Stability estimates for the X-ray transform of tensor fields and boundary rigidity

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Abstract

We study the boundary rigidity problem for domains in $\mathbf{R}^n$: is a Riemannian metric uniquely determined, up to an action of diffeomorphism fixing the boundary, by the distance function $\rho_g(x,y)$ known for all boundary points x and y? It was conjectured by Michel that this was true for simple metrics. In this paper, we study the linearized problem first which consists of determining a symmetric 2-tensor, up to a potential term, from its geodesic X-ray integral transform I_g. We prove that the normal operator $N_g = I_g^{\mathrm{T}} I_g$ is a pseudodifferential operator provided that g is simple, find its principal symbol, identify its kernel, and construct a microlocal parametrix. We prove hypoelliptic type of stability estimate related to the linear problem. Next we apply this estimate to show that unique solvability of the linear problem for a given simple metric g, up to potential terms, implies local uniqueness for the non-linear boundary rigidity problem near that g.

1 Introduction

Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with smooth boundary $\partial\Omega$ and let $g = \{g_{ij}\}$ be a Riemannian metric in $\bar{\Omega}$. Denote by ρ_g the boundary distance function which measures the geodesic distance between boundary points. We consider the inverse problem of whether $\rho_g(x,y)$, known for all x, y on $\partial\Omega$, determines the metric uniquely. This problem arose in geophysics in an attempt to determine the inner structure of the Earth by measuring the travel times of seismic waves. It goes back to Herglotz [H] and Wiechert and Zoeppritz[WZ]. Although the emphasis has been in the case that the medium is isotropic, the anisotropic case has received recent interest since it has been found that the inner core of the Earth exhibits anisotropic behavior [Cr]. In differential geometry this inverse problem has been studied because of rigidity questions and is known as the boundary rigidity problem. It is clear that one cannot determine the metric uniquely. Any isometry which is the identity at the boundary will give rise to the same measurements. Furthermore the boundary distance function only takes into account the travel times of the shortest geodesics and it is easy to find counterexamples to unique determination, so one needs to pose some restrictions on the metric. Michel [Mi], conjectured that a *simple* metric g is uniquely determined, up to an action of a diffeomorphism fixing the boundary, by the boundary distance function $\rho_g(x,y)$ known for all x and y on $\partial\Omega$. Loosely speaking, the metric g is called simple in Ω, if every two points x, y in $\bar{\Omega}$ can be connected by unique minimizing geodesics that depends smoothly on x and y, and Ω is strictly convex w.r.t. g. Such a metric can be extended as a simple one in some neighborhood of Ω.

Unique recovery of g (up to an action of a diffeomorphism) is known for simple metrics conformal to the Euclidean one [Mu1], [Mu2], [Mu-R], [BG], for flat metrics [Gr] and for metrics with negative curvature in

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two dimensions, see [C1], [O]. In [S-U], the authors proved this for metrics in a small neighborhood of the Euclidean one. This result was used in [LSU] to prove a semiglobal solvability result.

It is known [Sh1], that a linearization of the boundary rigidity problem near a simple metric g is given by the following integral geometry problem: show that if for a symmetric tensor of order 2, the geodesic X-ray transform

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt$$

vanishes for all geodesics γ in Ω, then $f = d^s v$ for some vector field v with $v|_{\partial\Omega} = 0$, where the symmetric differential d^s is defined below. We will refer to this property as *s-injectivity* of I_g. On the other hand, it is easy to see that $I_g d^s v = 0$ for any such v. This is the linear version of the fact that the ρ_g does not change on $\partial\Omega^2$ under an action of diffeomorphism as above. S-injectivity of I_g was proved in [PS] for metrics with negative curvature, in [Sh1] for metrics with small curvature and in [Sh-U] for Riemannian surfaces with no focal points. A conditional and non-sharp stability estimate is also established in [Sh1], see (2) in next section. This estimate was used in [CDS], [E] to get local uniqueness results for the boundary rigidity problem.

In this paper we consider a microlocal approach to the study of the linear geodesic X-ray transform for tensor fields and the non-linear boundary rigidity problem. The use of microlocal techniques in integral geometry goes back to Guillemin and Sternberg [GS]. We prove that the normal operator $N_g = I_g^{\mathrm{T}} I_g$, where I_g^{T} stands for the transpose of I_g, is a pseudo-differential operator of order -1, compute its principal symbol and identify its kernel. As a consequence, we construct a parametrix for N_g that allows to reconstruct the solenoidal part f^s up to smoothing operators and in section 6 we derive a stability estimate. The estimate is of hypoelliptic type with loss of one derivative. As a byproduct of our analysis of the linear problem, we prove sharp estimates about recovery of a 1-form $f = f_j dx^j$ and a function f from the associated $I_g f$ in sections 7 and 8. Finally, in section 9, we apply the results about I_g to prove local uniqueness for the boundary rigidity problem near any simple metric g with s-injective I_g.

2 Preliminaries

In this section we introduce some notation and recall some facts about integral geometry of tensors [Sh1]. Assume that g is a smooth Riemannian metric in the domain Ω with smooth boundary. We assume that g is simple in Ω, i.e., that Ω is strictly convex w.r.t. the metric g, and for any $x \in \bar\Omega$ the exponential map $\exp_x : \exp_x^{-1}(\bar\Omega) \to \bar\Omega$ is a diffeomorphism. We use the usual convention of raising and lowering indices and we will not make difference between covariant and contravariant tensors by considering them to be two representations of the same tensor. We will work with symmetric tensors only and we always consider them extended as 0 to $\mathbf{R}^n \setminus \Omega$. Everywhere in this paper, for $(x,\xi) \in T^*\mathbf{R}^n$, we denote $|x|^2 = g_{ij}x^i x^j$ and $|\xi|^2 = g^{ij}\xi_i\xi_j$.

We are going to work in the space $L^2(\Omega)$, and associated H^s spaces, of symmetric tensors $f = \{f_{ij}\}$ with inner product

$$(f,h) = \int_\Omega f_{ij}\bar h_{i'j'} g^{i'i} g^{j'j} (\det g)^{1/2} dx = \int_\Omega f_{ij}\bar h^{ij} (\det g)^{1/2} dx.$$

Given a symmetric 2-tensor $f = f_{ij}$, we define the 1-tensor $\delta^s f$ called *divergence* of f by

$$[\delta^s f]_i = g^{jk}\nabla_k f_{ij},$$

where ∇_i are the covariant derivatives. Given a 1-tensor (vector field) v, we denote by $d^s v$ the 2-tensor called symmetric differential of v:

$$[d^s v]_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right).$$

Operators d^s and $-\delta^s$ are formally adjoint to each other in $L^2(\Omega)$. It is easy to see that for each smooth v with $v = 0$ on $\partial\Omega$, we have $I_g(d^s v) = 0$. The natural conjecture is that $I_g f = 0$ implies $f = d^s v$ with some v vanishing on $\partial\Omega$ that we call s-injectivity.

It is known that each symmetric tensor f belonging to $L^2(\Omega)$ admits unique orthogonal decomposition $f = f^s + d^s v$ into a solenoidal part $\mathcal{S}f = f^s$ and a potential part $\mathcal{P}f = d^s v$, such that both terms are in $L^2(\Omega)$, f^s is solenoidal, i.e., $\delta^s f^s = 0$ in Ω, and $v \in H_0^1(\Omega)$ (i.e., $v = 0$ on $\partial\Omega$). In order to construct this decomposition, introduce the operator $\Delta^s = \delta^s d^s$ acting on tensors. This operator is elliptic in Ω, and the Dirichlet problem satisfies the Lopatinskii condition. Denote by Δ_D^s the Dirichlet realization of Δ^s in Ω. Then

$$v = \left(\Delta_D^s\right)^{-1} \delta^s f, \quad f^s = f - d^s \left(\Delta_D^s\right)^{-1} \delta^s f. \tag{1}$$

Operators $\mathcal{S}$ and $\mathcal{P}$ are orthogonal projectors. The problem about the s-injectivity of I_g then can be posed as follows: if $I_g f = 0$, show that $f^s = 0$, in other words, show that I_g is injective on the subspace $\mathcal{S}L^2$ of solenoidal tensors.

As mentioned in the Introduction, s-injectivity of I_g was proven by V. Sharafutdinov [Sh1] for metrics g with an explicit upper bound of the curvature which in particular includes metrics with negative curvature, see also [PS]. The method in [Sh1] is based on energy estimates in the spirit of Mukhometov's result in two dimensions and the s-injectivity result is a consequence of the following estimate:

$$\|f^s\|^2_{L^2(\Omega)} \quad \leq \quad C\Big(\|j_\nu f|_{\partial\Omega}\|_{L^2(\partial\Omega)} \, \|I_g f\|_{L^2(\Gamma_-)} + \|I_g f\|^2_{H^1(\Gamma_-)} \Big), \tag{2}$$

where Γ_- is defined below and the measure on Γ_- is $dS_z\, dS_\omega$ (see below), i.e., compared to $d\mu$, the factor $|\omega \cdot \nu|$ is not present. The term $j_\nu f$ is defined as $[j_\nu f]_j = f_{ij}\nu^i$, where ν is the unit normal to $\partial\Omega$. The map $I_g : H^s(\Omega) \to H^s(\Gamma_-)$ is bounded for any integer $s \geq 0$ and even though the estimate above implies s-injectivity, the stability for $f \in L^2(\Omega)$ is of conditional type because it requires an a priori estimate of the H^1-norm of f. One of the goals of this work is to prove an estimate of more conventional type.

3 Integral representation of N_g

Consider the Hamiltonian $H_g(x,\xi) = \frac{1}{2} g^{ij}\xi_i\xi_j$ and denote by $\Phi_g(t)$ the corresponding Hamiltonian flow. We will denote by $(x(t), \xi(t))$ the corresponding integral curves of H_g (bicharacteristics of the associated Laplace-Beltrami operator) on the energy level $H_g = 1/2$. We are going to use the following parameterization of those bicharacteristics. Denote

$$\Gamma_- := \{(z,\omega) \in T^*\Omega;\ z \in \partial\Omega,\ |\omega| = 1,\ \omega \cdot \nu(z) \leq 0\},$$

where $\nu(z)$ is the outer unit normal to $\partial\Omega$, $|\omega|^2 = g^{ij}\omega_i\omega_j$, and $\omega \cdot \nu = \omega_i \nu^i$. Introduce the measure

$$d\mu(z,\omega) = |\omega \cdot \nu(z)| dS_z dS_\omega \quad \text{on } \Gamma_-,$$

where dS_z and dS_ω are the surface measures on $\partial\Omega$ and $\{\omega \in T_x^*\Omega;\ |\omega| = 1\}$ in the metric, respectively. If $\partial\Omega$ is given locally by $x^n = 0$, then $dS_z = (\det g)^{1/2} dx^1 \dots dx^{n-1}$, and $dS_\omega = (\det g)^{-1/2} dS_{\omega_0}$, where dS_{ω_0} is the Euclidean measure on S^{n-1}. Define $(x(t), \xi(t)) = (x(t; z, \omega), \xi(t; z, \omega))$ to be the bicharacteristic issued from $(z,\omega) \in \Gamma_-$.

Let $\alpha(x,\xi)$ be a smooth weight function. We define the X-ray transform $I_g f$ of f more generally as weighted integrals of $f^{ij}\xi_i\xi_j$ over all bicharacteristics of H on the level $H = 1/2$, i.e.,

$$(I_g f)(z,\omega) = \int \alpha(x(t),\xi(t)) f^{ij}(x(t))\xi_i(t)\xi_j(t)\, dt, \quad (z,\omega) \in \Gamma_-, \tag{3}$$

where $((x(t), \xi(t)) = (x(t; z, \omega), \xi(t; z, \omega))$ as above is the maximal bicharacteristic in Ω issued from (z,ω).

Notice that if we regard (3) as integrals over the x-projections of the bicharacteristics (the geodesics) with $\xi_i = g_{ij}\dot{x}^j$, then we integrate over each geodesic twice — once in each direction. Moreover, t is the arc-length parameter.

Clearly, $I_g f \in L^2(\Gamma_-; d\mu)$ for smooth f. Moreover, $I_g : L^2(\Omega) \to L^2(\Gamma_-;\ d\mu)$ is bounded [Sh1]. Below we find a representation for $N_g = I_g^{\mathrm{T}} I_g$. Recall that $\rho(x,y)$ is the distance function.

Proposition 1 *For any symmetric 2-tensor $f \in C(\Omega)$ we have*

$$(N_g f)_{kl}(x) = \frac{1}{\sqrt{\det g}} \int A(x,y) \frac{f^{ij}(y)}{\rho(x,y)^{n-1}} \frac{\partial \rho}{\partial y^i} \frac{\partial \rho}{\partial y^j} \frac{\partial \rho}{\partial x^k} \frac{\partial \rho}{\partial x^l} \left| \det \frac{\partial^2(\rho^2/2)}{\partial x \partial y} \right| dy, \quad x \in \Omega, \tag{4}$$

with

$$A(x,y) = \bar{\alpha}(x, -\nabla_x \rho(x,y))\alpha(y, \nabla_y \rho(x,y)) + \bar{\alpha}(x, \nabla_x \rho(x,y))\alpha(y, -\nabla_y \rho(x,y)). \tag{5}$$

Proof. Pick another smooth tensor h supported in Ω. We have

$$\begin{aligned}
(I_g f, I_g h) &= \int_{\Gamma_-} (I_g f)(z,\omega)(I_g h)(z,\omega)\, d\mu(z,\omega) \\
&= \int_{\Gamma_-} \Big[\int \alpha(x(t),\xi(t)) f^{ij}(x(t))\xi_i(t)\xi_j(t)\alpha(t) dt \\
&\qquad \times \int \bar{\alpha}(x(s),\xi(s))\bar{h}^{kl}(x(s))\xi_k(s)\xi_l(s) ds \Big]\, d\mu(z,\omega) \\
&= I_+ + I_-,
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
I_\pm &= \int_{\Gamma_-} \int \int_0^\infty \alpha(x(s\pm t),\xi(s\pm t)) f^{ij}(x(s\pm t))\xi_i(s\pm t)\xi_j(s\pm t) \\
&\qquad \times \bar{\alpha}(x(s),\xi(s))\bar{h}^{kl}(x(s))\xi_k(s)\xi_l(s)\alpha(s)\, dt\, ds\, d\mu(z,\omega).
\end{aligned}$$

Here the bicharacteristics are parameterized by (z,ω) as above and all functions are assumed to be extended as 0 outside $\bar{\Omega}$. Notice that for any ω with $|\omega| = 1$, (z,s) are global coordinates in Ω. Here $z \in \partial\Omega$ is such that $(z,\omega) \in \Gamma_-$ and $s > 0$. Next, the Jacobian of the change of variables $(z,s) \mapsto x$ is $|\omega \cdot \nu(z)|$ on the boundary thus $dx = |\omega \cdot \nu(z)| dz\, ds$ there. Introduce new variable $\xi = t\omega$ on the boundary. Then at the boundary, we can pass to variables (x,ξ) and $dx\, d\xi = t^{n-1} dt\, ds\, d\mu(z,\omega)$. Since the Hamiltonian flow preserves the measure, we have the same in the domain Ω, i.e. for any (x,ξ). Set $x = x(s)$, $\xi/|\xi| = \xi(s)$, where $|\xi|$ is the length of the covector ξ in the metric g. Then $t = |\xi|$ and $(x(s+t),\xi(s+t)) = \Phi(1)(x,\xi)$. Therefore, $x(s+t) = \exp_x \xi =: y$. It is fairly easy to see that $\xi(s+t) = \nabla_y \rho(x,y)$. We treat I_- in the same way. We get

$$(N_g f, h) = \int\int A(x,y) f^{ij}(\exp_x \xi) \frac{\partial \rho}{\partial y^i} \frac{\partial \rho}{\partial y^j} \bar{h}^{kl}(x) \frac{\xi_k}{|\xi|} \frac{\xi_l}{|\xi|} \frac{d\xi}{|\xi|_x^{n-1}}\, dx, \tag{7}$$

where $\rho = \rho(x,y)$, $y = \exp_x \xi$ and A is given by

$$A(x,y) = \bar{\alpha}(x,\xi)\alpha(y, \nabla_y \rho(x,y)) + \bar{\alpha}(x,-\xi)\alpha(y, -\nabla_y \rho(x,y)). \tag{8}$$

Let us perform the change $y = \exp_x \xi$ in that integral. Since $x \in \operatorname{supp} f$, $y \in \operatorname{supp} h$, this map is a diffeomorphism by assumption. In the same way as before we get $\xi/|\xi| = -\nabla_x \rho(x,y)$ and $\xi = -\frac{1}{2}\nabla_x \rho^2(x,y)$. Thus, the Jacobian $|\det(d\xi/dy)|$ is $\frac{1}{2}|\det(\partial^2 \rho^2/\partial x \partial y)|$ and this completes the proof of the proposition. $\square$

Observe that if we extend g smoothly into a small strictly convex neighborhood Ω_1 of Ω as a simple metric, and $\operatorname{supp} f \subset \Omega$, then $N_g f$ remains the same for $x \in \Omega$ and is defined for $x \in \Omega_1$. We will use this in next sections.

From now on we assume that $\alpha = 1$.

4 The Euclidean case

In this section we explicitly compute the normal operator and the parametrix in the Euclidean case. Several of the calculations below can be found in [Sh1] for $g = e = \{\delta_{ij}\}$ and can be easily generalized to constant g by transforming g into e, for example by the symplectic transform $y = g^{1/2}x$, $\eta = g^{-1/2}\xi$, then $ds^2 = \sum (dy^i)^2$.

Let g be a constant coefficients metric. Then we parameterize the geodesics (lines) by the direction ξ and by the point z on the hyperplane $z^i\xi_i = 0$ where the line crosses that hyperplane. Then

$$I_g f(z,\xi) = \int f_{ij}(z+t\xi)\xi^i\xi^j\, dt.$$

Here f is viewed as a function on the whole $\mathbf{R}^n$, extended as 0 outside Ω. Any $f \in L^2(\Omega)$ can then be orthogonally decomposed uniquely into a solenoidal and potential part (different from the decomposition above!)

$$f = f^s_{\mathbf{R}^n} + d^s v_{\mathbf{R}^n} \quad \text{in } \mathbf{R}^n,$$

such that $\delta^s f^s_{\mathbf{R}^n} = 0$ in $\mathbf{R}^n$ and $f^s_{\mathbf{R}^n}$, $dv_{\mathbf{R}^n}$ are in $L^2(\mathbf{R}^n)$. Similarly to (1), we have

$$v_{\mathbf{R}^n} = \left(\Delta^s\right)^{-1}\delta^s f, \quad f^s_{\mathbf{R}^n} = f - d^s\left(\Delta^s\right)^{-1}\delta^s f, \tag{9}$$

with $\Delta^s = \delta^s d^s$ acting in the whole $\mathbf{R}^n$, and the notation $v_{\mathbf{R}^n}$ indicates that v does not necessarily satisfy boundary conditions. A more detailed form of this decomposition can be explicitly done by means of Fourier transform. We have

$$(\widehat{f}^s_{\mathbf{R}^n})_{kl} = \lambda^{ij}_{kl}(\xi)\widehat{f}_{ij}(\xi), \tag{10}$$

where

$$\lambda^{ij}_{kl}(\xi) = \left(\delta^i_k - \frac{\xi_k\xi^i}{|\xi|^2}\right)\left(\delta^j_l - \frac{\xi_l\xi^j}{|\xi|^2}\right). \tag{11}$$

It is important to note that in general, $f^s_{\mathbf{R}^n}$ and $dv_{\mathbf{R}^n}$ are not compactly supported anymore. It follows from section 3 that for $f \in C_0^\infty$,

$$(N_e f)^{kl}(x) = 2f_{ij} * \frac{x^i x^j x^k x^l}{|x|^{n+3}}. \tag{12}$$

Taking into account that $\mathcal{F}|x|^\alpha = (c_n/2)(\det g)^{-1/2}|\xi|^{-\alpha-n}$ with c_n as below, and Fourier transforming the latter, we get

$$\mathcal{F}(N_e f)^{kl} = c_n \hat{f}_{ij}\frac{\partial^4}{\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l}|\xi|^3, \quad c_n = \frac{\pi^{(n+1)/2}}{3\Gamma(n/2+3/2)}, \tag{13}$$

and

$$\partial^4|\xi|^3/\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l = 3|\xi|^{-1}\sigma(\varepsilon^{ij}\varepsilon^{kl}), \quad \varepsilon^{ij}(\xi) = \delta^{ij} - \xi^i\xi^j/|\xi|^2. \tag{14}$$

Here $\sigma(\varepsilon^{ij}\varepsilon^{kl})$ is the symmetrization of $\varepsilon^{ij}\varepsilon^{kl}$, i.e., the mean of all similar products with all possible permutation of i,j,k,l, see [Sh1]. It is easy to see that $\delta^s N_e f = 0$ and that $f^s_{\mathbf{R}^n}$ can be recovered from $N_e f$ by the formula

$$[\widehat{f}^s_{\mathbf{R}^n}]_{ij} = \left(\delta^{kl}_{ij} - \lambda^{kl}_{ij}\right)\hat{f}_{kl} = a_{ijkl}\mathcal{F}(N_e f)^{kl} = a^{kl}_{ij}\mathcal{F}(N_e f)_{kl}, \tag{15}$$

where $a_{ijkl}(\xi)$ is a rational function, homogeneous of order 1 singular only at $\xi = 0$ with explicit form

$$a_{ijkl} = |\xi|\left(c_1\delta_{ik}\delta_{jl} + c_2(\delta_{ij} - |\xi|^{-2}\xi_i\xi_j)\delta_{kl}\right). \tag{16}$$

The coefficients c_1 and c_2 depend on n only [Sh1]. This immediately implies that $I_e f = 0 \Longrightarrow f^s_{\mathbf{R}^n} = 0 \Longrightarrow f = d^s v_{\mathbf{R}^n}$. Moreover, in this case, if f has compact support, so does $v_{\mathbf{R}^n}$, and in particular, if f vanishes outside (the convex) Ω, so does $v_{\mathbf{R}^n}$. This proves s-injectivity of I_g for $g = e$.

We would like to explicitly emphasize here that the decomposition of f in the whole $\mathbf{R}^n$ (in case $g = \text{const.}$) described in this section is different than the one in Ω described in section 2. Even if $g = e$, formulas (1) and (9) differ by the fact that the latter involves the resolvent $(\Delta^s)^{-1}$ in the whole space while (1) involves the solution of a boundary value problem $\Delta^s v = \delta^s f$ in Ω, $v = 0$ on $\partial\Omega$.

5 N_g as a ΨDO and construction of parametrix for N_g

In this section, we show that N_g is a ΨDO and construct a parametrix of order 1. In next section, we refine this parametrix to infinite order.

Lemma 1 *For x close to y we have*

$$\begin{aligned} \rho^2(x,y) &= G^{(1)}_{ij}(x,y)(x-y)^i(x-y)^j, \\ \frac{\partial \rho^2(x,y)}{\partial x^j} &= 2G^{(2)}_{ij}(x,y)(x-y)^i, \\ \frac{\partial^2 \rho^2(x,y)}{\partial x^j \partial y^j} &= 2G^{(3)}_{ij}(x,y), \end{aligned}$$

where $G^{(1)}_{ij}$, $G^{(2)}_{ij}$ $G^{(3)}_{ij}$ are smooth and on the diagonal we have

$$G^{(1)}_{ij}(x,x) = G^{(2)}_{ij}(x,x) = G^{(3)}_{ij}(x,x) = g_{ij}(x).$$

Proof. Choose the covector ξ so that $y = \exp_x \xi$. Then $\rho^2(x,y) = |\xi|^2 = g^{-1}\xi \cdot \xi$. From the Hamiltonian system we get $y - x = g^{-1}(x)\xi + O(|\xi|^2)$, thus $\xi = g(x)(y-x) + O(|y-x|^2)$. This yields the first formula. The second and the third one follow by differentiation. $\square$

We will show now that N_g is a ΨDO of order -1 and we will compute the principal symbol of this operator. Note that N_g is an integral operator with kernel $K(x,y)$ having a weak singularity of the kind $|x-y|^{-n+1}$ at the diagonal. Therefore, it is a ΨDO of order -1. More precisely,

$$K_{ijkl} = 2\frac{\left[G^{(2)}(x-y)\right]_i \left[G^{(2)}(x-y)\right]_j \left[\tilde{G}^{(2)}(x-y)\right]_k \left[\tilde{G}^{(2)}(x-y)\right]_l}{\left(G^{(1)}(x-y)\cdot(x-y)\right)^{\frac{n-1}{2}+2}} |\det G^{(3)}|$$

with $\tilde{G}^{(2)}_{ij}(x,y) = G^{(2)}_{ij}(y,x)$ and $G(x-y) = G(x,y)(x-y)$ stands for multiplication of the matrix G and the vector $x-y$. Denote $z := x-y$. Then

$$\begin{aligned} K_{ijkl} &= 2\left(G^{(1)}z\cdot z\right)^{\frac{-n+1}{2}-2}\left[G^{(2)}z\right]_i\left[G^{(2)}z\right]_j\left[\tilde{G}^{(2)}z\right]_k\left[\tilde{G}^{(2)}z\right]_l |\det G^{(3)}| \\ &= \hat{M}_{ijkl}(x,y,x-y) \end{aligned}$$

with

$$\begin{aligned} M_{ijkl}(x,y,\xi) &= 2(2\pi)^{-n}\int e^{i\xi\cdot z}\left(G^{(1)}z\cdot z\right)^{\frac{-n+1}{2}-2} \\ &\qquad \times \left[G^{(2)}z\right]_i\left[G^{(2)}z\right]_j\left[\tilde{G}^{(2)}z\right]_k\left[\tilde{G}^{(2)}z\right]_l \frac{|\det G^{(3)}|}{\sqrt{\det g}}\, dz. \end{aligned} \tag{17}$$

Therefore, N_g is a ΨDO with amplitude $M(x,y,\xi)$. Note that the integrand above belongs locally to L^1. Clearly, M is homogeneous in ξ of order -1 and therefore has singularity at $\xi = 0$. This is integrable singularity however, so we can cut M near the origin and this would give rise to a bounded smoothing operator in L^2. In order to get the principal symbol of M, it is enough to set $y = x$, thus by Lemma 1 we formally replace G, $G^{(1)}$, $G^{(2)}$, $\tilde{G}^{(2)}$, $G^{(3)}$ in (17) by $g(x)$ to get

$$\begin{aligned} \sigma_p(N_g)_{ijkl}(x,\xi) &= M_{ijkl}(x,x,\xi) \\ &= 2(2\pi)^{-n}\sqrt{\det g}\int e^{i\xi\cdot z}|gz\cdot z|^{-\frac{n+3}{2}}[gz]_i[gz]_j[gz]_k[gz]_l\, dz. \end{aligned}$$

Recall that $[gz]_i = g_{ij}z^j = z_i$. Therefore,

$$\begin{aligned}\sigma_p(N_g)^{ijkl}(x,\xi) &= M^{ijkl}(x,x,\xi)\\ &= 2(2\pi)^{-n}\sqrt{\det g}\int e^{i\xi\cdot z}|g(x)z\cdot z|^{-\frac{n+3}{2}}z^iz^jz^kz^l\,dz.\end{aligned}$$

Notice that in the right hand side above, for any fixed x, we got exactly the symbol of N_g in the case when the metric g has constant coefficients, see (14). Thus we have proved the following.

Proposition 2 *The principal symbol of N_g is given by*

$$\sigma_p(N_g)^{ijkl}(x,\xi) = c_n\,|\xi|^{-1}\sigma(\varepsilon^{ij}\varepsilon^{kl}), \quad \varepsilon^{ij} = \delta^{ij} - \xi^i\xi^j/|\xi|^2.$$

Let g be a simple metric in Ω. Extend g near Ω and let

$$\Omega_1 = \Omega_0 \cup \{0 \le \mathrm{dist}(x,\partial\Omega_0) < \varepsilon\}.$$

For $\varepsilon > 0$ small enough, Ω_1 is strictly convex as well and g is simple near Ω_1. We will work with f supported in $\bar\Omega$. We assume that they are extended as 0 outside Ω. Choose a smooth function χ supported in Ω_1 such that $\chi = 1$ near Ω. Inspired by (15), we start constructing a parametrix for f^s by the formula

$$(Bf)_{ij} = \chi a_{ijkl}(x,D)\chi(N_gf)^{kl}, \tag{18}$$

where $a_{ijkl}(x,\xi)$ are defined by (16). We will first show below that Bf is a parametrix for $f^s_{\Omega_1}$, the solenoidal part of f in Ω_1, in the sense that

$$L^2(\Omega) \ni f \longmapsto f^s_{\Omega_1} - Bf \in L^2(\Omega_1) \quad \text{is a compact operator.} \tag{19}$$

By (15), $a_{kli'j'}[\sigma_p(N_g)]^{i'j'ij} = \lambda^{ij}_{kl} =: \Lambda_0$ in the case of constant g. Therefore, by (18),

$$Bf = \chi\left(\Lambda_0(x,D)f + R_{-1}f\right),$$

where R_{-1} is a ΨDO of order -1. In view of (1), our compactness claim will be proved, if we show that

$$\chi\Lambda_0(x,D) - \left(Id - d^s(\Delta^s_{\Omega_1,D})^{-1}\delta^s\right) : L^2(\Omega) \longrightarrow L^2(\Omega_1) \tag{20}$$

is compact. Above, $L^2(\Omega)$ is considered as a subspace of $L^2(\Omega_1)$, and $\Delta^s_{\Omega_1,D}$ stands for the Dirichlet realization of Δ^s in Ω_1. To prove that, we are going to use the fact that Λ_0 is equal to the principal symbol of $Id - d^s(\Delta^s_{\Omega_1,D})^{-1}\delta^s$ inside Ω_1, if $(\Delta^s_{\Omega_1,D})^{-1}$ is replaced by any parametrix near Ω. Next, replacing $(\Delta^s_{\Omega_1,D})^{-1}$ by a parametrix results in an "error" given by a compact operator when we work with f with $\mathrm{supp}\, f \subset \Omega$.

More precisely, note first that the principal symbols of δ^s and d^s are given by

$$\left(\sigma_p(\delta^s)\hat f\right)_j = \xi^i\hat f_{ij}, \quad \left(\sigma_p(d^s)\hat v\right)_{ij} = \frac{1}{2}\left(\xi_j\hat v_i + \xi_i\hat v_j\right).$$

A straightforward calculation shows that

$$\left(\sigma_p(\Delta^s)\hat v\right)_i = \frac{1}{2}\left(|\xi|^2\delta^j_i + \xi_i\xi^j\right)\hat v_j, \quad \left(\sigma_p(\Delta^s)^{-1}\hat v\right)_i = \frac{1}{|\xi^2|}\left(2\delta^j_i - \frac{\xi_i\xi^j}{|\xi|^2}\right)\hat v_j.$$

Therefore, for $\Lambda_0 = \lambda^{ij}_{kl}$ defined originally by (11), we get

$$\Lambda_0 = \sigma_p\,(Id) - \sigma_p(d^s)\sigma_p(\Delta^s)^{-1}\sigma_p(\delta^s), \tag{21}$$

which confirms that Λ_0 is the principal symbol of the projection onto the subspace of solenoidal tensors (if we replace $(\Delta^s_D)^{-1}$ by $(\Delta^s)^{-1}$) not only in the Euclidean case . Next, for $u = \left(\sigma_p(\Delta^s)^{-1}(x,D) - (\Delta^s_{\Omega_1,D})^{-1}\right)\delta^s f$ we have

$$\begin{cases} \Delta^s u &= Kf \quad \text{in } \Omega_1, \\ u|_{\partial\Omega_1} &= \sigma_p(\Delta^s)^{-1}(x,D)\delta^s f|_{\partial\Omega_1}, \end{cases} \tag{22}$$

where K is of order 0. Assume now that $\operatorname{supp} f \subset \bar\Omega$. Then the map $f \mapsto \sigma_p(\Delta^s)^{-1}(x,D)\delta^s f|_{\partial\Omega_1}$ is smoothing by the pseudolocal property of ΨDOs. Therefore, if $f \in L^2$ in (22), then $u \in H^2$. Thus, for $f \in L^2(\Omega)$, we have $\Lambda_0(x,D)f - f^s_{\Omega_1} \in H^1(\Omega_1)$. We can multiply the first term by χ by the pseudolocal property of ΨDOs, and this proves (20) and therefore (19).

We have therefore proved the following.

Theorem 1 *Let g be a simple metric in Ω and let χ be as in (18). Then for any symmetric tensor $f \in L^2(\Omega)$,*

$$\chi a_{ijkl}(x,D)\chi(N_g f)^{kl} = f^s_{\Omega_1} + Kf, \tag{23}$$

where $K : L^2(\Omega) \to H^1(\Omega_1)$ is bounded.

6 Stability estimates for N_g

Theorem 1 gives a formula for the recovery of the most singular part of f^s from $N_g f$ in Ω, if f vanishes near $\partial\Omega$ (then we apply the theorem with $\Omega_1 = \Omega$). In general, it gives a parametrix of f^s related to the larger domain Ω_1. In this section, we will construct a parametrix to infinite order and in the same domain. The latter comes with the price of losing one derivative in the inversion, see Remark 2 at the end of this section.

First, we construct a parametrix of N_g to infinite order similar to B in (18). Notice that N_g is not elliptic and its principal symbol $\sigma_p(N_g)$ vanishes on the range of $\sigma_p(d^s)$. On the other hand, $\sigma_p(N_g)$ leaves invariant the subspace of symmetric tensors h_{ij} satisfying $\xi^i h_{ij} = 0$ (the solenoidal tensors). On this subspace, $\sigma_p(N_g)$ is elliptic with inverse given by a_{ijkl}, see (15). Next, $\sigma_p(d^s)\sigma_p(\Delta^s)^{-1}\sigma_p(\delta^s)$ is the projector onto the orthogonal complement of this subspace. Having this in mind, we construct first a parametrix in Ω_1 of the elliptic (as will become clear below) operator

$$M = |D|N_g + d^s(\Delta^s_{\Omega_2,D})^{-1}\delta^s$$

of order 0. Here $\Omega_2 \supset\supset \Omega_1$ is a small strictly convex neighborhood of Ω_1 and the metric is extended there smoothly. Inside Ω_2, and therefore on $\bar\Omega_1$, $(\Delta^s_{\Omega_2,D})^{-1}$ is a ΨDO with full symbol equal to the parametrix of Δ^s modulo operators of order $-\infty$. The principal symbol $\sigma_p(M)$ of M is given by $\sigma_p(M) = |\xi|\sigma_p(N_g) + Id - \Lambda_0$, where $\Lambda_0 = \lambda^{ij}_{kl} = \sigma_p(Id - d^s(\Delta^s_{\Omega_2,D})^{-1}\delta^s)$, see (21). Denote also by $\Lambda = \sigma(\mathcal{S}_{\Omega_2})$ the full symbol of the projector $\mathcal{S}_{\Omega_2} = Id - d^s(\Delta^s_{\Omega_2,D})^{-1}\delta^s$ in $L^2(\Omega_2)$. Notice that $\sigma_p(Id) = \delta^{ij}_{kl}$. Next, Λ_0 and $Id - \Lambda_0$ are projectors (onto the principal parts of Fourier transforms of solenoidal and potential tensors, respectively), while $|\xi|\sigma_p(N_g)$ can be “inverted” as in (15) by $|\xi|^{-1}A_0 = |\xi|^{-1}a_{ijkl}$. We therefore have

$$\left(|\xi|^{-1}A_0 + (Id - \Lambda_0)\right)\sigma_p(M) = \Lambda_0 + (Id - \Lambda_0) = Id.$$

Thus $\sigma_p(M)$ is elliptic. There exists a symbol L of order 0, such that

$$L \circ \sigma(M) \sim Id$$

for $x \in \Omega_1$, where $\sim$ stands for equivalence modulo symbols of order $-\infty$. This yields

$$\Lambda \circ L \circ \sigma(M) \circ \Lambda \ \sim\ \Lambda.$$

On the other hand, $\sigma(M) \circ \Lambda = \sigma(|D|N_g)$, so

$$(\Lambda \circ L) \circ \sigma(|D|N_g) \ \sim\ \Lambda. \tag{24}$$

The symbol $\Lambda\circ L$ is the parametrix that we need. Notice that the principal part of $\Lambda\circ L$ is $|\xi|^{-1}a_{ijkl}$.

Therefore, there exists a first order ΨDO A in Ω_2 with principal symbol $A_0=a_{ijkl}$, such that $AN_gf=f^s_{\Omega_2}+Kf$ in Ω_2, where K is smoothing acting on functions supported in Ω_1. Note that Kf may become singular at $\partial\Omega_2$. As above we can achieve that

$$AN_gf=f^s_{\Omega_1}+Kf \quad \text{in } \Omega_1,\ \forall f\in L^2(\Omega), \tag{25}$$

with a modified K with kernel in $C^\infty(\bar\Omega_1\times\bar\Omega)$. Since $f=0$ outside Ω, this implies that

$$d^sv_{\Omega_1}=-AN_gf+Kf \quad \text{in } \Omega_1\setminus\Omega. \tag{26}$$

We will use (26) and the fact that $v_{\Omega_1}=0$ on $\partial\Omega_1$ to estimate v_{Ω_1} in $\Omega_1\setminus\Omega$.

For $y\in\Omega_1\setminus\Omega$ in a small neighborhood $|y-y_0|<\varepsilon$ of a fixed $y_0\in\partial\Omega$, and for a unit ξ such that the geodesic $\gamma(t)=\gamma(t;y,\xi)$ in $\Omega_1\setminus\Omega$ issued from (y,ξ) meets $\partial\Omega_1$ before it meets $\partial\Omega$ at a positive time that we denote by $\tau=\tau(y,\xi)$, we have

$$[v_{\Omega_1}(\gamma(s))]_i\,\dot\gamma^i(s)=-\int_s^\tau [d^sv_{\Omega_1}(\gamma(t))]_{ij}\dot\gamma^i(t)\dot\gamma^j(t)\,dt, \tag{27}$$

(see [Sh1, Ch. 3.3]). Clearly, $|v_{\Omega_1}(x)|^2$ can be estimated by $C\sum_{k=1}^n\left|[v_{\Omega_1}(x)]_i\xi^i_{(k)}\right|^2$, with some constant C if $\xi_{(k)}$ are linearly independent and x is close to a fixed point. Using this, we estimate v_{Ω_1} first locally and then globally to get the following Poincaré estimate

$$\|v_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}\le C\|d^sv_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}. \tag{28}$$

We will estimate next the H^1 norm of v_{Ω_1} in $\Omega_1\setminus\Omega$. We have

$$\xi^i\nabla_i\left[v_{\Omega_1}(y)\right]_j\xi^j=[d^sv_{\Omega_1}(y)]_{ij}\xi^i\xi^j.$$

To estimate $\eta^i\nabla_i\left[v_{\Omega_1}(y)\right]_j\xi^j$ for η not parallel to ξ, we differentiate (27). Fix again $y_0\in\partial\Omega$ and choose local coordinates x' on $\partial\Omega$ near y_0. Fix a unit ξ_0 close to the unit normal to $\partial\Omega$ at y_0. Each point in $p\in\Omega_1\setminus\Omega$ near y_0 can be uniquely expressed as $p=\gamma(t;x',\xi_0)$, where the latter is the geodesic issued from a point on $\partial\Omega$ with coordinates x' in the direction ξ_0 (a more precise notation would be $\gamma(t;(x',0),\xi_0)$). Choose $x_n=t$ as an n-th coordinate. In those coordinates, we get from (27)

$$[v_{\Omega_1}(x',x_n)]_n=-\int_{x_n}^\tau [d^sv_{\Omega_1}(x',t)]_{nn}\,dt.$$

Let χ be a smooth cut-off function such that $\chi=1$ near $\partial\Omega$ and $\chi=0$ near $\partial\Omega_1$ and outside Ω_1. Then $K_1:f\mapsto(1-\chi)d^sv_{\Omega_1}$ is a smoothing operator by (26) and therefore,

$$[v_{\Omega_1}(x',x_n)]_n=-\int_{x_n}^\infty \chi[d^sv_{\Omega_1}(x',t)]_{nn}\,dt+K_2f,$$

where K_2 is also smoothing. We now differentiate the equality above w.r.t. x' and x_n. Note that in the r.h.s. we will get only derivatives w.r.t. x'. Writing the result in invariant form, we get in some neighborhood U of x_0:

$$\left\|[v_{\Omega_1}]_i\,\xi^i\right\|_{H^1(U)}\le C\left(\sum_{k=1}^{n-1}\|X^{(k)}d^sv_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}+\|d^sv_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)}+\|K_2f\|\right), \tag{29}$$

where the vector fields $X^{(k)}$ are tangent to $\partial\Omega$ and ξ is the tangent vector field to $\gamma(t;x',\xi_0)$. Introduce the space $\tilde H^1(\Omega_1\setminus\Omega)$ with norm equal to the L^2 norm outside a neighborhood of $\partial\Omega$ and near $\partial\Omega$ (but outside Ω) having the following form in normal local coordinates:

$$\|f\|^2_{\tilde H^1(\Omega_1\setminus\Omega)}=\int_{\Omega_1\setminus\Omega}\left(\sum_{i=1}^{n-1}|\partial_if|^2+|x^n\partial_nf|^2+|f|^2\right)dx, \quad \operatorname{supp} f\subset U. \tag{30}$$

Here U is a small neighborhood of a point on $\partial\Omega$ and the norm in $\tilde H^1(\Omega_1 \setminus \Omega)$ is defined by using partition of unity. We now repeat the construction above leading to (29) with n linearly independent choices of ξ_0 and use partition of unity to get

$$\|v_{\Omega_1}\|_{H^1(\Omega_1\setminus\Omega)} \le C\left(\|d^s v_{\Omega_1}\|_{\tilde H^1(\Omega_1\setminus\Omega)} + \|K_2 f\|\right). \tag{31}$$

Of course, this implies

$$\|v_{\Omega_1}\|_{H^1(\Omega_1\setminus\Omega)} \le C\left(\|d^s v_{\Omega_1}\|_{H^1(\Omega_1\setminus\Omega)} + \|K_2 f\|\right),$$

but we need the more precise estimate (31) because it does not involve transversal derivatives to $\partial\Omega$. By (26), (31), and the trace theorem,

$$\|v_{\Omega_1}\|_{H^{1/2}(\partial\Omega)} \le C\|AN_g f\|_{\tilde H^1(\Omega_1\setminus\Omega)} + C_s\|f\|_{H^{-s}(\Omega_1)}, \quad \forall s. \tag{32}$$

We are ready now to compare v_Ω and v_{Ω_1}. For $w = v_{\Omega_1} - v_\Omega$ we have

$$\begin{cases} \Delta^s w = 0 & \text{in } \Omega, \\ w = v_{\Omega_1} & \text{on } \partial\Omega. \end{cases} \tag{33}$$

By standard elliptic estimates we get that $\|w\|_{H^1(\Omega)}$ can be estimated by the r.h.s. of (32). Therefore, for $f^s_\Omega = f^s_{\Omega_1} + d^s w$ we get from this and (25),

$$\|f^s_\Omega\|_{L^2(\Omega)} \le C\left(\|AN_g f\|_{\tilde H^1(\Omega_1\setminus\Omega)} + \|AN_g f\|_{L^2(\Omega_1)}\right) + C_s\|f\|_{H^{-s}(\Omega_1)}, \quad \forall s.$$

Note that a sufficient condition for the norm in the r.h.s. above to be finite is f, extended as 0 outside Ω, to be in H^1, i.e., $f \in H^1_0(\Omega)$. It is not hard to see however that since the $\tilde H^1$ norm that we use involves tangential derivatives at $\partial\Omega$ only, we can take $f \in H^1(\Omega)$ above. Indeed, in boundary normal coordinates the commutators $[\partial_k, AN_g]$, $k = 1, \dots, n-1$, and $[x^n\partial_n, AN_g]$ are of order 0, and $\partial_k f \in L^2(\Omega)$, $k = 1, \dots, n-1$, and $x^n\partial_n f \in L^2(\Omega)$ for any $f \in H^1(\Omega)$ (without the assumption $f = 0$ on $\partial\Omega$).

Introduce the norm

$$\|N_g f\|_{\tilde H^2(\Omega_1)} = \sum_{i=1}^n \|\partial_i N_g f\|_{\tilde H^1(\Omega_1)} + \|N_g f\|_{H^1(\Omega_1)}.$$

The $\tilde H^1$ norm above is defined as in (30) with the integral taken in a small two sided neighborhood of $\partial\Omega$, not only outside Ω as in (30). The norm above defines a Hilbert space $\tilde H^2(\Omega_1)$. We have therefore proved part (a) of the following theorem. Recall that $\mathcal{S}$ is the projection onto the space of solenoidal tensors.

Theorem 2 *Assume that g is simple metric in Ω and extend g as a simple metric in $\Omega_1 \supset\supset \Omega$.*
(a) The following estimate holds for each symmetric 2-tensor f in $H^1(\Omega)$:

$$\|f^s_\Omega\|_{L^2(\Omega)} \le C\|N_g f\|_{\tilde H^2(\Omega_1)} + C_s\|f\|_{H^{-s}(\Omega_1)}, \quad \forall s > 0.$$

(b) $\operatorname{Ker} I_g \cap \mathcal{S}L^2(\Omega)$ *is finite dimensional and included in* $C^\infty(\bar\Omega)$*.*
(c) Assume that I_g is s-injective in Ω, i.e., that $\operatorname{Ker} I_g \cap \mathcal{S}L^2(\Omega) = \{0\}$*. Then for any symmetric 2-tensor f in $H^1(\Omega)$ we have*

$$\|f^s\|_{L^2(\Omega)} \le C\|N_g f\|_{\tilde H^2(\Omega_1)}. \tag{34}$$

Remark 1. We would like to note that in fact we actually constructed f^s from $N_g f$ up to smoothing operators. The first step in this is to construct the parametrix of N_g as in (24) and with its aid, we construct $f^s_{\Omega_1}$ modulo smoothing operators as in (25). Since $f = 0$ outside Ω, this gives us $d^s v_{\Omega_1} = f^s_{\Omega_1}$ in $\Omega_1 \setminus \Omega$, see (26). Integrating this along certain geodesics, see (27), we get $v_{\Omega_1} = f^s_{\Omega_1}$ in $\Omega_1 \setminus \Omega$. Using the so obtained boundary value of v_{Ω_1} on $\partial\Omega$, we solve the Dirichlet problem (33) to get $w = v_{\Omega_1} - v_\Omega$ in Ω. Finally, we construct $f^s_\Omega = f^s_{\Omega_1} + d^s w$.

Next we prove part (b) of the theorem. Let $f \in L^2(\Omega)$, $I_g f = 0$ and $\delta^s f = 0$ in Ω. Then $N_g f = 0$ and $f^s = f$. Part (a) immediately yields the finiteness assertion (see also [Sh2]). By the remark above, $f = f^s$ is smooth in $\bar{\Omega}$ (see also [Ch]).

Part (c) of the theorem follows from the following simple lemma (see also [T, Proposition V.3.1], which also implies the lemma below):

Lemma 2 *Let X, Y, Z be Banach spaces, let $A : X \to Y$ be a closed linear operator with domain $\mathcal{D}(A)$, and $K : X \to Z$ be a compact linear operator. Let*

$$\|f\|_X \le C\left(\|Af\|_Y + \|Kf\|_Z\right), \quad \forall f \in \mathcal{D}(A). \tag{35}$$

Assume that A is injective. Then

$$\|f\|_X \le C'\|Af\|_Y, \quad \forall f \in \mathcal{D}(A).$$

Proof. We show first that one can assume that A is bounded. Indeed, let $\|\cdot\|_{\mathcal{D}(A)}$ denotes the graph norm. Then (35) implies

$$\|f\|_{\mathcal{D}(A)} \le C\left(\|Af\|_Y + \|Kf\|_Z\right), \quad \forall f \in \mathcal{D}(A).$$

Assuming the lemma for bounded operators, we get $\|f\|_{\mathcal{D}(A)} \le C\|Af\|_Y$ and this implies the estimate we want to prove.

For bounded A, assume the opposite. Then there exists a sequence f_n in X with $\|f_n\|_X = 1$ and $Af_n \to 0$ in Y. Since $K : X \to Z$ is compact, there exists a subsequence, that we will still denote by f_n, such that Kf_n converges in Z, therefore is a Cauchy sequence in Z. Applying (35) to $f_n - f_m$, we get that $\|f_n - f_m\|_X \to 0$, as $n \to \infty$, $m \to \infty$, i.e., f_n is a Cauchy sequence in X. Therefore, there exists $f \in X$ such that $f_n \to f$ and we must have $\|f\|_X = 1$. Then $Af_n \to Af = 0$. This contradicts the injectivity of A thus proving the lemma. $\square$

To complete the proof of Theorem 2(c), we need to redefine N_g as a closed operator on a certain space. Let $X = \mathcal{S}L^2(\Omega)$. Set $Y = \tilde{H}^2(\Omega_1)$ and $Z = H^{-s}(\Omega_1)$ with some fixed $s > 0$. Consider the unbounded operator $N_g : X \to Y$ an let $\mathcal{D}$ be the closure of $\mathcal{S}H^1(\Omega)$ under the graph norm $\|f\|_{\mathcal{D}} = \|f\|_{L^2(\Omega)} + \|N_g f\|_{\tilde{H}^2(\Omega_1)}$. Given $f \in \mathcal{D}$, there exists a sequence $\mathcal{S}H^1(\Omega) \ni f_n \to f$ in X such that $N_g f_n$ is a Cauchy sequence in Y, thus $N_g f_n \to h \in Y$ in Y for some $h \in Y$. We set $N_g f = h$ thus defining N_g on $\mathcal{D}$. Since N_g is a ΨDO, $N_g f_n \to N_g f$ in $L^2(\Omega_1)$, so this definition agrees with the action of N_g on any element in $L^2(\Omega)$. We will show that N_g, with domain $\mathcal{D}$, is closed. Let as above $\mathcal{S}H^1(\Omega) \ni f_n \to f \in X$ in X and $N_g f_n \to h \in Y$ in Y. We saw that this means that $h = N_g f$ and by our definition of $\mathcal{D}$, we have $f \in \mathcal{D}$. Therefore, N_g is closed. On the other hand, $N_g f = 0$ in Y for some $f \in \mathcal{S}L^2(\Omega)$ (then f actually has to be smooth by (b)) implies $(N_g f, f) = \|I_g f\|^2 = 0$, thus $f = 0$ so N_g is injective. An application of Lemma 2 then yields part (c) of the theorem.

Remark 2. The r.h.s. of the inequality in Theorem 2(a) above can be estimated by $C\|f^s\|_{H^1(\Omega)}$ (actually, we need the derivatives only near the boundary). On the other hand, in the l.h.s. we have $\|f^s\|_{L^2(\Omega)}$. We believe that this is not only a technical difficulty and is related to the nature of the problem. It remains an open question however to find other reasonable norms of f^s and $N_g f$ above so that the estimates above are sharp, as in Theorems 3 and 4 below.

Remark 3. It follows from the proof that without assuming s-injectivity of I_g, estimate (34) holds for any f orthogonal to $\operatorname{Ker} I_g \cap \mathcal{S}L^2(\Omega)$.

7 Recovery of a function from integrals along geodesics

Let $If(\gamma) = \int_\gamma f\, dt$ be the geodesic X-ray transform of functions $f(x)$, $x \in \Omega$, that can be written also as:

$$I_g f(z,\omega) = \int f(\gamma(t;x,\omega))\, dt, \quad (z,\omega) \in \Gamma_-.$$

The analysis above applies in this case as well with obvious modifications. For $N_g = I_g^{\mathrm{T}} I_g$ we get similarly to (4),

$$N_g f(x) = \frac{1}{\sqrt{\det g(x)}} \int_\Omega \frac{f(y)}{\rho(x,y)^{n-1}} \left| \det \frac{\partial^2(\rho^2(x,y)/2)}{\partial x \partial y} \right| dy.$$

We have $\sigma_p(N_g) = |\xi|^{-1} = (g^{ij}\xi_i\xi_j)^{-1/2}$. As in Theorem 1, we get

$$c_n^{-1}|D|\chi N_g f = f + Kf \quad \text{in } \Omega_1, \quad c_n = \frac{4\pi^{(n+1)/2}}{\Gamma(n/2 - 1/2)}. \tag{36}$$

The operator $|D|$ is defined as $|D| = \mathrm{Op}(|\xi|)$. The operator K is a ΨDO of order -1.

It is known [Mu2], [Mu-R], [BG], [Sh1] that for simple metrics that we consider here, I_g is injective on H^1, i.e., $I_g f = 0$ for some $f \in H^1(\Omega)$ implies $f = 0$ with non-sharp stability estimates. We will use the injectivity to get sharp estimate for N_g.

Now, (36) implies

$$\|f\| \le C \left(\|N_g f\|_{H^1(\Omega_1)} + \|Kf\|_{L^2(\Omega)} \right), \quad \forall f \in L^2(\Omega).$$

If $N_g f = 0$ in Ω_1 with some $f \in L^2(\Omega)$, then by (36), $f \in H_0^1(\Omega)$, and by the injectivity of I_g on H^1 we get $f = 0$. Therefore, $N_g : L^2(\Omega) \to H^1(\Omega_1)$ is injective (and bounded). By Lemma 2 we get:

Theorem 3 *Let g be a simple metric in Ω and assume that g is extended smoothly as a simple metric near the convex domain $\Omega_1 \supset\supset \Omega$. Then for any function $f \in L^2(\Omega)$,*

$$\|f\|/C \le \|N_g f\|_{H^1(\Omega_1)} \le C\|f\|.$$

Moreover, in Ω, $f = c_n^{-1}|D|\chi N_g f \mod H^1(\Omega)$.

The assumption that g is smooth can be relaxed a bit. Since N_g depends continuously on $g \in C^k$ with some finite $k = k(n)$, the constant C in Theorem 3 above can be chosen locally uniform for simple metrics $g \in C^k$.

8 Recovery of a differential form from the geodesic X-ray transform

Consider the geodesic X-ray transform for one component tensors f_i in Ω. They can be identified with 1-differential forms $f = f_i dx^i$. Then $I_g f$ is defined by $I_g f(\gamma) = \int_\gamma f$, i.e.,

$$I_g f(z,\omega) = \int f_i(\gamma(t,z,\omega))\dot\gamma^i(t,z,\omega)\, dt, \quad (z,\omega) \in \Gamma_-.$$

As above, it is easy to see that $I_g(d\psi) = 0$ for any smooth function ψ in Ω with $\psi|_{\partial\Omega} = 0$. Here $d\psi = (\partial\psi/\partial x^i)dx^i$ is the differential of ψ. As before, we define a divergence operator δf sending 1-forms to functions by the formula $\delta f = g^{ij}\nabla_j f_i$. Any form $f \in L^2(\Omega)$ can be decomposed orthogonally as

$$f = f^s + d\psi,$$

where $\psi = 0$ on $\partial\Omega$, ψ is given by $\psi = (-\Delta_D)^{-1}\delta f$ and $\delta f^s = 0$. Here Δ_D is the Laplace-Beltrami operator related to g with Dirichlet boundary conditions. It is known [AR] that I_g is injective on the space of solenoidal forms satisfying $\delta f = 0$ for simple metrics g with a non-sharp stability estimate. In other words, $f \in H^1(\Omega)$ and $I_g f = 0$ implies $f^s = 0$, i.e., $f = d\psi$ with some ψ vanishing on $\partial\Omega$. Our goal here is to formulate a sharp stability estimate.

In the case g is a constant coefficient metric, the symbol of N_g is given by (compare with Proposition 2)

$$\sigma_p(N_g)^{ij} = c_n \frac{\partial^2}{\partial\xi_i\partial\xi_j}|\xi| = c_n|\xi|^{-1}\left(\delta^{ij} - \xi^i\xi^j/|\xi|^2\right), \quad c_n = \frac{2\pi^{(n+1)/2}}{\Gamma(n/2+1/2)},$$

As before, we see that this formula remains true (with $|\xi|^2 = g^{ij}(x)\xi_i\xi_j$ and $\xi^i = g^{ij}(x)\xi_j$) for metrics with variable coefficients, then the second equality is to be considered modulo symbols of order -2. The expression $\delta^{ij} - \xi^i\xi^j/|\xi|^2$ equals the principal symbol of $\mathcal{S}$. Therefore the parametrix of N_g in this case is simply equal to $c_n^{-1}|\xi|$ as in the preceding section. Similarly to Theorem 1 we get

$$\chi|D|\chi N_g f = f^s_{\Omega_1} + Kf, \quad \operatorname{supp} f \subset \bar{\Omega}, \tag{37}$$

where Ω_1 and χ are as before and K is operator of order -1.

Next we will show how to construct f_Ω following the approach in section 6. In this case however, we will get sharp estimates. Denote $A = \chi|D|\chi N_g$. Then $f - d\psi_{\Omega_1} + Kf = Af$ as in (25) and in particular, $d\psi_{\Omega_1} = (-A+K)f$ in $\Omega_1 \setminus \Omega$. Since $\psi_{\Omega_1} = 0$ on $\partial\Omega_1$, we have $\psi_{\Omega_1}(x) = \int_\gamma d\psi_{\Omega_1}$, where γ is any curve in $\Omega_1 \setminus \Omega$ that connects x and a point on $\partial\Omega_1$. Let us choose $\gamma = \gamma_x(s)$, $0 \le s \le T(x)$ to be the geodesic such that $\gamma_x(0) = x$, and the maximal extension of γ in $\Omega_1 \setminus \Omega$ is orthogonal (in the metric) to $\partial\Omega$. In local normal coordinates,

$$\psi_{\Omega_1}(x) = -\int_{x^n}^{\varepsilon} [d\psi_{\Omega_1}]_n(x', s)\, ds, \quad 0 \le x^n \le \varepsilon,$$

where $[d\psi_{\Omega_1}]_n = \partial\psi_{\Omega_1}/\partial x^n$. Similarly to (28), this easily implies the following Poincaré type of inequality

$$\|\psi_{\Omega_1}\|_{L^2(\Omega_1\setminus\Omega)} \le C\|d\psi\|_{L^2(\Omega_1\setminus\Omega)}.$$

Therefore, $\|\psi_{\Omega_1}\|_{H^1(\Omega_1\setminus\Omega)} \le C\|(A-K)f\|_{L^2(\Omega_1\setminus\Omega)}$ and the trace theorem guarantees that $\psi_{\Omega_1}|_{\partial\Omega}$ is well defined in $H^{1/2}(\partial\Omega)$. We have

$$\|\psi_{\Omega_1}\|_{H^{1/2}(\partial\Omega)} \le C\|(A-K)f\|_{L^2(\Omega_1\setminus\Omega)} \le C\left(\|Af\|_{L^2(\Omega_1\setminus\Omega)} + \|Kf\|_{L^2(\Omega_1)}\right).$$

For $\phi = \psi_{\Omega_1} - \psi_\Omega$ with $\psi_\Omega = \Delta^{-1}_{\Omega,D}\delta f$ we have $\Delta\phi = 0$ in Ω, $\phi = \psi_{\Omega_1}$ on $\partial\Omega$, compare with (33). Therefore, the H^1 norm of ϕ in Ω can be estimated by the r.h.s. of the estimate above. This allows us to compare $f^s_{\Omega_1}$ and f^s_Ω by writing $f^s_\Omega = f^s_{\Omega_1} + d\phi$ and we get

$$\|f^s_\Omega\|_{L^2(\Omega)} \le C\left(\|N_g f\|_{H^1(\Omega_1)} + \|Kf\|_{L^2(\Omega_1)}\right), \quad \operatorname{supp} f \subset \bar{\Omega}. \tag{38}$$

Recall that $N_g f = N_g f^s_{\Omega_1}$, $\forall f \in L^2(\Omega_1)$, where $f^s_{\Omega_1}$ is the projection of f onto the subspace of solenoidal forms in Ω_1. Let $N_g f = N_g f^s_{\Omega_1} = 0$ with some $f \in L^2$ with $\operatorname{supp} f \subset \bar{\Omega}$. Then by (37), $f^s_{\Omega_1} \in H^1(\Omega_1)$. By the injectivity of N_g, we have $f^s_{\Omega_1} = 0$. Then $f = d\phi_{\Omega_1}$ with $\phi_{\Omega_1} = 0$ on $\partial\Omega_1$ and since $\operatorname{supp} f \subset \bar{\Omega}$, we get $\operatorname{supp}\phi_{\Omega_1} \subset \bar{\Omega}$, thus $f^s_\Omega = 0$ as well. Therefore, $N_g : \mathcal{S}L^2(\Omega) \to H^1(\Omega_1)$ is injective.

We apply Lemma 2 to $A = N_g$ with $X = \mathcal{S}L^2(\Omega)$, $Y = H^1(\Omega_1)$, $Z = L^2(\Omega_1)$ to get the following.

Theorem 4 *Assume that g is simple metric in Ω and extend g as a simple metric in $\Omega_1 \supset\supset \Omega$. Then for any 1-form $f = f_i dx^i$ in $L^2(\Omega)$ we have*

$$\|f^s\|_{L^2(\Omega)}/C \le \|N_g f\|_{H^1(\Omega_1)} \le C\|f^s\|_{L^2(\Omega)}.$$

Moreover, in Ω, we have $f^s = c_n^{-1}|D|\chi N_g f \mod H^1(\Omega)$.

Similarly to section 7, the estimate above is locally uniform for $g \in C^k(\Omega)$ with some $k \gg 1$.

9 Local uniqueness for the boundary rigidity problem

In this section we apply the results we obtained for the linear X-ray geodesic transform I_g in section 6 to show that s-injectivity of I_g for a fixed simple metric g in Ω implies local uniqueness of the non-linear boundary rigidity problem near the same g.

Theorem 5 *Let g be a simple metric in the domain Ω. Assume that I_g is s-injective. Then there exists $\varepsilon > 0$ and $k > 0$, such that if for another metric $\tilde{g}$ in Ω we have $\|\tilde{g} - g\|_{C^k} \le \varepsilon$ and $\rho_{\tilde{g}} = \rho_g$ on $\partial\Omega^2$, then there exists a diffeomorphism $\psi : \Omega \to \Omega$ with $\psi|_\Omega = Id$, such that $\tilde{g} = \psi_* g$.*

Proof. We will first pass to semi-geodesic coordinates.

As above, we can extend the metric in a neighborhood Ω_1 of Ω such that Ω_1 is strictly convex with smooth boundary and g is simple in Ω_1. Assume now that g and $\tilde{g} = g + f$ are two simple smooth metrics in Ω with the same distance function. By [LSU], we can choose diffeomorphic copies of g and $\tilde{g}$, that we will still denote by g and $\tilde{g}$, such that $f = \tilde{g} - g$ vanishes at $\partial\Omega$ of any order.

Fix $x_0 \in \partial\Omega_1$ and consider the map $\exp_{x_0}^{-1} : \Omega_1 \to \exp_{x_0}^{-1}(\Omega_1)$ that is a diffeomorphism according to our assumptions. Introduce polar coordinates $\xi = r\theta$ in $\exp_{x_0}^{-1}(\Omega_1)$, where $r > 0$, $g^{ij}(x_0)\theta_i\theta_j = 1$. Choose a Cartesian coordinate system in which $\{\xi_n = 0\}$ is the plane tangent to the boundary of $\exp_{x_0}^{-1}(\Omega_1)$ at $\xi = 0$. Then by the convexity assumptions, the function θ_n has positive lower bound in the closure of $\exp_{x_0}^{-1}(\Omega)$. Clearly, so does r. Define new coordinates (y', y_n), where $y' \in \mathbf{R}^{n-1}$, $y^n > 0$, by $y' = \theta'/\theta_n$, $y_n = r$. The map $\xi \mapsto (y', y_n)$ is a diffeomorphism between $\exp_{x_0}^{-1}(\Omega_1)$ and its image with inverse map given by $\xi = r\theta = y_n\theta_n(y', 1)$ with $\theta_n = (1 + |y'|^2)^{-1/2}$.

In the coordinates $\xi = r\theta$, the lines $\theta =$ const. are geodesics with r natural parameter. Moreover, those geodesics are perpendicular to the geodesic spheres $r =$ const. In the y–coordinates those geodesics take the form $y' =$ const. and y_n is an arc-length parameter. Moreover, they are orthogonal to the planes $y_n =$ const. This shows that $(\psi_* g)^{in} = \delta^{in}$, $i = 1, \dots, n$, where δ^{in} is the Kronecker symbol, where ψ is the diffeomorphism $y \mapsto x$. We also have $(\psi^* g)_{in} = \delta_{in}$.

We repeat the same construction with $\tilde{g}$. First, we extend $\tilde{g}$ in Ω_1 by setting $g = \tilde{g}$ in $\Omega_1 \setminus \Omega$. This extension is smooth because $f = \tilde{g} - g \in C^\infty_{(0)}(\Omega)$. The so extended $\tilde{g}$ is simple in Ω_1 as well. Moreover, the exponential map $\xi \mapsto \exp_x \xi$ is the same for both metrics for $x \in \Omega_1 \setminus \Omega$ and has domain and image the same for both metrics as well. Therefore, for the diffeomorphism ψ constructed above and $\tilde{\psi}$ similar to ψ but related to $\tilde{g}$, we have $\tilde{\psi}(\Omega_1 \setminus \Omega) = \psi(\Omega_1 \setminus \Omega)$. In particular, $\tilde{\psi}^* \tilde{g} = \psi^* g$ in $\psi(\Omega_1 \setminus \Omega)$. Therefore, in what follows we may assume that

$$\begin{aligned} g, \tilde{g} \in C^\infty(\bar{\Omega}_1), &\quad \operatorname{supp}(\tilde{g} - g) \subset \bar{\Omega}, \\ g_{in} = \delta_{in} &\quad \text{for } i = 1, \dots, n. \end{aligned} \tag{39}$$

In particular, for $f = \tilde{g} - g$ we have

$$f \in C^\infty(\Omega_1), \quad \operatorname{supp} f \subset \bar{\Omega}, \quad f_{in} = f_{ni} = 0, \quad i = 1, \dots, n. \tag{40}$$

Let $\tilde{g}$, g be as in the theorem, in particular, $\tilde{g}$ is also simple provided that $\varepsilon \ll 1$. Linearizing near g, we get as in [E],

$$\rho_{\tilde{g}}(x, y) - \rho_g(x, y) = I_g f(x, y) + R_g(f)(x, y), \quad \forall (x, y) \in \partial\Omega^2, \tag{41}$$

where, with some abuse of notation, $I_g f(x, y)$ stands for $I_g f(x, \xi)$ with $\xi = \exp_x^{-1} y / |\exp_x^{-1} y|$. The remainder term $R_g(f)$ is non-linear and satisfies the estimate [E]

$$|R_g(f)(x, y)| \le C|x - y| \|f\|^2_{C^1(\bar{\Omega})}, \quad \forall (x, y) \in \partial\Omega^2 \tag{42}$$

with $C > 0$ uniform in $\tilde{g}$ if $0 < \varepsilon \ll 1$. By the assumptions of the theorem, the l.h.s. of (41) vanishes, thus

$$|I_g f(x, y)| \le C|x - y| \|f\|^2_{C^1(\bar{\Omega})}, \quad \forall (x, y) \in \partial\Omega^2. \tag{43}$$

Apply I^{T} to both sides and use the estimate $\|I^{\mathrm{T}}u\|_{L^\infty(\Omega_1)} \le C\|u\|_{L^\infty(\Omega)}$ to get

$$\|N_g f\|_{L^\infty(\Omega_1)} \le C\|f\|^2_{C^1(\bar{\Omega})}. \tag{44}$$

Since f extends smoothly as zero into the whole $\mathbf{R}^n$, we will denote $\|f\|_{C^1(\bar{\Omega})} = \|f\|_{C^1}$, and similarly for the other norms of f below. Note that f^s does not need to vanish at $\partial\Omega$. On the other hand, since f vanishes on $\partial\Omega$ with all derivatives, and N_g is an ΨDO of order -1, we have

$$\|N_g f\|_{H^{k+1}(\bar{\Omega}_1)} \le C_k\|f\|_{H^k}, \quad \forall k. \tag{45}$$

Applying the interpolation inequality $\|f\|_{\alpha_1 s_1+\alpha_2 s_2} \le \|f\|^{\alpha_1}_{s_1}\|f\|^{\alpha_2}_{s_2}$, $\alpha_1+\alpha_2 = 1$, $\alpha_i \ge 0$, with $s_2 = 0$, we get

$$\|N_g f\|_{\tilde{H}^2(\Omega_1)} \le C\|N_g f\|^{1-2/s_1}_{L^2(\Omega_1)}\|N_g f\|^{2/s_1}_{H^{s_1}(\Omega_1)} \le CA_1\|f\|^{2-4/s_1}_{C^1}, \tag{46}$$

by (44), provided that $s_1 > 2$. By (45), A_1 is such that $\|f\|^{2/s_1}_{H^{s_1-1}} \le A_1$. Applying Theorem 2, we get

$$\|f^s\|_{L^2(\Omega)} \le CA_1\|f\|^{2-4/s_1}_{C^1}. \tag{47}$$

We use the fact next that for tensors satisfying (40), we have the inequalities [E]

$$\|d^s v\|_{L^2(\Omega)} \le C\|f^s\|_{H^2(\Omega)} \quad \Longrightarrow \quad \|f\|_{L^2} \le C\|f^s\|_{H^2(\Omega)}. \tag{48}$$

The proof of (48) is based on the observation that $\partial_n v_n = [d^s v]_{nn} = -f^s_{nn}$ for such tensors and this allows us to estimate v_n. Next, we estimate v_j and its derivatives for $j = 1,\dots,n-1$ by writing $\nabla_n v_j = -2f^s_{jn} - \nabla_j v_n$. Using interpolation estimates again, we get with the aid of (48)

$$\|f\|_{C^1} \le C\|f\|_{H^{n/2+1+\epsilon}} \le C\|f\|^{\alpha}_{L^2}\|f\|^{1-\alpha}_{H^{s_2}} \le CA_2\|f\|^{\alpha}_{L^2} \le CA_2\|f^s\|^{\alpha}_{H^2(\Omega)}. \tag{49}$$

with $\alpha = 1-(n/2+1+\epsilon)/s_2$, $\epsilon > 0$, and $s_2 \gg 0$ such that $\alpha > 0$. Here $A_2 \ge \|f\|^{1-\alpha}_{H^{s_2}}$. Using interpolation estimates for the third time, we get

$$\|f^s\|_{H^2(\Omega)} \le CA_3\|f^s\|^{1-2/s_3}_{L^2(\Omega)}, \quad A_3 \ge \|f^s\|^{2/s_3}_{H^{s_3}(\Omega)} \tag{50}$$

for $s_3 > 2$. Combine (47), (49) and (50) to get

$$\|f^s\|_{L^2(\Omega)} \le C\|f\|^{2-4/s_1}_{C^1} \le C\|f^s\|^{\alpha(2-4/s_1)}_{H^2(\Omega)} \le C\|f^s\|^{\alpha(2-4/s_1)(1-2/s_3)}_{L^2(\Omega)}.$$

The conditions imposed on s_j, $j = 1,2,3$ are satisfied for s_j large enough and $\alpha \to 1$, as $s_2 \to \infty$. Therefore, there exists a choice of those three constants such that $\alpha(2-4/s_1)(1-2/s_3) > 1$. By the equality above then we get that for f^s small enough in L^2, we have $f^s = 0$, and by (48) we conclude that $f = 0$. For this to be true, it is enough $\|f\|_{L^2}$ to be small and A_1, A_2, A_3 above to be finite. This is satisfied if f is small enough in C^k with some finite $k > 0$. This is equivalent to the closeness of $\tilde{g}$ to g in C^k in semigeodesic coordinates. On the other hand, if $\|\tilde{g} - g\|_{C^{k+2}} \ll 1$ in the original coordinates, this implies the same for their pull-backs in the semigeodesic coordinates [S-U]. Notice that k can be estimated explicitly by optimizing the choice of s_j above. This completes the proof of the theorem. □

Remark 4. Theorem 5 implies a local result for the boundary rigidity problem as long as the linearized operator I_g is s-injective. In particular this gives as a corollary the result in [CDS]. A new result implied by Theorem 5 is the case of slightly anisotropic media. This means that in Theorem 5 we take $g = \frac{1}{c^2(x)}e$ with c a smooth positive function. In this case g is conformal to the Euclidean metric and $\tilde{g}$ is a small perturbation of g.

Remark 5. As it is clear from the proof, Theorem 5 admits the following more general formulation: $\exists s > 0$ such that for any $K > 0$ there is $\varepsilon = \varepsilon(K) > 0$ with the property that $\|\tilde{g}\|_{H^s} \leq K$ and $\|\tilde{g} - g\|_{L^2} \leq \varepsilon$ implies uniqueness, . i.e., the smallness is needed only in L^2 if we have an a priori bound in some H^s.

Remark 6. The same method can be used to prove a Hölder type stability estimate.

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