\|^2 \quad \forall\, s \in \mathbb{R}.$$

Now let Ω_1 be open and $\bar{\Omega} \subset \Omega_1 \subset \Omega_0$. The estimate (2.6) also holds for $V \in C_0^\infty(\Omega_1)$. Then as done in [3], we can obtain that

$$\frac{h^2}{\varepsilon}\|V\|^2_{H_h^2(\mathbb{R}^3)} \lesssim \|(A_\varepsilon + iB_\varepsilon) < hD >^2 V\|^2_{H_h^{-2}(\mathbb{R}^3)}. \tag{2.7}$$

To add the first order perturbation $hA_{1,\varepsilon}V + h^2 A_0 V = hA_1(x, hD + i\nabla\varphi_\varepsilon)V + h^2 A_0 V$ into (2.7), we note that

$$\|(hA_{1,\varepsilon} + h^2 A_0) < hD >^2 V\|^2_{H_h^{-2}(\mathbb{R}^3)} \lesssim h^2 \|V\|^2_{H_h^1(\mathbb{R}^3)}. \tag{2.8}$$

In view of (2.8), we get from (2.7) that

$$\|(A_\varepsilon + iB_\varepsilon + hA_{1,\varepsilon} + h^2 A_0) < hD >^2 V\|^2_{H_h^{-2}(\mathbb{R}^3)} \gtrsim h^2 \| < hD >^2 V\|^2 \quad (2.9)$$

provided $\varepsilon \ll 1$. Transforming back to the original operator, (2.9) is equivalent to

$$\| < hD >^2 V\| \lesssim h\|e^{\phi_\varepsilon/h} P e^{-\varphi_\varepsilon/h} < hD >^2 V\|_{H_h^{-2}(\mathbb{R}^3)} \quad (2.10)$$

for $V \in C_0^\infty(\Omega_1)$.

Let $\chi \in C_0^\infty(\Omega_1)$ with $\chi = 1$ on Ω and $W \in C_0^\infty(\Omega)$. Substituting $V = \chi < hD >^{-2} W$ into (2.10) and using the property that

$$\|(1-\chi) < hD >^{-2} W\|_{H_h^s} = O(h^\infty)\|W\|$$

for any $s \in \mathbb{R}$, we get that

$$\|W\| \lesssim h\|e^{\phi_\varepsilon/h} P e^{-\varphi_\varepsilon/h} W\|_{H_h^{-2}(\mathbb{R}^3)}. \quad (2.11)$$

Now since $e^{\varphi_\varepsilon/h} = e^{\varphi^2/\varepsilon} e^{\varphi/h}$ and $e^{\varphi^2/\varepsilon} = O(1)$, (2.11) becomes

$$\|W\| \lesssim h\|e^{\phi/h} P e^{-\varphi/h} W\|_{H_h^{-2}(\mathbb{R}^3)}. \quad (2.12)$$

Note that (2.12) also holds when φ is replaced by $-\varphi$. Therefore, by the Hahn-Banach theorem, we have the following existence theorem.

Theorem 2.1 *For h sufficiently small, for any $F \in L^2(\Omega)$, there exists $V \in H_h^2(\Omega)$ such that*

$$e^{\varphi/h} P_h(e^{-\varphi/h} V) = F$$

and $h\|V\|_{H_h^2(\Omega)} \lesssim \|F\|_{L^2(\Omega)}$.

3 Construction of complex spherical waves

In this section we will construct complex spherical waves for the elasticity system (1.3). We apply the method in [3] and [15] to our system here. We will work with the reduced system (2.1). Let ψ be a solution of the eikonal equation

$$a(x, \nabla\psi) = b(x, \nabla\psi) = 0, \quad \forall\ x \in \Omega,$$

i.e.,

$$\begin{cases} (\nabla\psi)^2 = (\nabla\varphi)^2 \\ \nabla\varphi \cdot \nabla\psi = 0, \end{cases} \quad \forall\ x \in \Omega. \quad (3.1)$$

Since $\{a, b\} = 0$ on $a = b = 0$, there exists a solution to (3.1). To construct complex spherical waves, we choose the limit Carleman weight

$$\varphi(x) = \log|x - x_0| \quad \text{for} \quad x_0 \notin \overline{\mathrm{ch}(\Omega)},$$

then a solution of (3.1) is

$$\psi(x) = \frac{\pi}{2} - \arctan \frac{\omega \cdot (x - x_0)}{\sqrt{(x-x_0)^2 - (\omega \cdot (x-x_0))^2}} = d_{\mathbb{S}^2}(\frac{x - x_0}{|x - x_0|}, \omega)$$

where $\mathrm{ch}(\Omega) :=$ convex hull of Ω and $\omega \in \mathbb{S}^2$ such that $\omega \neq (x - x_0)/|x - x_0|$ for all $x \in \overline{\Omega}$ [3]. We can be more explicitly in the choices of φ and ψ. In fact, by suitable translation and rotation, we can take $x_0 = 0$, $\omega = (1, 0, 0)$ and set $z = x_1 + i|x'|$ with $x' = (x_2, x_3)$, then $\varphi + i\psi = \log z$ (see [3, Remark 3.1]). Having found ψ, we look for $U = e^{-(\varphi+i\psi)/h}(L + R)$ satisfying

$$(-h^2\Delta + h^2 A_1(x, D) + h^2 A_0(x))U = 0 \quad \text{in} \quad \Omega.$$

Equivalently, we need to solve

$$e^{(\varphi+i\psi)/h} P_h(e^{-(\varphi+i\psi)/h}(L + R)) = 0 \quad \text{in} \quad \Omega.$$

We can compute that

$$\begin{aligned}
&e^{(\varphi+i\psi)/h} P_h e^{-(\varphi+i\psi)/h} \\
&= ((hD - \nabla\psi)^2 - (\nabla\varphi)^2) + i(\nabla\varphi \cdot (hD - \nabla\psi) + (hD - \nabla\psi) \cdot \nabla\varphi) \\
&\quad + h^2 A_1(x, D) + hA_1(x, i\nabla\varphi - \nabla\psi) + h^2 A_0 \\
&= h(-\nabla\psi \cdot D - D \cdot \nabla\psi + i\nabla\varphi \cdot D + iD \cdot \nabla\varphi + A_1(x, i\nabla\varphi - \nabla\psi)) + P_h \\
&= hQ + P_h
\end{aligned}$$

where $Q = -\nabla\psi \cdot D - D \cdot \nabla\psi + i\nabla\varphi \cdot D + iD \cdot \nabla\varphi + A_1(x, i\nabla\varphi - \nabla\psi)$. Hence we want to find L, independent of h, so that

$$QL = 0 \quad \text{in} \quad \Omega. \tag{3.2}$$

The equation (3.2) is a system of the Cauchy-Riemann type. In fact, in view of the choices of φ and ψ above, (3.2) is equivalent to

$$\partial_{\bar{z}} L + \widetilde{A}(z, \theta)L = 0 \quad \text{in} \quad \Omega \tag{3.3}$$

where $\widetilde{A}(z, \theta)$ is a C^2 matrix-valued function. Here we have used the cylindrical coordinates for $\mathbb{R}^3$, i.e. $x = (x_1, r, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^1$, and $z = x_1 + ir$. Using the results in [4], [6], or [17], one can find an invertible 4×4 matrix $G(x) \in C^2(\overline{\Omega})$ satisfying (3.2). For the sake of clarity, we outline the proof of the existence of G. We refer to, for example, [6, page 59-60] for more detailed arguments. It suffices to consider (3.3). Let $M > 0$ satisfy $\bar{\Omega} \subset \{(x_1, r, \theta) : |x_1| \leq M,\ 0 \leq r \leq M,\ \theta \in \mathbb{S}^1\} := \mathcal{U}$. Without restriction, we can assume that (3.3) holds in $\mathcal{U}$ by suitably extending the matrix $\widetilde{A}$. By using cut-off functions with sufficiently small supports, one can show that G exists near $z_0 = x_1^0 + ir_0$ with $|x_1^0| < M, 0 < r_0 < M$ and depends C^2 smoothly on θ for all $\theta \in \mathbb{S}^1$. To construct a global invertible G in $\mathcal{U}$, we simply patch local solutions together with the help of Cartan's lemma.

So L can be chosen from columns of G. Then, R is required to satisfy

$$e^{\varphi/h}P_h(e^{-(\varphi+i\psi)/h}R) = -e^{-i\psi/h}P_hL. \tag{3.4}$$

Note that $\|e^{-i\psi/h}P_hL\| \lesssim h^2$. Thus Theorem 2.1 implies that

$$\|e^{-i\psi/h}R\|_{H_h^2(\Omega)} \lesssim h, \tag{3.5}$$

which leads to

$$\|\partial^\alpha R\|_{L^2(\Omega)} \lesssim h^{1-|\alpha|} \quad \text{for} \quad |\alpha| \le 2. \tag{3.6}$$

So if we write $L = \begin{pmatrix} \ell \\ d \end{pmatrix}$ and $R = \begin{pmatrix} r \\ s \end{pmatrix}$ with $\ell, r \in \mathbb{C}^3$, then

$$w = e^{-(\varphi+i\psi)/h}(\ell + r) \quad \text{and} \quad g = e^{-(\varphi+i\psi)/h}(d + s)$$

where r and s satisfy the estimate (3.6). Therefore, $u = \mu^{-1/2}w + \mu^{-1}\nabla g - g\nabla\mu^{-1}$ is the complex spherical wave for (1.3).

Remark 3.1 *Even though the four-vector* $\begin{pmatrix} \ell \\ d \end{pmatrix}$ *is nonzero in* Ω*, we can not conclude that both* ℓ *and* d *never vanish in* Ω*. However, for any point* $y \in \Omega$*, it is easy to show that there exists a small ball* $B_\delta(y)$ *of* y *with* $B_\delta(y) \subset \Omega$ *such that one can find a pair of* ℓ *and* d *which do not vanish in* $B_\delta(y)$*. We will use this fact in studying our inverse problem in the next section.*

4 Probing for inclusions

In this section we shall apply complex spherical waves we constructed above to the problem of identifying the inclusion embedded inside an elastic body with isotropic medium. We now begin to set up the problem. Let D be an open subset of Ω with Lipschitz boundary satisfying that $D \subset\subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. Assume that $\lambda_0(x) \in C^2(\overline{\Omega})$ and $\mu_0(x) \in C^4(\overline{\Omega})$ satisfy the strong convexity condition, i.e.,

$$3\lambda_0(x) + 2\mu_0(x) > 0 \quad \text{and} \quad \mu_0(x) > 0 \quad \forall \ x \in \overline{\Omega}. \tag{4.1}$$

It is obvious that (4.1) implies (1.1). On the other hand, we assume that $\widetilde{\lambda}(x)$, $\widetilde{\mu}(x)$ be two essentially bounded functions such that either

$$\widetilde{\mu} \ge 0 \quad \text{and} \quad 3\widetilde{\lambda} + 2\widetilde{\mu} \ge 0 \quad \text{a.e. in} \quad D$$

or

$$\widetilde{\mu} \le 0 \quad \text{and} \quad 3\widetilde{\lambda} + 2\widetilde{\mu} \le 0 \quad \text{a.e. in} \quad D.$$

For our inverse problem here, we shall also assume appropriate jump conditions across ∂D:

$$
\text{For } y \in \partial D, \text{ there exists a ball } B_\epsilon(y) \text{ such that one of the following conditions holds:}
$$

$$
\begin{cases} (\mu+): & \widetilde{\mu} > \epsilon,\ 3\widetilde{\lambda} + 2\widetilde{\mu} \geq 0 \\ (\lambda+): & \widetilde{\mu} = 0,\ \widetilde{\lambda} > \epsilon \\ (\mu-): & \widetilde{\mu} < -\epsilon,\ 3\widetilde{\lambda} + 2\widetilde{\mu} \leq 0 \\ (\lambda-): & \widetilde{\mu} = 0,\ \widetilde{\lambda} < -\epsilon \end{cases}, \ \forall\, x \in B_\epsilon(y) \cap D. \tag{4.2}
$$

To make sure that the forward problem is well-posed, we suppose that $\lambda = \lambda_0 + \chi_D \widetilde{\lambda}$ and $\mu = \mu_0 + \chi_D \widetilde{\mu}$ satisfy (4.1) a.e. in Ω, where χ_D is the characteristic function of D. Therefore, for any $f \in H^{1/2}(\partial\Omega)$, there exists a unique (weak) solution u to

$$
\begin{cases} \mathcal{L}_D u = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega. \end{cases}
$$

Here the elastic operator $\mathcal{L}_D$ is defined in terms of λ and μ. The Dirichlet-to-Neumann map related to $\mathcal{L}_D$ is now defined as

$$
\Lambda_D : f \to \sigma(u)\nu|_{\partial\Omega}
$$

where ν is the unit outer normal of $\partial\Omega$ and for $x \in \partial\Omega$

$$
\sigma(u) = \lambda(\nabla \cdot u)I + 2\mu \operatorname{Sym}(\nabla u) = \lambda_0(\nabla \cdot u)I + 2\mu_0 \operatorname{Sym}(\nabla u).
$$

Now assume that all parameters are known except $\widetilde{\lambda}$, $\widetilde{\mu}$, and D. The inverse problem is to determine D by Λ_D. This inverse problem was studied by Ikehata [10] with the so-called probe method. However, as we mentioned in the Introduction, this method relies on Runge's approximation property, which is difficult to realize in practice. In this paper we approach this inverse problem from a different viewpoint. We would like to get partial information of D by local measurements. Our main tool is the use of complex spherical waves to probe for the inclusions. One of the advantages of our method is that we do not need Runge's property and we can quickly determine roughly where the inclusion is located by only a few measurements that can be advantageous in practical applications.

We first derive some integral inequalities that we need. Let Λ_0 be the Dirichlet-to-Neumann map related to $\mathcal{L}_0$, where $\mathcal{L}_0$ is the elastic operator defined in terms of λ_0 and μ_0. Assume that u_0 is the solution of

$$
\begin{cases} \mathcal{L}_0 u_0 = 0 & \text{in} \quad \Omega \\ u_0 = f & \text{on} \quad \partial\Omega. \end{cases} \tag{4.3}
$$

Then we have the following inequalities

$$
\begin{aligned}
&\int_D \{\frac{3\lambda_0+2\mu_0}{3(3\lambda+2\mu)}(3\widetilde{\lambda}+2\widetilde{\mu})|\nabla\cdot u_0|^2+2\frac{\mu_0}{\mu}\widetilde{\mu}|\mathrm{Sym}(\nabla u_0)-\frac{\nabla\cdot u_0}{3}I|^2\}dx \\
\le\quad & <(\Lambda_D-\Lambda_0)f,\overline{f}> \\
\le\quad & \int_D \frac{3\widetilde{\lambda}+2\widetilde{\mu}}{3}|\nabla\cdot u_0|^2+2\widetilde{\mu}|\mathrm{Sym}(\nabla u_0)-\frac{\nabla\cdot u_0}{3}I|^2\}dx \qquad (4.4)
\end{aligned}
$$

(see [10, Proposition 5.1]). The plan now is to plug complex spherical waves u_0 given in Ω with parameters $h>0$ and $t>0$, denoted by $u_{0,h,t}$, into (4.4). For brevity, we will suppress the subscript 0 and denote $u_{0,h,t}=u_{h,t}$. We set $u_{h,t}=e^{\log t/h}v$ and $v_h=\mu_0^{-1/2}w+\mu_0^{-1}\nabla g-g\nabla\mu_0^{-1}$ with $w=e^{-(\varphi+i\psi)/h}(\ell+r)$ and $g=e^{-(\varphi+i\psi)/h}(d+s)$, where $W=\begin{pmatrix} w\\ g\end{pmatrix}$ satisfies $PW=0$ in Ω with λ, μ being replaced by λ_0, μ_0 (see (2.1)). Recall that r and s satisfy (3.6). Furthermore, for any $x\in\Omega$, we can choose a neighborhood of x such that $\ell(x)$ and $d(x)$ never vanish in such neighborhood. In view of (4.4), we need to compute $\nabla\cdot u_{h,t}$ and $\mathrm{Sym}(\nabla u_{h,t})$ in detail. We note that

$$
\Delta g=-\mu_0^{1/2}\frac{\lambda_0+\mu_0}{\lambda_0+2\mu_0}\nabla\cdot w+b_0\cdot w+c_0 g, \qquad (4.5)
$$

where (b_0,c_0) is the bottom row of A_0. From (4.5) we have

$$
\begin{aligned}
&\nabla\cdot v_h\\
&=\nabla\mu_0^{-1/2}\cdot w+\mu_0^{-1/2}\nabla\cdot w+\nabla\mu_0^{-1}\cdot\nabla g+\mu_0^{-1}\Delta g-\nabla g\cdot\nabla\mu_0^{-1}-g\Delta\mu_0^{-1}\\
&=\nabla\mu_0^{-1/2}\cdot w+\mu_0^{-1/2}\nabla\cdot w+\mu_0^{-1}\Delta g-g\Delta\mu_0^{-1}\\
&=(\nabla\mu_0^{-1/2}+\mu_0^{-1}b_0)\cdot w+\mu_0^{-1/2}(1-\tfrac{\lambda_0+\mu_0}{\lambda_0+2\mu_0})\nabla\cdot w+(\mu_0^{-1}c_0-\Delta\mu_0^{-1})g\\
&=e^{-(\varphi+i\psi)/h}\{(\nabla\mu_0^{-1/2}+\mu_0^{-1}b_0)\cdot(\ell+r)-\mu_0^{-1/2}(1-\tfrac{\lambda_0+\mu_0}{\lambda_0+2\mu_0})\tfrac{\nabla\varphi+i\nabla\psi}{h}\cdot(\ell+r)\\
&\quad+\mu_0^{-1/2}(1-\tfrac{\lambda_0+\mu_0}{\lambda_0+2\mu_0})\nabla\cdot(\ell+r)+(\mu_0^{-1}c_0-\Delta\mu_0^{-1})(d+s)\}. \qquad (4.6)
\end{aligned}
$$

Next we observe that

$$
\mathrm{Sym}(\nabla v_h)=\mathrm{Sym}(\nabla\mu_0^{-1/2}\otimes w)+\mu_0^{-1/2}\mathrm{Sym}(\nabla w)+\mu_0^{-1}\nabla^2 g-g\nabla^2\mu_0^{-1}
$$

and hence

$$
\begin{aligned}
&\mathrm{Sym}(\nabla v_h)\\
&=e^{-(\varphi+i\psi)/h}\{\mathrm{Sym}(\nabla\mu_0^{-1/2}\otimes(\ell+r))-\tfrac{1}{h}\mu_0^{-1/2}\mathrm{Sym}((\nabla\varphi+i\nabla\psi)\otimes(\ell+r))\\
&\quad+\mu_0^{-1/2}\mathrm{Sym}(\nabla(\ell+r))+\mu_0^{-1}\nabla^2(d+s)-\mu_0^{-1}\tfrac{1}{h}(d+s)\nabla^2(\varphi+i\psi)\\
&\quad-\mu_0^{-1}\tfrac{2}{h}\mathrm{Sym}(\nabla(\varphi+i\psi)\otimes\nabla(d+s))+\mu_0^{-1}\tfrac{1}{h^2}\nabla(\varphi+i\psi)\otimes\nabla(\varphi+i\psi)(d+s)\\
&\quad-(d+s)\nabla^2\mu_0^{-1}\} \qquad (4.7)
\end{aligned}
$$

where $(a\otimes b)_{jk}=(a_jb_k)$ for $1\le j,k\le 3$.

We are now at a position to discuss the inverse problem. Recall that $\varphi=\log|x-x_0|$ with $x_0\notin\overline{\mathrm{ch}(\Omega)}$. Let $f_{h,t}$ be the boundary value of $u_{h,t}$ on $\partial\Omega$ and denote

$$E(h,t) = |<(\Lambda_D - \Lambda_0)f_{h,t}, \overline{f_{h,t}}>|.$$

Our main result for the inverse problem is

Theorem 4.1 *Assume that the jump condition (4.2) holds. For $t > 0$ and sufficiently small h, we have that*
(i). *If* $\operatorname{dist}(D, x_0) =: d_0 > t$, *then* $E(h,t) \le Ca^{1/h}$ *for some constants* $C > 0$ *and* $a < 1$;
(ii). *If* $d_0 < t$, *then* $E(h,t) \ge Cb^{1/h}$ *for some constants* $C > 0$ *and* $b > 1$ *with appropriate choices of* $f_{h,t}$;
(iii). *If* $\overline{D} \cap \overline{B_t(x_0)} = y$, *then*

$$\begin{cases} C'h^{-1} \le E(h,t) \le Ch^{-3} \text{ if } (\mu\pm) \text{ holds near } y, \\ C'h \le E(h,t) \le Ch^{-1} \text{ if } (\lambda\pm) \text{ holds near } y, \end{cases} \tag{4.8}$$

provided ℓ *and* d *of* $u_{h,t}$ *do not vanish near* y.

Proof. To prove the theorem, we simply substitute $u_{h,t}$ into (4.4). The key observation comes from (4.6) and (4.7). We only consider the cases $(\mu+)$ and $(\lambda+)$ of (4.2) here. The same arguments work for $(\mu-)$ and $(\lambda-)$ of (4.2). The only change is to use integral inequalities obtained by multiplying "$-$" on (4.4). If $(\mu+)$ holds, then the leading terms of two integrals in (4.4) come from $\operatorname{Sym}(\nabla u_{h,t})$ and are determined by

$$\frac{1}{h^4}(\frac{t}{|x-x_0|})^{2/h}((\nabla\varphi)^2 + (\nabla\psi)^2)^2|d|^2 = \frac{4}{h^4}(\frac{t}{|x-x_0|})^{2/h}(\nabla\varphi)^4|d|^2. \tag{4.9}$$

On the other hand, if $(\lambda+)$ holds, then the leading terms in those integrals in (4.4) come from $\nabla \cdot u_{h,t}$ and are governed by

$$\frac{2}{h^2}(\frac{t}{|x-x_0|})^{2/h}(\nabla\varphi)^2|\ell|^2. \tag{4.10}$$

Using (4.4), (4.9), and (4.10), the proof of (i) follows easily from

$$E(h,t) \le C\frac{1}{h^4}(\frac{t}{d_0})^{2/h} \quad \text{when } (\mu+) \text{ holds,}$$

and

$$E(h,t) \le C\frac{1}{h^2}(\frac{t}{d_0})^{2/h} \quad \text{when } (\lambda+) \text{ holds.}$$

For the proof of (ii), we pick a small ball $B_\delta \subset\subset B_t(x_0) \cap D$ such that the jump conditions $(\mu+)$ or $(\lambda+)$ hold in B_δ and $\ell(x)$, $d(x)$ of $u_{h,t}$ never vanish in $B_\delta(x)$. The latter property is guaranteed by Remark 3.1. For such choice of ℓ and d, the Dirichlet data is a priori given by

$$f_{h,t} = u_{h,t}|_{\partial\Omega} = e^{(\log t - \varphi - i\psi)/h}(\ell + d)|_{\partial\Omega}.$$

Thus, argued as above, we have either

$$E(h,t) \geq C\frac{1}{h^4}(\frac{t}{d_0})^{2/h} \quad \text{when } (\mu+) \text{ holds,}$$

or

$$E(h,t) \geq C\frac{1}{h^2}(\frac{t}{d_0})^{2/h} \quad \text{when } (\lambda+) \text{ holds.}$$

which implies (ii).

Now let $y \in \overline{D} \cap \overline{B_t(x_0)}$ and choose a ball $B_\epsilon(y)$ such that (4.2) holds and $\ell(x)$, $d(x)$ of $u_{h,t}$ never vanish in $B_\epsilon(y) \cap D$. Pick a small cone with vertex at y, say Γ, so that there exists an $\eta > 0$ satisfying

$$\Gamma_\eta := \Gamma \cap \{0 < |x-y| < \eta\} \subset B_\epsilon(y) \cap D.$$

We observe that if $x \in \Gamma_\eta$ with $|x-y| = \rho < \eta$ then $|x - x_0| \leq \rho + t$, i.e.

$$\frac{1}{|x-x_0|} \geq \frac{1}{\rho+t}.$$

Thus, for the case $(\mu+)$, we get that from (4.4) and (4.9)

$$\begin{aligned} E(h,t) &\geq C\tfrac{1}{h^4}\textstyle\int_D \widetilde{\mu}^2(\nabla\varphi)^4|d|^2(\tfrac{t}{|x-x_0|})^{2/h}dx \\ &\geq C\epsilon\tfrac{1}{h^4}\textstyle\int_0^\eta(\tfrac{t}{\rho+t})^{2/h}\rho^2 d\rho \\ &\geq C\epsilon h^{-1}. \end{aligned} \tag{4.11}$$

On the other hand, we can choose a cone $\widetilde{\Gamma}$ with vertex at x_0 such that $\overline{D} \subset \widetilde{\Gamma} \cap \{|x-x_0| > t\}$. Hence, we can estimate

$$\begin{aligned} E(h,t) &\leq C\tfrac{1}{h^4}\textstyle\int_{\widetilde{\Gamma}\cap\{t<|x-x_0|<t+\eta\}}(\tfrac{t}{|x-x_0|})^{2/h}dx \\ &\quad +C\tfrac{1}{h^4}\textstyle\int_{\widetilde{\Gamma}\cap\{t+\eta\leq|x-x_0|\}}(\tfrac{t}{|x-x_0|})^{2/h}dx \\ &\leq C\tfrac{1}{h^4}\textstyle\int_t^{t+\eta}(\tfrac{t}{r})^{2/h}r^2dr + O((\tfrac{t}{t+\eta})^{2/h}) \\ &\leq Ch^{-3}. \end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12) yields the first estimate of (4.8). Using similar arguments we can get the second estimate of (4.8) for the case $(\lambda+)$. □

Remark 4.1 (1). *Using Theorem 4.1, we can clearly determine whether the probing front $\{|x-x_0| = t\}$ intersects the inclusion. In view of* (iii) *of the theorem, it is also possible to determine whether we have material jumps in μ or λ when the front touches the boundary of the inclusion.*
(2). *In the proofs of* (ii) *and* (iii) *we need to choose ℓ and d which are nonvanishing in small subdomains of Ω. Since ℓ and d depend only on the known background medium, they can be chosen to be nonvanishing near any point in Ω at our disposal. If fact, it suffices to take ℓ and d which are nonvanishing near the probe front $\{|x-x_0| = t\}$. Different choices of ℓ and d will give rise to different Dirichlet data $f_{h,t}$ and therefore different measurements.*
(3). *In real applications, we believe that the concerns in* (ii) *and* (iii) *can be ignored.*

Taking advantage of the decay of the complex spherical waves in the region $\{|x-x_0|>t\}$, we can localize the measurements, which is of great practical value. Let $\phi_{\delta,t}(x) \in C_0^\infty(\mathbb{R}^3)$ satisfy

$$\phi_{\delta,t}(x) = \begin{cases} 1 & \text{on} \quad B_{t+\delta/2}(x_0) \\ 0 & \text{on} \quad \mathbb{R}^3 \setminus \overline{B_{t+\delta}(x_0)} \end{cases}$$

where $\delta > 0$ is sufficiently small. Now we are going to use the measurements $f_{\delta,h,t} = \phi_{\delta,t} f_{h,t} = \phi_{\delta,t} u_{h,t}|_{\partial\Omega}$. Clearly, the measurements $f_{\delta,h,t}$ are localized on $B_{t+\delta}(x_0) \cap \partial\Omega$. Let us define

$$E_\delta(h,t) = | < (\Lambda_D - \Lambda_0) f_{\delta,h,t}, \overline{f_{\delta,h,t}} > |.$$

Theorem 4.2 *The statements of Theorem 4.1 are valid for* $E_\delta(h,t)$.

Proof. The main idea is to prove that the error caused by the remaining part of the measurement $(1-\phi_{\delta,t})f_{h,t} =: g_{\delta,h,t}$ is exponentially small. Let $w_{\delta,h,t}$ be the solution of (4.3) with boundary value $g_{\delta,h,t}$. We now want to compare $w_{\delta,h,t}$ with $(1-\phi_{\delta,t})u_{h,t}$. To this end, we first observe that

$$\begin{cases} \mathcal{L}_0((1-\phi_{\delta,t})u_{h,t} - w_{\delta,h,t}) = \mathcal{L}_0((1-\phi_{\delta,t})u_{h,t}) \\ (1-\phi_{\delta,t})u_{h,t} - w_{\delta,h,t} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Since

$$\|\mathcal{L}_0((1-\phi_{\delta,t})u_{h,t})\|_{L^2(\Omega)} \le C\beta^{1/h}$$

for some $0 < \beta < 1$, we have that

$$\|(1-\phi_{\delta,t})u_{h,t} - w_{\delta,h,t}\|_{H^1(\Omega)} \le C\beta^{1/h}. \tag{4.13}$$

Using (4.4) for $< (\Lambda_D - \Lambda_0) g_{\delta,h,t}, \overline{g_{\delta,h,t}} >$ with u_0 being replaced by $w_{\delta,h,t}$, we get from (4.13) and decaying property of $u_{h,t}$ that

$$| < (\Lambda_D - \Lambda_0) g_{\delta,h,t}, \overline{g_{\delta,h,t}} > | \le C\widetilde{\beta}^{1/h}$$

for some $0 < \widetilde{\beta} < 1$.

Now we first consider (i) of Theorem 4.1 for $E_\delta(h,t)$. We begin with the case $(\mu+)$ of (4.2). In view of the first inequality of (4.4), we see that

$$0 \le < (\Lambda_D - \Lambda_0)(\zeta f_{\delta,h,t} \pm \zeta^{-1} g_{\delta,h,t}), \overline{\zeta f_{\delta,h,t} \pm \zeta^{-1} g_{\delta,h,t}} >$$

for any $\zeta > 0$, which leads to

$$\begin{aligned} & | < (\Lambda_D - \Lambda_0) f_{\delta,h,t}, \overline{g_{\delta,h,t}} > + < (\Lambda_D - \Lambda_0) g_{\delta,h,t}, \overline{f_{\delta,h,t}} > | \\ & \le \zeta^2 < (\Lambda_D - \Lambda_0) f_{\delta,h,t}, \overline{f_{\delta,h,t}} > + \zeta^{-2} < (\Lambda_D - \Lambda_0) g_{\delta,h,t}, \overline{g_{\delta,h,t}} > . \end{aligned} \tag{4.14}$$

It now follows from $f_{h,t} = f_{\delta,h,t} + g_{\delta,h,t}$ and (4.14) with $\zeta = 1/\sqrt{2}$ that

$$\begin{aligned} & \tfrac{1}{2} < (\Lambda_D - \Lambda_0) f_{\delta,h,t}, \overline{f_{\delta,h,t}} > \\ & \le < (\Lambda_D - \Lambda_0) g_{\delta,h,t}, \overline{g_{\delta,h,t}} > + < (\Lambda_D - \Lambda_0) f_{h,t}, \overline{f_{h,t}} > \\ & \le C\widetilde{\beta}^{1/h} + < (\Lambda_D - \Lambda_0) f_{h,t}, \overline{f_{h,t}} > . \end{aligned} \tag{4.15}$$

So from (i) of Theorem 4.1, the same statement holds for $E_\delta(h,t)$. Other cases of (4.2) are treated similarly.

Next we consider (ii) and (iii) of Theorem 4.1 for $E_{\delta,h,t}$. As before, we only treat ($\mu+$) of (4.2). Choosing $\zeta = 1$ in (4.14) we get that

$$\begin{aligned}&\tfrac{1}{2} < (\Lambda_D - \Lambda_0) f_{h,t}, \overline{f_{h,t}} > \\ &\leq < (\Lambda_D - \Lambda_0) g_{\delta,h,t}, \overline{g_{\delta,h,t}} > + < (\Lambda_D - \Lambda_0) f_{\delta,h,t}, \overline{f_{\delta,h,t}} > \\ &\leq C\widetilde{\beta}^{1/h} + < (\Lambda_D - \Lambda_0) f_{\delta,h,t}, \overline{f_{\delta,h,t}} > .\end{aligned} \tag{4.16}$$

Therefore, (ii) of Theorem 4.1 and (4.16) implies that the same fact is true for $E_\delta(h,t)$. Finally, combining (4.15) and (4.16) yields the statement (iii) for $E_\delta(h,t)$. The proof is now complete. □

Remark 4.2 *With the help of Theorem 4.2, when parts of ∂D are near the boundary $\partial\Omega$, it is possible to detect some points of ∂D from only a few measurements taken from a very small region of $\partial\Omega$.*

To end this section, we provide an algorithm of the method.

Step 1. Pick a point x_0 near ch(Ω). Construct complex spherical waves $u_{h,t}$.

Step 2. Draw two balls $B_t(x_0)$ and $B_{t+\delta}(x_0)$. Set the Dirichlet data $f_{\delta,h,t} = \phi_{\delta,t} u_{h,t}|_{\partial\Omega}$. Measure the Neumann data $\Lambda_D f_{\delta,h,t}$ over the region $B_{t+\delta}(x_0) \cap \partial\Omega$.

Step 3. Calculate $E_\delta(h,t) =< (\Lambda_D - \Lambda_0) f_{\delta,h,t}, f_{\delta,h,t} >$. If $E(h,t)$ tends to zero as $h \to 0$, then the probing front $\{|x - x_0| = t\}$ does not intersect the inclusion. Increase t and compute $E_\delta(h,t)$ again.

Step 4. If $E_{\delta,h,t}$ increases to ∞ as $h \to 0$, then the front $\{|x - x_0| = t\}$ intersects the inclusion. Decrease t to make more accurate estimate of ∂D.

5 Conclusion

In this work we have constructed complex spherical waves or complex geometrical optics solutions for the elasticity system with isotropic inhomogeneous medium. We used these special solutions to investigate the inverse problem of identifying inclusions with localized measurements. Numerical realization of this method would be an interesting project. The same method should work for identifying cavities.

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