DIFFRACTION FROM CONORMAL SINGULARITIES

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Abstract. In this paper we show that for metrics with conormal singularities that correspond to class $C^{1,\alpha}$, $\alpha > 0$, the reflected wave is more regular than the incident wave in a Sobolev sense. This is helpful in the analysis of the multiple scattering series since higher order terms can be effectively 'peeled off'.

1. Introduction

In this paper we show that for metrics with conormal singularities that correspond to class $C^{1,\alpha}$, $\alpha > 0$, the reflected wave is more regular than the incident wave in a Sobolev sense for a range of background Sobolev spaces. That is, informally, for suitable $s \in \mathbb{R}$ and $\epsilon_0 > 0$, depending on the order of the conormal singularity (thus on α), if a solution of the wave equation is microlocally in the Sobolev space $H^{s-\epsilon_0}_{\mathrm{loc}}$ prior to hitting the conormal singularity of the metric in a normal fashion, then the reflected wave front is in H^s_{loc}, while the transmitted front is just in the a priori space $H^{s-\epsilon_0}_{\mathrm{loc}}$. (This assumes that along the backward continuation of the reflected ray, one has H^s_{loc} regularity, i.e. there is no incident H^s_{loc} singularity for which transmission means propagation along our reflected ray.) Such a result is helpful in the analysis of the multiple scattering series, i.e. for waves iteratively reflecting from conormal singularities, since higher order terms, i.e. those involving more reflections, can be effectively 'peeled off' since they have higher regularity.

Here the main interest is in $\alpha < 1$, for in the $C^{1,1}$ setting one has at least a partial understanding of wave propagation *without* a geometric structure to the singularities of the metric, such as conormality (though of course one does need *some* geometric structure to obtain a theorem analogous to ours), as then the Hamilton vector field is Lipschitz, and automatically has unique integral curves; see Smith's paper [17] where a parametrix was constructed, and also the work of Geba and Tataru [2]. We also recall that, in a different direction, for even lower regularity coefficients, Tataru has shown Strichartz estimates [18]; these are not microlocal in the sense of distinguishing reflected vs. transmitted waves as above.

In order to state the theorem precisely we need more notation. First suppose X is a $\dim X = n$-dimensional C^∞ manifold, and Y is a smooth embedded submanifold of codimension

$$\operatorname{codim} Y = k.$$

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With Hörmander's normalization [7], the class of Lagrangian distributions associated to the conormal bundle N^*Y of Y (also called distributions conormal to Y), denoted by $I^\sigma(N^*Y)$, arises from symbols in $S^{\sigma+(\dim X-2k)/4}$ when parameterized via a partial inverse Fourier transform in the normal variables. That is, if one has local coordinates (x,y), such that Y is given by $x=0$, then $u\in I^\sigma(N^*Y)$ can be written, modulo $C^\infty(\mathbb{R}^n)$, as

$$(2\pi)^{-k}\int e^{ix\cdot\xi}a(y,\xi)\,d\xi,\qquad a\in S^{\sigma+(n-2k)/4}.$$

For us it is sometimes convenient to have the orders relative to delta distributions associated to Y, which arise as the partial inverse Fourier transforms of symbols of order 0, as in [4], thus we let

$$I^{[-s_0]}(Y)=I^{-s_0-(\dim X-2k)/4}(N^*Y),$$

so elements of $I^{[-s_0]}(Y)$ are s_0 orders more regular than such a delta distribution. For any C^∞ vector bundle over X one can then talk about conormal sections (e.g. via local trivialization of the bundle); in particular, one can talk conormal metrics.

Thus, if X is a C^∞ manifold, Y an embedded submanifold, and g a symmetric 2-cotensor which is in $I^{[-s_0]}(Y)$ with $s_0>k=\operatorname{codim}Y$ (here we drop the bundle from the notation of conormal spaces), then g is continuous. We say that g is Lorentzian if for each $p\in X$, g defines a symmetric bilinear form on T_pX of signature $(1,n-1)$, $n=\dim X$. (One would say g is Riemannian if the signature is $(n,0)$. Another possible normalization of Lorentzian signature is $(n-1,1)$.) We say that Y is time-like if the pull-back of g to Y (which is a C^∞ 2-cotensor) is Lorentzian, or equivalently if the dual metric G restricted to N^*Y is negative definite.

A typical example, with Y time-like, is if $X=X_0\times\mathbb{R}_t$, where X_0 is the 'spatial' manifold, $Y=Y_0\times\mathbb{R}$, $g=dt^2-g_0$, g_0 is (the pull-back of) a Riemannian metric on X_0 which is conormal to Y_0, in the class $I^{[-s_0]}(Y_0)$, where $s_0>\operatorname{codim}_X Y=\operatorname{codim}_{X_0}Y_0$. In this case, one may choose local coordinates (x,y') on X_0 such that Y_0 is given by $x=0$; then with $y=(y',t)$, (x,y) are local coordinates on X in which Y is given by $x=0$. Thus, the time variable t is one of the y variables in this setting.

Before proceeding, recall that there is a propagation of singularities result in the manifolds with corners setting [19], which requires only minimal changes to adapt to the present setting. This states that for solutions of the wave equation lying in $H^{1,r}_{\mathrm{b}}(X)$ for some $r\in\mathbb{R}$, $\mathrm{WF}^{1,m}_{\mathrm{b}}$ propagates along generalized broken bicharacteristics. Thus, for a ray normally incident at Y, if all of the incoming rays that are incident at the same point in Y and that have the same tangential momentum carry H^{m+1} regularity, then the outgoing rays from this point in Y with this tangential momentum will carry the same regularity. In other words, in principle (and indeed, when one has boundaries, or transmission problems with jump singularities of the metric, this is typically the case) H^{m+1} singularities can jump from a ray to another ray incident at the same point with the same tangential momentum (let us call these *related rays*), i.e. one has a whole cone (as the magnitude of the normal momentum is conserved for the rays) of reflected rays carrying the H^{m+1} singularity. Here we recall that for $r\geq 0$, $H^{1,r}_{\mathrm{b}}(X)$ is the subspace of $H^1(X)$ consisting of elements possessing r b (i.e. tangential to Y) derivatives in $H^1(X)$; for $r<0$ these are distributions obtained from $H^1(X)$ by taking finite linear combinations of up to $-r$ derivatives of elements of $H^1(X)$. In particular, one can have arbitrarily large singularities; one can always represent these by taking tangential derivatives, in particular time derivatives. Via standard functional analytic duality arguments, these estimates (which also hold for the inhomogeneous equation) also give *solvability*, provided there is a global time function t. Phrased in terms of these spaces, and for convenience for the inhomogeneous equation with vanishing initial data, for $f \in H_{\mathrm{b}}^{-1,r+1}(X)$ supported in $t > t_0$ there exists a unique $u \in H_{\mathrm{b}}^{1,r}(X)$ solving the equation $\Box_g u = f$ such that $\operatorname{supp} u \subset \{t > t_0\}$.

The object of this paper is to improve on this propagation result by showing that, when $s_0 > k+1$ (thus $I^{[-s_0]}(Y) \subset C^{1+\alpha}$ for $\alpha < s_0 - k - 1$) in fact this jump to the related rays does not happen in an appropriate range of Sobolev spaces. As above, let (x, y) denote local coordinates on X, Y given by $x = 0$, and let (ξ, η) denote dual variables. Let $\Sigma \subset T^*X$ denote the characteristic set of the wave operator $\Box = \Box_g$; this is the zero-set of the dual metric G in T^*X.

Theorem 1.1. *Suppose* $\operatorname{codim} Y = k = 1$, $k + 1 + 2\epsilon_0 < s_0$ *and* $0 < \epsilon_0 \le s < s_0 - \epsilon_0 - 1 - k/2$. *Suppose that* $u \in L^2_{\mathrm{loc}}$, $\Box u = 0$,

$$q_0 = (0, y_0, \xi_0, \eta_0) \in \Sigma,\ \xi_0 \neq 0,$$

and the backward bicharacteristics from related *points* $(0, y_0, \xi, \eta_0) \in \Sigma$ *are disjoint from* $\mathrm{WF}^{s-\epsilon_0}(u)$, *and the backward bicharacteristic from the point* q_0 *is disjoint from* $\mathrm{WF}^s(u)$. *Then the forward bicharacteristic from* $(0, y_0, \xi_0, \eta_0)$ *is disjoint from* $\mathrm{WF}^s(u)$.

Remark 1.2. The theorem is expected to be valid for all values of k, and the limitation on k in the statement is so that it fits conveniently into the existing (b-microlocal) framework for proving the basic propagation of singularities (law of reflection) without too many technical changes. This is discussed in Section 4, and is to some extent 'orthogonal' to the actual main ideas of the paper; it is only used to microlocalize the 'background regularity', $H^{s-\epsilon_0}$. If one does not want to microlocalize the background regularity, i.e. assumes u is in $H^{s-\epsilon_0}$ at least locally, we prove the result for all codimensions, see Theorem 1.4.

Thus, the limiting Sobolev regularity s that one can obtain, if s_0 is slightly greater than $1 + k$, i.e. 2 in the case of a hypersurface, which is the minimum allowed by the first constraint, is just above $k/2$. On the other hand, if $s_0 > 1 + k$ then for any $0 \le s < s_0 - 1 - k/2$, one can choose $\epsilon_0 > 0$ sufficiently small so that all the inequalities are satisfied, so the theorem always provides interesting information on wave propagation for a range of values of s, providing at least some improvement over the basic propagation of singularities result (which would not allow better regularity than that on backward rays from $(0, y_0, \xi, \eta_0) \in \Sigma$, i.e. $H^{s-\epsilon_0}$).

Corollary 1.3. *Under assumptions as in the theorem, the terms of the multiple scattering series have higher regularity, in the sense of Sobolev wave front sets, with each iteration, until the limiting regularity,* $H^{s_0-1-k/2}$, *is reached.*

In view of the propagation of singularities along *generalized broken bicharacteristics*, i.e. that singularities can spread at most to related rays, Theorem 1.1 is in fact equivalent to the weaker version where one assumes $H^{s-\epsilon_0}_{\mathrm{loc}}$ regularity *not just on related rays*. Thus, as we show in Section 4, it suffices to prove the following theorem, which is what we prove in Section 8:

Theorem 1.4. *Suppose that* $\epsilon_0 > 0$, $k+1+2\epsilon_0 < s_0$ *and* $-k/2 < s < s_0 - \epsilon_0 - 1 - k/2$. *Then for* $u \in H^{s-\epsilon_0}_{\mathrm{loc}}$, $\Box u \in H^{s-1}_{\mathrm{loc}}$, $\mathrm{WF}^s(u)$ *is a union of maximally extended bicharacteristics in* Σ.

Note that if $s_0 > 1 + k$, then first taking $-k/2 < s < s_0 - 1 - k/2$, and then $\epsilon_0 > 0$ sufficiently small, all the inequalities in the Theorem are satisfied.

This theorem is proved by a positive commutator, or microlocal energy, estimate. They key issue is that as the wave operator does not have C^∞ coefficients, the commutator of a pseudodifferential microlocalzier with it is *not* a pseudodifferential operator; instead it is a sum of paired Lagrangian distributions associated to various Lagrangian submanifolds of $T^*(X \times X)$. Thus, the main technical task is to analyze these Lagrangian pairs, including their Sobolev boundness properties.

The plan of the paper is the following. In Section 2 we recall the structure of positive commutator estimates, in particular the robust version due to Melrose and Sjöstrand [11, 12], used in their proof of propagation of singularities at glancing rays on manifolds with a smooth boundary. In Section 3 we describe the structure of the bicharacteristics, in particular their uniqueness properties. In Section 4 we recall the already mentioned b-Sobolev spaces and the 'standard' propagation of singularities theorem based on these, also discussing how these can be used to reduce Theorem 1.1 to Theorem 1.4. Section 5 is the technical heart of the paper in which we analyze paired Lagrangian distributions relevant to the positive commutator estimates in our setting. Section 6 gives microlocal elliptic regularity in this setting, and is used as a warm-up towards the positive commutator estimate. Section 7 gives the proof of the key analytic estimate towards the proof of the propagation of singularities, which is completed in Section 8 in the form of Theorem 1.4.

2. The structure of positive commutator estimates

In order to motivate our proof, we recall the structure of the standard positive commutator estimate, in the formulation of Hörmander [9], Melrose and Sjöstrand [11, 12], giving propagation of singularities for the wave operator $\Box$ on a C^∞ Lorentzian manifold (X, g) (and indeed more generally for pseudodifferential operators of real principal type).

We state at the outset that since all results are local, *one may always arrange that the Schwartz kernels of various operators we consider have proper support, or even compact support, and we do not comment on support issues from this point on.* Similarly, *all Sobolev spaces in which distributions are assumed to lie are local*, and we do not always show this in the notation explicitly.

One arranges that for an appropriate operator $A \in \Psi^{2s-1}(X)$ that

$$i[\Box, A] = B^*B + E + F,\ B \in \Psi^s(X),\ E \in \Psi^{2s}(X),\ F \in \Psi^{2s-2\epsilon_0}(X),$$

with $\epsilon_0 > 0$ (typically $\epsilon_0 = 1/2$), where the solution is a priori known to lie in H^s on $\mathrm{WF}'(E)$ (this is where we propagate the estimate from), and lie in $H^{s-\epsilon_0}$ on $\mathrm{WF}'(F)$ (which is typically equal to $\mathrm{WF}'(A)$). Then one gets for u with $\Box u = 0$ (or even $\Box u = f$),

$$\langle iAu, \Box u\rangle - \langle i\Box u, A^*u\rangle = \langle i[\Box, A]u, u\rangle = \|Bu\|^2 + \langle Eu, u\rangle + \langle Fu, u\rangle, \tag{2.1}$$

provided that u is sufficiently nice for the pairings and the adjoint (integration by parts) to make sense; then one can estimate Bu in L^2, and thus u on the elliptic set of B in H^s in terms of u on $\mathrm{WF}'(E)$ in H^s, u on $\mathrm{WF}'(F)$ in $H^{s-\epsilon_0}$ and u itself in any Sobolev space H^{-N} globally (the latter is to deal with smoothing errors). A standard regularization argument gives that $u \in H^s$ actually on the elliptic set $\mathrm{Ell}(B)$ of B even without stronger a priori assumptions.

The desired commutator then is arranged by choosing some symbol a in S^{2s-1}, such that, with p denoting the dual metric function, which is the principal symbol of $\Box$,

$$H_p a = -b^2 + e, \text{ modulo } S^{2s-2\epsilon_0}, \tag{2.2}$$

and letting a, b, e be the principal symbols of A, B and E respectively. We recall how to do this in a robust manner, following the presentation of [21, Section 7], though with the more convenient notation of constants of [19] and [20]. Fix ρ to be a positive elliptic symbol of order 1 locally in the region where we are considering, e.g. $\rho = \langle\xi\rangle$ in canonical coordinates (x, ξ) based on local coordinates x on the base space X. Let

$$\mathsf{H}_p = \rho^{-m+1} H_p,$$

so H_p is homogeneous of degree zero. Homogeneous degree zero functions can be regarded as functions on S^*X, and correspondingly H_p can be considered a vector field on S^*X. One can actually arrange local coordinates $(q_1, q_2, \ldots, q_{2n-1})$ on S^*X such that $\mathsf{H}_p = \frac{\partial}{\partial q_1}$ – this is not necessary, but is a useful guide. First let $\tilde{\eta} \in C^\infty(S^*X)$ be a function with

$$\tilde{\eta}(\bar{q}) = 0, \ \mathsf{H}_p \tilde{\eta}(\bar{q}) > 0. \tag{2.3}$$

Thus, $\tilde{\eta}$ measures propagation along bicharacteristics; e.g. $\tilde{\eta} = q_1$ works, but so do many other choices. We will use a function ω to localize near putative bicharacteristics. This statement is deliberately vague; at first we only assume that $\omega \in C^\infty(S^*X)$ is the sum of the squares of C^∞ functions σ_j, $j = 1, \ldots, 2n-2$, with non-zero differentials at $\bar{q}$ such that $d\tilde{\eta}$ and $d\sigma_j$, $j = 1, \ldots, 2n-2$, span $T_{\bar{q}} S^*X$, and such that

$$\mathsf{H}_p \sigma_j(\bar{q}) = 0. \tag{2.4}$$

Such a function ω is non-negative and it vanishes quadratically at $\bar{q}$, i.e. $\omega(\bar{q}) = 0$ and $d\omega(\bar{q}) = 0$. Moreover, $\omega^{1/2} + |\tilde{\eta}|$ is equivalent to the distance from $\bar{q}$ with respect to any distance function given by a Riemannian metric on S^*X. An example is $\omega = q_2^2 + \ldots + q_{2n-1}^2$ with the notation from before, but again there are many other possible choices; with this choice $\mathsf{H}_p \omega = 0$. We now consider a family symbols, parameterized by constants $\delta \in (0, 1)$, $\epsilon \in (0, 1]$, $\beta \in (0, 1]$, of the form

$$a = \chi_0 \left(\digamma^{-1} \Big(2\beta - \frac{\phi}{\delta} \Big) \right) \chi_1 \left(\frac{\tilde{\eta} + \delta}{\epsilon\delta} + 1 \right), \tag{2.5}$$

where

$$\phi = \tilde{\eta} + \frac{1}{\epsilon^2 \delta} \omega,$$

$\chi_0(t) = 0$ if $t \leq 0$, $\chi_0(t) = e^{-1/t}$ if $t > 0$, $\chi_1 \in C^\infty(\mathbb{R})$, $\chi_1 \geq 0$, $\sqrt{\chi_1} \in C^\infty(\mathbb{R})$, $\operatorname{supp} \chi_1 \subset [0, +\infty)$, $\operatorname{supp} \chi_1' \subset [0, 1]$, and $\digamma > 0$ will be taken large. Here $\digamma$ is used to deal with technical issues such as weights and regularization, so at first reading one may consider it fixed. We also need weights such as ρ^{2s-1} where $s \in \mathbb{R}$ is as above; in a product type Lorentzian setting these can be arranged Hamilton commute with p by taking $\rho = |\tau|$ and thus can be ignored, otherwise taking $\digamma$ large will deal with them in any case. Thus, the actual principal symbol of A is

$$\sigma_{2s-1}(A) = \rho^{2s-1}\chi_0\left(\digamma^{-1}\Big(2\beta - \frac{\phi}{\delta}\Big)\right)\chi_1\left(\frac{\tilde\eta+\delta}{\epsilon\delta}+1\right), \tag{2.6}$$

We analyze the properties of a step by step. First, note that $\phi(\bar q) = 0$, $\mathsf{H}_p\phi(\bar q) = \mathsf{H}_p\tilde\eta(\bar q) > 0$, and $\chi_1(\frac{\tilde\eta+\delta}{\epsilon\delta}+1)$ is identically 1 near $\bar q$, so $\mathsf{H}_p a(\bar q) < 0$. Thus, $\mathsf{H}_p a$ has the correct sign, and is in particular non-zero, at $\bar q$.

Next,

$$q \in \operatorname{supp} a \Rightarrow \phi(q) \le 2\beta\delta \text{ and } \tilde\eta(q) \ge -\delta - \epsilon\delta.$$

Since $\epsilon \le 1$, we deduce that in fact $\tilde\eta = \tilde\eta(q) \ge -2\delta$. But $\omega \ge 0$, so $\phi = \phi(q) \le 2\beta\delta$ implies that $\tilde\eta = \phi - \epsilon^{-2}\delta^{-1}\omega \le \phi \le 2\beta\delta \le 2\delta$. Hence, $\omega = \omega(q) = \epsilon^2\delta(\phi - \tilde\eta) \le 4\epsilon^2\delta^2$. Since ω vanishes quadratically at $\bar q$, it is useful to rewrite the estimate as $\omega^{1/2} \le 2\epsilon\delta$. Combining these, we have seen that on $\operatorname{supp} a$,

$$-\delta - \epsilon\delta \le \tilde\eta \le 2\beta\delta \text{ and } \omega^{1/2} \le 2\epsilon\delta. \tag{2.7}$$

Moreover, on $\operatorname{supp} a \cap \operatorname{supp}\chi_1'$,

$$-\delta - \epsilon\delta \le \tilde\eta \le -\delta \text{ and } \omega^{1/2} \le 2\epsilon\delta.$$

Note that given any neighborhood U of $\bar q$, we can thus make a supported in U by choosing δ sufficiently small (and keeping $\epsilon, \beta \le 1$). Note that $\operatorname{supp} a$ is a parabola shaped region, which is very explicit in case $\tilde\eta = q_1$ and $\omega = q_2^2 + \ldots + q_{2n-1}^2$. Note that as $\epsilon \to 0$, but δ fixed, the parabola becomes very sharply localized at $\omega = 0$; taking β small makes a localized very close to the segment $\tilde\eta \in [-\delta, 0]$.

So we have shown that a is supported near $\bar q$. We define

$$e = \chi_0\left(\digamma^{-1}\Big(2\beta - \frac{\phi}{\delta}\Big)\right)\mathsf{H}_p\left(\chi_1\Big(\frac{\tilde\eta+\delta}{\epsilon\delta}+1\Big)\right), \tag{2.8}$$

so the crucial question in our quest for (2.2) is whether $\mathsf{H}_p\phi \ge 0$ on $\operatorname{supp} a$. Note that choosing $\delta_0 \in (0,1)$ sufficiently small, one has for $\delta \in (0,\delta_0]$, $\epsilon \in (0,1]$, $\beta \in (0,1]$, $\mathsf{H}_p\tilde\eta \ge c_0 > 0$ where $|\tilde\eta| \le 2\delta_0$, $\omega^{1/2} \le 2\delta_0$. So $\mathsf{H}_p\phi \ge \frac{c_0}{2} > 0$ on $\operatorname{supp} a$ if $\delta < \delta_0$, $\epsilon, \beta \le 1$, provided that $|\mathsf{H}_p\omega| \le \frac{c_0}{2}\epsilon^2\delta$ there, which is automatically the case if one arranges

$$\mathsf{H}_p q_j = 0 \text{ for } j \ge 2, \text{ and } \sigma_j = q_{j+1}, \tag{2.9}$$

i.e. any $\epsilon > 0$ works. Note that if $H_p\phi \ge \frac{c_0}{2}$ on $\operatorname{supp} a$ then one can let

$$b = \digamma^{-1/2}\delta^{-1/2}\sqrt{\mathsf{H}_p\phi}\sqrt{\chi_0'\left(\digamma^{-1}\Big(2\beta - \frac{\phi}{\delta}\Big)\right)}\sqrt{\chi_1\left(\frac{\tilde\eta+\delta}{\epsilon\delta}+1\right)}; \tag{2.10}$$

thus (2.2) holds with $s = 1/2$ and $\epsilon_0 = 1/2$.

However, we do not need such a strong relationship to H_p, which cannot be arranged (with smooth σ_j) if one makes p have conormal singularities at a submanifold. Suppose instead that we merely get ω 'right' at $\bar q$, in the sense that

$$\omega = \sum \sigma_j^2,\ \mathsf{H}_p\sigma_j(\bar q) = 0. \tag{2.11}$$

Then, $\mathsf{H}_p\sigma_j$ being a C^∞, thus locally Lipschitz, function,

$$|\mathsf{H}_p\sigma_j| \le C_0(\omega^{1/2} + |\tilde\eta|), \tag{2.12}$$

so $|\mathsf{H}_p\omega| \leq C\omega^{1/2}(\omega^{1/2}+|\tilde\eta|)$. Using (2.7), we deduce that $|\mathsf{H}_p\omega| \leq \frac{c_0}{2}\epsilon^2\delta$ provided that $\frac{c_0}{2}\epsilon^2\delta \geq C''(\epsilon\delta)\delta$, i.e. that $\epsilon \geq C'\delta$ for some constant C' independent of ϵ, δ (and of β). Now the size of the parabola at $\tilde\eta = -\delta$ is roughly $\omega^{1/2} \sim \delta^2$, i.e. we have localized along a single direction, namely the direction of H_p at $\bar q$.

By a relatively simple argument, also due to Melrose and Sjöstrand [11, 12] in the case of smooth boundaries, one can piece together such estimates (i.e. where the direction is correct 'to first order') and deduce the propagation of singularities. We explain this in more detail in the last section of the paper.

This argument would go through if one manages to arrange this with F having just the property that $F : H^{s-\epsilon_0} \to H^{-s+\epsilon_0}$, i.e. the ps.d.o. behavior of F does not matter as long as one has $H^{s-\epsilon_0}$ background regularity – indeed, one only needs the $H^{s-\epsilon_0}$ regularity on the wave front set of F.

We finally indicate how one deals with regularizers and weights. Let ρ is a positive elliptic symbol of order 1 as above. It is convenient to write

$$\breve a = \rho^{s-1/2}\sqrt{a} \in S^{s-1/2}$$

with a as in (2.5), and let $\breve A \in \Psi^{s-1/2}$ have principal symbol $\breve a$, $\mathrm{WF}'(\breve A)$ contained in the conic support of $\breve a$, and be formally self-adjoint (e.g. take $\breve A_0$ to be a quantization of $\breve a$ in local coordinates, and then take the self-adjoint part, $\breve A = (\breve A + \breve A^*)/2$), and let $A = \breve A^2$. We also let Λ_r, $r \in [0,1]$, be such that the family is uniformly bounded in $\Psi^0(X)$, $\Lambda_r \in \Psi^{-1}$ for $r > 0$, and $\Lambda_r \to \mathrm{Id}$ in Ψ^ϵ for $\epsilon > 0$, and Λ_r formally self-adjoint. For instance, one can take Λ_r to be a (symmetrized) quantization of $\phi_r = (1+r\rho)^{-1}$. Let

$$A_r = \Lambda_r A \Lambda_r,\ \ a_r = \phi_r^2\rho^{2s-1}a.$$

Then the principal symbol of $i[\Box, A_r]$, as a family with values in Ψ^{2s}, is

$$\phi_r^2\rho^{2s}\mathsf{H}_p a + a\phi_r^2\rho^{2s}((2s-1) - r\phi_r\rho)(\rho^{-1}\mathsf{H}_p\rho).$$

Now, $|r\phi_r\rho| \leq 1$ while $\rho^{-1}\mathsf{H}_p\rho$ is bounded, being a symbol of order 0, so the second term is bounded in absolute value by $Ca\phi_r^2\rho^{2s}$. Now, given $M > 0$, for sufficiently large $\digamma$, not only is $\mathsf{H}_p a$ of the form $-b^2 + e$, but

$$\phi_r^2\rho^{2s}\big(\mathsf{H}_p a + ((2s-1) - r\phi_r\rho)(\rho^{-1}\mathsf{H}_p\rho)a\big) = -b_r^2 - M^2\rho a_r + e_r,$$

with $e_r = \phi_r^2\rho^{2s}e$, e as before. This is due to $\chi_0(t) = t^2\chi_0'(t)$ for $t \in \mathbb{R}$, so

(2.13)

$$\begin{aligned}
&\digamma^{-1}\delta^{-1}(\mathsf{H}_p\phi)\chi_0'\left(\digamma^{-1}\Big(2\beta - \frac{\phi}{\delta}\Big)\right)\\
&\qquad - \Big(\big((2s-1) - r\phi_r\rho\big)(\rho^{-1}\mathsf{H}_p\rho) + M^2\Big)\chi_0\left(\digamma^{-1}\Big(2\beta - \frac{\phi}{\delta}\Big)\right)\\
&= \digamma^{-1}\delta^{-1}\left((\mathsf{H}_p\phi) - \Big(\big((2s-1) - r\phi_r\rho\big)(\rho^{-1}\mathsf{H}_p\rho) + M^2\Big)\digamma^{-1}\delta\Big(2\beta - \frac{\phi}{\delta}\Big)^2\right)\\
&\qquad\qquad \times \chi_0'\left(\digamma^{-1}\Big(2\beta - \frac{\phi}{\delta}\Big)\right),
\end{aligned}$$

and $|2\beta - \frac{\phi}{\delta}| \leq 4$ on $\operatorname{supp} a$, so for sufficiently large $\digamma$ (independent of $\delta, \epsilon, \beta \in (0,1]$ as long as $\epsilon \geq C'\delta$, C' as above), the factor in the large parentheses on the right hand side is positive, with a positive lower bound, and thus its square root c_r satisfies that $c_r \in S^0$ uniformly, $c_r \in S^{-1}$ for $r > 0$, and c_r is elliptic where χ_0 and χ_1 are both positive. Now with $E_r = \Lambda_r E \Lambda_r$, E as before with wave front set in the conic support of a, and taking B_r a family, uniformly bounded in Ψ^s, with (uniform, or family) wave front set in the conic support of a and with principal symbol

$$b_r = \phi_r \rho^s c_r \sqrt{\chi_0'\left(F^{-1}\left(2\beta - \frac{\phi}{\delta}\right)\right)}\sqrt{\chi_1\left(\frac{\tilde{\eta}+\delta}{\epsilon\delta}+1\right)}, \tag{2.14}$$

we have

$$i[\Box, A_r] = -B_r^* B_r - M^2(\breve{A}_r)^* Q^* Q \breve{A}_r + E_r + F_r,\ \ \breve{A}_r = \breve{A}\Lambda_r,$$

with $Q \in \Psi^{1/2}$ with symbol ρ (thus elliptic), with F_r uniformly bounded in Ψ^{2s-1}, and with uniform wave front set in the conic support of a. Now for $r > 0$ applying this expression to u and pairing with u, as in (2.1), makes sense provided $\mathrm{WF}^{s-1/2}(u)$ is disjoint from the conic support of a, and we obtain

$$\|B_r u\|^2 + M^2\|Q\breve{A}_r u\|^2 \leq 2|\langle A_r u, \Box u\rangle| + |\langle E_r u, u\rangle| + |\langle F_r u, u\rangle|. \tag{2.15}$$

Further, with G a parametrix for Q with $GQ = \mathrm{Id} + R$, $R \in \Psi^{-\infty}$,

$$\begin{aligned} 2|\langle A_r u, \Box u\rangle| &\leq 2|\langle Q\breve{A}_r u, G\breve{A}_r \Box u\rangle| + 2|\langle R\breve{A}_r u, \breve{A}_r \Box u\rangle| \\ &\leq \|Q\breve{A}_r u\|^2 + \|G\breve{A}_r \Box u\|^2 + 2|\langle R\breve{A}_r u, \breve{A}_r \Box u\rangle|, \end{aligned} \tag{2.16}$$

and the first term on the left hand side now can be absorbed into $M^2\|Q\breve{A}_r u\|^2$ (if we chose $M \geq 1$). Letting $r \to 0$ we get a uniform bound for $\|B_r u\|$, and thus by the weak compactness of the unit ball in L^2 plus that $B_r u \to B_0 u$ in distributions, we conclude that $B_0 u \in L^2$, completing the proof that the elliptic set of B_0, i.e. where χ_0 and χ_1 are positive, is disjoint from $\mathrm{WF}^s(u)$.

One completes the proof of the propagation estimate by an inductive argument in s, raising the order s by $1/2$ in each step. During this process one needs to shrink the support of a so that, denoting the replacement of a given in the next step of the iteration by a', at every point of $\operatorname{supp} a'$ either b is elliptic (the b corresponding to the original a), or one has a priori regularity there (which is the case on $\operatorname{supp} e$). This can be done by reducing β which shrinks the support as desired. We refer to [8, Section 24.5], in particular to last paragraph of the proof of Proposition 24.5.1, for further details.

3. Bicharacteristics

Since g is not C^∞, we need to discuss the behavior of bicharacteristics, i.e. integral curves of H_p, in some detail. When $g \in I^{[-s_0]}(Y)$ and $\operatorname{codim} Y + 1 + \alpha < s_0 < \operatorname{codim} Y + 2$ (with $0 < \alpha < 1$), which is the main case of interest for us, then g is $C^{1,\alpha}$, and thus H_p is a $C^{0,\alpha}$. Thus, the standard ODE theory ensures the existence of bicharacteristics, but does not ensure their uniqueness (as Hölder-α, $\alpha < 1$, is insufficient for this; Lipschitz would suffice). Nevertheless, for normally incident rays at a codimension one hypersurface Y one has local uniqueness. In this setting, locally, H_p is transversal to $T^*_Y X$, and using local coordinates (x, y) such that $Y = \{x = 0\}$ and dual coordinates (ξ, η), H_p is continuous in x and C^∞ in (y, ξ, η), so the following lemma gives this conclusion:

Lemma 3.1. *If $I \subset \mathbb{R}_{x_n}$ is an open interval containing 0, $O \subset \mathbb{R}^{n-1}_{x'}$ open containing 0, $V = \sum_{j=1}^n V_j(x)\partial_j$ is a continuous real vector field on $O \times I$ with*

$V_j \in C(I; C^{0,1}(O))$ and with $V_n(0) \neq 0$ then there exists $\Omega \subset O \times I$ open containing 0 and $\delta > 0$ such that the given $x^{(0)} \in \Omega$, there is a unique C^1 integral curve $x : (-\delta, \delta) \to O \times I$ with $x(0) = x^{(0)}$.

Proof. Since the other sign works similarly, we may assume that $V_n(0) > 0$, and also at the cost of shrinking I and O then $V_n > c > 0$ on $O \times I$.

Being an integral curve means that $\frac{dx_j}{dt}(t) = V_j(x(t))$. We consider an other system of ODE, namely writing $Z(s) = (z'(s), s)$, with $(-\delta', \delta') \subset I$, $s_0 \in (-\delta', \delta')$, $z' \in C^1((-\delta', \delta'); O)$, $z_n \in C^1((-\delta', \delta'); \mathbb{R})$, $z = (z', z_n)$.

$$\frac{dz}{ds}(s) = F(z'(s), s),\ z(s_0) = z^{(0)} \in O' \times I', \tag{3.1}$$

with

$$\begin{aligned} F_j(y) &= \frac{V_j(y)}{V_n(y)},\ j = 1, \ldots, n-1, \\ F_n(y) &= \frac{1}{V_n(y)}, \end{aligned} \tag{3.2}$$

so $F \in C((-\delta', \delta')_s; C^{0,1}(O))$. The key point here is that $F(z'(s), s)$ on the right hand side of (3.1) is independent of $z_n(s)$, i.e. (3.1) is of the type $\frac{dz}{ds}(s) = \Phi(z(s), s)$, with Φ continuous in the last variable and Lipschitz in the first. Thus, the standard ODE existence and uniqueness theorem applies, giving the local existence and uniqueness of solutions to (3.1), provided $O' \times I'$ is a sufficiently small neighborhood of 0.

Now if $x = x(t)$ is a C^1 integral curve of V, and we let T be the inverse function of $x_n = x_n(t)$ near 0, which exists and is C^1 by the inverse function theorem as $\frac{dx_n}{dt}(t) = V_n(x(t)) \geq c > 0$, with $T'(s) = \frac{1}{V_n(x(T(s)))}$, then $z = (z', z_n)$ with $z' = x' \circ T$, $z_n = T$, satisfies (3.1) with $s_0 = x_n(0) = (x^{(0)})_n$, $z^{(0)} = (x'(0), 0) = ((x^{(0)})', 0)$. Indeed, z is C^1 as x and T are such, and

$$\begin{aligned} \frac{dz_j}{ds} &= (\frac{dx_j}{dt} \circ T)T' = \frac{V_j \circ x \circ T}{V_n \circ x \circ T},\ j = 1, \ldots, n-1, \\ \frac{dz_n}{ds} &= \frac{1}{V_n \circ x \circ T}, \end{aligned}$$

which, as $x_n \circ T(s) = s$, is a rewriting of (3.1). One can also proceed backwards, starting with a solution of (3.1), by letting x_n be the inverse function of z_n, and then letting $x_j = z_j \circ x_n$ for $j = 1, \ldots, n-1$.

Thus, if one has two solutions $x(t)$ and $\tilde{x}(t)$ of $\frac{dx_j}{dt}(t) = V_j(x(t))$ with $x(0) = x^{(0)}$, then defining T, resp. $\tilde{T}$, as the inverse functions of x_n, resp. $\tilde{x}_n$, we have solutions z, resp. $\tilde{z}$ of (3.1) with initial conditions $((x^{(0)})', 0)$ and time $(x^{(0)})_n$. Thus, by the uniqueness part of the ODE theorem, $z = \tilde{z}$. The nth components give then $T = \tilde{T}$, hence $x_n = \tilde{x}_n$, and thus the other components yield $x_j = \tilde{x}_j$, completing the proof. □

As mentioned, an immediate consequence is, if one lets $\mathcal{G}$ be the glancing set, i.e. where H_p is tangent to $T^*_Y X$:

Corollary 3.2. *Suppose Y has codimension 1. Then the integral curves of H_p in $\Sigma \setminus \mathcal{G}$ through a given point are unique.*

Proof. Suppose there are two solutions $x(t)$ and $\tilde{x}(t)$ with the same initial condition $x^{(0)}$ at time 0. Assuming that $x(t) \neq \tilde{x}(t)$ for some $t > 0$, let t_0 be the infimum of positive times such that $x(t) \neq \tilde{x}(t)$, so any neighborhood I of t_0 contains $t \in I$ such that $x(t) \neq \tilde{x}(t)$ but, as x and $\tilde{x}$ are continuous $x(t_0) = \tilde{x}(t_0)$. (The last assertion is clear if $t_0 = 0$; if $t_0 > 0$ it follows as $x(t) = \tilde{x}(t)$ for $t \in [0, t_0)$ by definition of t_0.) Then the local uniqueness result stated above yields a contradiction. Since negative times are dealt with similarly, this completes the proof. □

4. Law of reflection: standard propagation of singularities

We now recall from [19] the basic law of reflection. In [19] this is shown in the setting of manifolds with corners with Dirichlet or Neumann boundary conditions. However, the same arguments go through in our setting, where we consider the quadratic form domain $H^1_{\mathrm{loc}}(X)$. Generalized broken bicharacteristics (GBB) are defined in this setting to allow reflected rays as follows.

For simplicity consider Y of codimension 1 (this is all that is needed for Theorem 1.1, and Theorem 1.4 does not need this at all). Since the results are local, we may assume that Y separates X into two manifolds $X_\pm$ with boundary Y. Each of $X_\pm$ comes equipped with the so-called b-cotangent bundle, ${}^{\mathrm{b}}T^*X_\pm$. This is the dual bundle of the b-tangent bundle, whose smooth sections are C^∞ vector fields on $X_\pm$ tangent to Y, denoted by $\mathcal{V}_{\mathrm{b}}(X_\pm)$. Over $C^\infty(X_\pm)$, these are locally spanned by $x\partial_x$ and ∂_{y_j}, $j = 1, \dots, n-1$, and correspondingly, a local basis for smooth sections of ${}^{\mathrm{b}}T^*X_\pm$ is $\frac{dx}{x}$ and dy_j, $j = 1, \dots, n-1$. One may thus write smooth sections of ${}^{\mathrm{b}}T^*X_\pm$ as

$$\sigma(x,y)\,\frac{dx}{x} + \sum_j \eta_j(x,y)\,dy_j; \tag{4.1}$$

so (x, y, σ, η) are local coordinates on ${}^{\mathrm{b}}T^*X_\pm$. As $\mathcal{V}_{\mathrm{b}}(X_\pm) \subset \mathcal{V}(X_\pm)$, there is a dual map $\pi_\pm : T^*X_\pm \to {}^{\mathrm{b}}T^*X_\pm$; the kernel at $p \in Y$ is given by N^*_pY, and the range can be naturally identified with $T^*_pY = T^*_pX_\pm / N^*_pY$. Concretely, if one uses canonical dual coordinates (x, y, ξ, η) on T^*X, writing one-forms as

$$\xi(x,y)\,dx + \sum_j \eta_j(x,y)\,dy_j,$$

then

$$\pi_\pm(x, y, \xi, \eta) = (x, y, x\xi, \eta),$$

corresponding to the identification $\xi\,dx = (x\xi)\,\frac{dx}{x}$. The same constructions can be performed directly on X, working with C^∞ vector fields tangent to Y, which we denote by $\mathcal{V}_{\mathrm{b}}(X;Y)$. The so obtained cotangent bundle ${}^{\mathrm{b}}T^*X$, which is a C^∞ vector bundle, when restricted to $X_\pm$, gives ${}^{\mathrm{b}}T^*X_\pm$, and again comes with a natural map $\pi : T^*X \to {}^{\mathrm{b}}T^*X$.

In particular, one can now consider the characteristic set $\Sigma \subset T^*X$ of $\Box$, and its image $\dot{\Sigma} \subset {}^{\mathrm{b}}T^*X$ under π; this is called the compressed characteristic set. A GBB $\tilde{\gamma}$ is defined to be a continuous map from an interval to $\dot{\Sigma}$ satisfying a Hamilton vector field condition, namely that for all $f \in C^\infty({}^{\mathrm{b}}T^*X)$ real valued,

$$\limsup_{s \to s_0} \frac{f(\tilde{\gamma}(s)) - f(\tilde{\gamma}(s_0))}{s - s_0} \leq \sup\{(\mathsf{H}_p \pi^* f)(q) :\ q \in \Sigma,\ \pi(q) = \tilde{\gamma}(s_0)\}.$$

Thus, C^1 integral curves of H_p in $\Sigma \subset T^*X$ are certainly generalized broken bicharacteristics (i.e. their image under π is), but more generally, any two integral curve segments of H_p, say γ_+ defined on $[0, s_0)$ and γ_- on $(-s_0', 0]$, can be combined into a single GBB provided $\pi(\gamma_+(0)) = \pi(\gamma_-(0))$.

For a Lorentzian metric g, T^*Y can be regarded as a subset of T^*X, identified as the orthocomplement of the spacelike N^*Y. In fact, one may arrange that the dual metric G is

$$G = A(x, y)\partial_x^2 + \sum_j 2C_j(x, y)\partial_x\partial_{y_j} + \sum_{ij} B_{ij}(x, y)\partial_{y_i}\partial_{y_j},$$

with

$$C_j(0, y) = 0,\ A(0, y) < 0,\ B(0, y) \text{ Lorentzian on } T_y^*Y,$$

see [20, Section 2]. We write

$$B(0, y)\eta \cdot \eta = \sum_{ij} B_{ij}(0, y)\eta_i\eta_j$$

for the dual metric function of B. Then T^*Y is identified with points with $x = 0$ and $\xi = 0$. We recall from [19] and [20] that $\dot{\Sigma} = \mathcal{H} \cup \mathcal{G}$ is the union of the hyperbolic and the glancing sets at ${}^{\mathrm{b}}T_Y^*X$ with

$$\mathcal{H} \cap {}^{\mathrm{b}}T_Y^*X = \pi(\Sigma \setminus T^*Y),\ \mathcal{G} \cap {}^{\mathrm{b}}T_Y^*X = \pi(\Sigma \cap T^*Y).$$

Concretely, in coordinates on a chart $\mathcal{U}$, using the b-coordinates (x, y, σ, η),

$$\mathcal{H} \cap {}^{\mathrm{b}}T_{\mathcal{U}\cap Y}^*X = \{(0, y_0, 0, \eta_0) \in {}^{\mathrm{b}}T_{\mathcal{U}\cap Y}^*X :\ B(0, y_0)\eta_0 \cdot \eta_0 > 0\},$$
$$\mathcal{G} \cap {}^{\mathrm{b}}T_{\mathcal{U}\cap Y}^*X = \{(0, y_0, 0, \eta_0) \in {}^{\mathrm{b}}T_{\mathcal{U}\cap Y}^*X :\ B(0, y_0)\eta_0 \cdot \eta_0 = 0\}$$

If $q_0 = (0, y_0, \xi_0, \eta_0) \in \Sigma$ is not a glancing point, then locally all GBB $\tilde{\gamma}$ with $\tilde{\gamma}(0) = q_0$ are of the form discussed above, i.e. the concatenation of two integral curves of H_p. Indeed, such GBB stay outside ${}^{\mathrm{b}}T_Y^*X$ for a punctured time interval, i.e. there is $\epsilon > 0$ such that $\tilde{\gamma}(s) \notin {}^{\mathrm{b}}T_Y^*X$ for $s \in (-\epsilon, \epsilon) \setminus \{0\}$, so $\gamma_+ = \tilde{\gamma}|_{(0,\epsilon)}$, $\gamma_- = \tilde{\gamma}|_{(-\epsilon,0)}$ are integral curves of H_p; see [20, Lemma 2.1]. In view of the kernel of the map $T^*X \to {}^{\mathrm{b}}T^*X$ at Y, this means exactly that GBBs allow the standard law of reflection, i.e. the incident and reflected rays differ by a covector in N^*Y.

In order to state the propagation of singularities theorem, we need a notion of wave front set in ${}^{\mathrm{b}}T^*X \setminus o$. This is a simple extension of $\mathrm{WF}_{\mathrm{b}}^{1,m}(u)$ introduced in [19] for manifolds with corners to a manifold with a codimension one hypersurface Y replacing the boundary, as above. This wave front set in turn is based on the so-called b-pseudodifferential operators. In the setting of manifolds with boundaries, or indeed, corners, such as $X_\pm$, these are just the totally characteristic, or b-, pseudodifferential operators introduced by Melrose [14], see also [15], and discussed by Melrose and Piazza [16, Section 2]. We also refer to [19] for a concise description of the background. In our setting, to work on X with these operators, we recall that the Schwartz kernels of $\Psi_{\mathrm{b}}(X_+), \Psi_{\mathrm{bc}}(X_+)$ are tempered distributions on $X_+ \times X_+$ which are conormal on the blow-up $[X_+ \times X_+; \partial X_+ \times \partial X_+]$ to the front face and the lifted diagonal, in the sense of being either the partial Fourier transforms of symbols in the case of $\Psi_{\mathrm{bc}}(X_+)$, or those of classical (one-step polyhomogeneous) symbols in the case of $\Psi_{\mathrm{b}}(X_+)$, which extend smoothly across the front face (to which the diagonal is transversal, and thus this makes sense), and vanishing to infinite order on the side faces, i.e. the lifts of $X_+ \times \partial X_+$ and $\partial X_+ \times X_+$. Concretely, fixing $\phi \in C_c^\infty(\mathbb{R})$, identically 1 near 0, supported in $(-1/2, 1/2)$ and a coordinate chart

(x,y), a large subset of elements of $\Psi^m_{\mathrm{bc}}(X_+)$ and $\Psi^m_{\mathrm{b}}(X_+)$ (and indeed, all modulo smoothing operators, i.e. elements of $\Psi^{-\infty}_{\mathrm{b}}(X_+)=\Psi^{-\infty}_{\mathrm{bc}}(X_+)$) have the form

$$\begin{aligned}&(A_+v)(x,y)\\&=(2\pi)^{-n}\int e^{i\left(\sigma\frac{x-x'}{x'}+\sum_j\eta_j(y_j-y_j')\right)}\,\phi\Big(\frac{x-x'}{x'}\Big)\,a_+(x,y,\sigma,\eta)\,v(x',y')\,\frac{dx'\,dy'}{x'},\end{aligned}$$

where

$$a_+\in S^m([0,\infty)_x\times\mathbb{R}^{n-1}_y;\mathbb{R}^n_{\sigma,\eta}),\ \text{resp. } a_+\in S^m_{\mathrm{cl}}([0,\infty)_x\times\mathbb{R}^{n-1}_y;\mathbb{R}^n_{\sigma,\eta})$$

if $A_+\in\Psi^m_{\mathrm{bc}}(X_+)$, resp. $A_+\in\Psi^m_{\mathrm{b}}(X_+)$. (Here the symbol notation denotes symbolic behavior in the variables after the semicolon.) Note that ϕ is identically 1 near the diagonal lifted to $[X_+^2;(\partial X_+)^2]$, i.e. it does not affect the diagonal singularity at all; its role is to localize away from the side faces. Here the image of a_+ in S^m/S^{m-1}, or if a_+ is classical, the homogeneous degree m summand in its asymptotic expansion, is the principal symbol $\sigma_{\mathrm{b},m}(A_+)$ of A_+; this is naturally a function (or equivalence class of functions) on ${}^{\mathrm{b}}T^*X_+\setminus o$ (with o the zero section) regarding (σ,η) as fiber coordinates on this bundle as in (4.1).

We then *define* $\Psi_{\mathrm{b}}(X,Y)$ to consist of operators A acting on $C^\infty_{\mathrm{piece}}(X)$, continuous piecewise C^∞ functions, i.e. continuous functions v on X with $v|_{X_\pm}$ being C^∞, via Schwartz kernels on X^2 supported in $(X_+)^2\times(X_-)^2$, conormal on $[X^2;Y^2]$ such that the normal operators are the same. Such an operator can be identified with a pair of operators (A_+,A_-) given by the restriction to $\dot C^\infty(X_+)$, $\dot C^\infty(X_-)$, which are then in $\Psi_{\mathrm{b}}(X_+)$, resp. $\Psi_{\mathrm{b}}(X_-)$. Thus, modulo $\Psi^{-\infty}_{\mathrm{b}}(X,Y)$, with ϕ as above, these operators are of the form

$$\begin{aligned}&(Av)(x,y)\\&=(2\pi)^{-n}\int e^{i\left(\sigma\frac{x-x'}{x'}+\sum_j\eta_j(y_j-y_j')\right)}\,\phi\Big(\frac{x-x'}{x'}\Big)\,a(x,y,\sigma,\eta)\,v(x',y')\,\frac{dx'\,dy'}{x'},\end{aligned}$$

where

$$a\in S^m(\mathbb{R}_x\times\mathbb{R}^{n-1}_y;\mathbb{R}^n_{\sigma,\eta}),\ \text{resp. } a\in S^m_{\mathrm{cl}}(\mathbb{R}_x\times\mathbb{R}^{n-1}_y;\mathbb{R}^n_{\sigma,\eta})$$

if $A\in\Psi_{\mathrm{bc}}(X,Y)$, resp. $A\in\Psi_{\mathrm{b}}(X,Y)$. Note that the support condition on ϕ implies that $\frac12\le\frac{x}{x'}\le\frac32$ on $\operatorname{supp}\phi$, so in particular x and x' have the same sign, which means that A preserves the class of distributions supported in X_+, as well as those in X_-.

The key property of $\Psi^0_{\mathrm{bc}}(X,Y)$ is given in the following lemma:

Lemma 4.1. *(cf.* [19, Lemma 3.2]*) Any $A\in\Psi^0_{\mathrm{bc}}(X,Y)$ of compactly support is bounded on $H^1(X)$, with norm bounded by a seminorm in $\Psi^0_{\mathrm{bc}}(X,Y)$. By duality, the analogous statement holds on $H^{-1}(X)$ as well.*

Proof. If $u\in C^\infty_{\mathrm{comp}}(X)$ (which is a dense subspace of $H^1(X)$), then the compactly supported Au restricts to a C^∞ function on both X_+ and X_-, namely $A_\pm u|_{X_\pm}$, whose restriction to the boundary is the indicial operator $\hat N(A_\pm)(0)$ applied to $u|_Y$, and thus these two C^∞ functions coincide at Y. As first derivatives of such a continuous piecewise C^∞ function are given by the (no longer necessarily continuous, but still locally bounded) C^∞ functions given by differentiating the restrictions to each half-space separately, and as $\|A_\pm u|_{X_\pm}\|_{H^1}\le C\|u|_{X_\pm}\|_{H^1}$ by [19, Lemma 3.2], with C bounded by a continuous seminorm on $\Psi^0_{\mathrm{bc}}(X_\pm)$, the claim follows. □

We in fact need to generalize the coefficients of $\Psi_{\mathrm{bc}}(X,Y)$ to allow conormal singularities if g_{ij} are not simply piecewise smooth, i.e. have C^∞ restrictions to $X_\pm$. The key point is that one can allow more general conormal behavior at the front faces, i.e. allow a to satisfy symbolic bounds in x:

$$\left|\big((xD_x)^\ell D_y^\alpha D_{(\sigma,\eta)}^\beta a\big)(x,y,\sigma,\eta)\right| \leq C_{\ell\alpha\beta}\langle(\sigma,\eta)\rangle^{m-|\beta|};$$

denote by $\Psi_{\mathrm{bcc}}(X,Y)$ the resulting space. With such coefficients, in general, $A \in \Psi^0_{\mathrm{bcc}}(X,Y)$ no longer preserves H^1, though if one requires $A = A_0 + A_1$ with $A_0 \in \Psi^0_{\mathrm{bc}}(X,Y) + x\Psi^0_{\mathrm{bcc}}(X,Y)$, the H^1 bounds remain valid. However, L^2 bounds are valid in general, and $\Psi_{\mathrm{bcc}}(X,Y)$ is closed under composition with

$$\begin{aligned} &A \in \Psi^m_{\mathrm{bcc}}(X,Y),\ B \in \Psi^{m'}_{\mathrm{bcc}}(X,Y) \\ &\qquad \Longrightarrow AB \in \Psi^{m+m'}_{\mathrm{bcc}}(X,Y),\ [A,B] \in \Psi^{m+m'-1}_{\mathrm{bcc}}(X,Y), \end{aligned}$$

with principal symbols given by

$$\sigma_{\mathrm{b},m+m'}(AB) = \sigma_{\mathrm{b},m}(A)\sigma_{\mathrm{b},m'}(B),\ \sigma_{\mathrm{b},m+m'}([A,B]) = \frac{1}{i}\{\sigma_{\mathrm{b},m}(A), \sigma_{\mathrm{b},m'}(B)\},$$

with $\{.,.\}$ being the Hamilton bracket lifted to ${}^{\mathrm{b}}T^*X$. Note that if $f \in I^{[-s]}(Y)$ then the operator of multiplication by f is in $\Psi^0_{\mathrm{bc}}(X,Y)$ provided $s > 1$.

The propagation of singularities theorem is then the following:

Theorem 4.2. *Suppose $r, m \in \mathbb{R}$, $u \in H^{1,r}_{\mathrm{b,loc}}(X)$ and $\Box u \in H^{-1,m+1}_{\mathrm{b,loc}}(X)$. Then $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ is a union of maximally extended GBB.*

This theorem is proved by using b-ps.d.o's, $A \in \Psi_{\mathrm{bc}}(X)$ (so no conormal coefficients allowed), as microlocalizers, gaining regularity relative to $H^1_{\mathrm{loc}}(X)$. One works with the quadratic form as was done in [19] for the Neumann boundary condition and in [20] for differential forms. This requires commuting A past D_i, which works exactly as in these papers, as well as commuting A through $g_{ij} \in I^{[-s_0]}$. However, the commutator $[A, g_{ij}] \in \Psi_{\mathrm{bcc}}(X,Y)$ need not be further commuted through the derivatives D_i in view of the arguments of [20, Proposition 3.10] and its uses in Propositions 5.1 and Propositions 6.1 there, thus the proof of Theorem 4.2 can be completed as there.

Remark 4.3. Note that in particular Theorem 4.2 holds for transmission problems; indeed, these do not even require the introduction of $\Psi_{\mathrm{bcc}}(X,Y)$, i.e. are in this sense technically a bit easier than our, more regular, problem!

Thus, if $q_0 = (0, y_0, \xi_0, \eta_0) \in \mathrm{WF}^{1,m}_{\mathrm{b}}(u)$, then there is a GBB $\tilde\gamma$ with $\tilde\gamma(0) = q_0$ which is in $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$. If q_0 is not glancing, this states that for small $\epsilon > 0$, one of the backward integral curve segments of H_p, defined over $(-\epsilon, 0]$, is in $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$. Since $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ is just $\mathrm{WF}^{m+1}(u)$ outside Y, we thus have that if $q_0 \in \mathrm{WF}^{1,m}_{\mathrm{b}}(u)$, then there is a backward integral curve segment from q_0 which is in $\mathrm{WF}^{m+1}(u)$ over $(-\epsilon, 0)$.

As a corollary we can now prove that Theorem 1.4 implies Theorem 1.1:

Proof of Theorem 1.1 given Theorem 1.4. By assumption, for some $\delta > 0$, u is in $H^{s-\epsilon_0}$ along the backward bicharacteristics from q_0, i.e. $\mathrm{WF}^{s-\epsilon_0}(u) \cap \tilde\gamma|_{(-\delta,0)} = \emptyset$ for all $\tilde\gamma$ with $\tilde\gamma(0) = q_0$; note that for $\delta > 0$ sufficiently small, these are disjoint from ${}^{\mathrm{b}}T^*_Y X$. The wave front set being closed, there is a neighborhood U of these bicharacteristic segments disjoint from $\mathrm{WF}^{s-\epsilon_0}(u)$. Let t be a global time function, which thus has a derivative with a definite sign along H_p depending on the component of the characteristic set. Since the other case of similar, we assume that t is increasing along H_p in the component of q_0. Now let $t_0 = t(q_0)$, and let

$$T_2 = \sup\{t(\tilde\gamma(-3\delta/4)) : \ \tilde\gamma \text{ a GBB}, \ \tilde\gamma(0) = q_0\} < t_0,$$

and let $T_1 \in (T_2, t_0)$. Let

$$K = \{\tilde\gamma(s) : \ t(\tilde\gamma(s)) \in [T_2, T_1], \ \tilde\gamma \text{ a GBB}, \ \tilde\gamma(0) = q_0\},$$

which is thus compact, and if $\tilde\gamma(s) \in K$ then $s \in (-\delta, 0)$, so $\tilde\gamma(s) \notin \mathrm{WF}^{s-\epsilon_0}(u)$, so $K \cap \mathrm{WF}^{s-\epsilon_0}(u) = \emptyset$ and U is a neighborhood of K. Let $\chi_0 \in C^\infty(\mathbb{R})$ be such that $\chi_0 \equiv 1$ near $(-\infty, T_2]$, and $\chi_0 \equiv 0$ near $[T_1, \infty)$, and let $\chi = \chi_0 \circ t$. Let $\Box_+^{-1}$ denote the forward solution operator for $\Box$, i.e. given f supported in $t > t_1$, $v = \Box_+^{-1} f$ is the unique solution of $\Box v = f$ with $t > t_1$ on $\operatorname{supp} v$. Then

$$u = \chi u - \Box_+^{-1}[\Box, \chi]u,$$

since both sides solve $\Box w = 0$ and the difference is supported in $t \geq T_2$. Similarly, with $\Box_-^{-1}$ the backward solution operator,

$$u = (1-\chi)u - \Box_-^{-1}[\Box, 1-\chi]u = (1-\chi)u + \Box_-^{-1}[\Box, \chi]u,$$

so

$$u = (\Box_-^{-1} - \Box_+^{-1})[\Box, \chi]u.$$

Moreover, for any f, $v = (\Box_-^{-1} - \Box_+^{-1})f$ solves $\Box v = f$, and as $\mathrm{WF}_{\mathrm{b}}^{1,m}(\Box_+^{-1})(f)$ is contained in points from which some backward GBB enters $\mathrm{WF}_{\mathrm{b}}^{-1,m-1}(f)$, and analogously $\mathrm{WF}_{\mathrm{b}}^{1,m}(\Box_-^{-1})(f)$ is contained in points from which some forward GBB enters $\mathrm{WF}_{\mathrm{b}}^{-1,m-1}(f)$, $\mathrm{WF}_{\mathrm{b}}^{1,m}(v)$ is contained in GBB through $\mathrm{WF}_{\mathrm{b}}^{-1,m-1}(f)$.

So now let $Q \in \Psi^0(X)$ be such that $\mathrm{WF}'(Q) \subset U$ and $\mathrm{WF}'(\mathrm{Id} - Q) \cap K = \emptyset$, and let

$$u_0 = (\Box_-^{-1} - \Box_+^{-1})Q[\Box, \chi]u, \ \ u_1 = (\Box_-^{-1} - \Box_+^{-1})(\mathrm{Id} - Q)[\Box, \chi]u.$$

We treat u_0 and u_1 separately.

We start with u_1. We note that backward bicharacteristics from q_0 cannot enter $\mathrm{WF}'(\mathrm{Id} - Q) \cap T^*_{\operatorname{supp} d\chi} X$, for if $\tilde\gamma$ is such a backward bicharacteristic from q_0 and $\tilde\gamma(s) \in T^*_{\operatorname{supp} d\chi} X$, then $t(\tilde\gamma(s)) \in [T_2, T_1]$, so $\tilde\gamma(s) \in K$, which is disjoint from $\mathrm{WF}'(\mathrm{Id} - Q)$. Correspondingly

$$q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u_1),$$

and $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u_1)$ is disjoint from forward bicharacteristic segments from q_0, in particular, for sufficiently small $s > 0$, for which $\tilde\gamma(s) \notin {}^{\mathrm{b}}T^*_Y X$, $\tilde\gamma(s) \notin \mathrm{WF}(u_1)$.

Now we turn to u_0. As $\mathrm{WF}^{s-\epsilon_0-1}([\Box, \chi]u) \subset \mathrm{WF}^{s-\epsilon_0}(u) \cap T^*_{\operatorname{supp} d\chi} X$ is disjoint from U, we deduce that $Q[\Box, \chi]u \in H^{s-\epsilon_0-1}$, and thus

$$u_0 = (\Box_-^{-1} - \Box_+^{-1})Q[\Box, \chi]u \in H^{1, s-\epsilon_0-1}_{\mathrm{b,loc}}(X).$$

In particular, $u_0 \in L^2_{\mathrm{loc}}$ as $s - \epsilon_0 \geq 0$. By Corollary 8.4, $u_0 \in H^{s-\epsilon_0}_{\mathrm{loc}}$. Moreover, with γ_0 denoting the integral curve of H_p through $(0, y_0, \xi_0, \eta_0)$, $\gamma_0|_{(-\delta,0)}$ is disjoint from $\mathrm{WF}^s(u_0)$ since the analogous statement is true for u. Thus, Theorem 1.4 is applicable to u_0, giving that all of γ_0 is disjoint from $\mathrm{WF}^s(u_0)$. Combining with the result on u_1, Theorem 1.1 is proved. $\Box$

5. Paired Lagrangian distributions

The class of distributions that plays the starring role below is that of paired Lagrangian distributions associated to two cleanly intersecting Lagrangians with the intersection having codimension k; these were introduced by Guillemin and Uhlmann [6] following the codimension 1 work of Melrose and Uhlmann [13]. In the model case where these Lagrangians are $\tilde{\Lambda}_0 = T^*_0\mathbb{R}^n$ and $\tilde{\Lambda}_1 = N^*\{x'' = 0\}$ in $T^*\mathbb{R}^n$ where the coordinates on $\mathbb{R}^n$ are $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, these (compactly supported) elements of $I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ are defined in [6], modulo $C^\infty_c(\mathbb{R}^n)$, by oscillatory integrals of the form

$$\int e^{i[(x'-s)\zeta' + x''\zeta'' + s\sigma]} a(x, s, \zeta, \sigma)\, ds\, d\zeta\, d\sigma, \tag{5.1}$$

a being a *product type* symbol $a \in S^{M,M'}(\mathbb{R}^{n+k}_{x,s}, \mathbb{R}^n_\zeta, \mathbb{R}^k_\sigma)$ with $M = p - n/4 + k/2$, $M' = l - k/2$ and with compact support in x, s, and in general via reduction to this model Lagrangian pair via a Fourier integral operator. Here $a \in S^{M,M'}(\mathbb{R}^{n+k}_{x,s}, \mathbb{R}^n_\zeta, \mathbb{R}^k_\sigma)$ means that

$$|(D^\alpha_{x,s} D^\beta_\zeta D^\gamma_\sigma a)(x, s, \zeta, \sigma)| \le C_{\alpha\beta\gamma} \langle\zeta\rangle^{M-|\beta|} \langle\sigma\rangle^{M'-|\gamma|}.$$

Such a distribution is, microlocally away from $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1$, in $I^p(\tilde{\Lambda}_1 \setminus \tilde{\Lambda}_0)$ and in $I^{p+l}(\tilde{\Lambda}_0 \setminus \tilde{\Lambda}_1)$. It is important to realize that these distributions are *not* a simple extension of these two classes of Lagrangian distributions, and in particular it is *not* the case that $I^{p+l}(\tilde{\Lambda}_0) \subset I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ for all p, l, though this inclusion of course holds away from $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1$. In fact, what is true is

$$I^p(\tilde{\Lambda}_0) \subset I^{p-k/2,k/2}(\tilde{\Lambda}_0, \tilde{\Lambda}_1);$$

we show this below in Lemma 5.2. On the other hand, $I^p(\tilde{\Lambda}_1) \subset I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$, so there is a fundamental asymmetry between the two Lagragians.

Indeed, this model can be simplified as follows. A distribution u is in $I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$, modulo $C^\infty_c(\mathbb{R}^n)$, if it can be written as

$$\int e^{i[x'\zeta' + x''\zeta'']} b(x, \zeta)\, d\zeta,$$

i.e. is essentially the inverse Fourier transform of b, with b satisfying the following estimates with $M = p - n/4 + k/2$, $M' = l - k/2$ as before: First, in the region $|\zeta'| \le C'|\zeta''|$, $|\zeta''| \ge 1$, the conditions on b amount to

$$|(Qb)(x, \zeta)| \le C\langle\zeta''\rangle^M \langle\zeta'\rangle^{M'}$$

whenever Q is a finite product of differential operators of the form

$$D_{\zeta'_m},\ \zeta'_j D_{\zeta'_m},\ \zeta''_j D_{\zeta''_m},$$

i.e. standard product-type regularity, when localized to this region. (Note that by localizing to the region where ζ''_q, for instance, dominates the other ζ''_j, one may simply replace ζ''_j by ζ''_q, as may be convenient on occasion.) On the other hand, in the region where $|\zeta''| \le C''|\zeta'|$, $|\zeta'| \ge 1$, which maps to $\tilde{\Lambda}_0$ away from the intersection of $\tilde{\Lambda}_0$ and $\tilde{\Lambda}_1$ and is not of too much interest, one has standard symbolic regularity, i.e.

$$|(Qb)(x, \zeta)| \le C\langle\zeta'\rangle^{M+M'}$$

whenever Q is a finite product of differential operators of the form

$$\zeta'_j D_{\zeta'_m},\ \zeta'_j D_{\zeta''_m}.$$

Alternatively, altogether, without any localization, one has bounds

$$|(Qb)(x,\zeta)| \leq C\langle\zeta\rangle^M \langle\zeta'\rangle^{M'} \tag{5.2}$$

whenever Q is a finite product of differential operators of the form

$$D_{\zeta'_m},\ \zeta'_j D_{\zeta'_m},\ D_{\zeta''_m},\ \zeta''_j D_{\zeta''_m},\ \zeta'_j D_{\zeta''_m}. \tag{5.3}$$

One direction of this equivalence claim is easily shown by starting from (5.1) by taking

$$b(x,\zeta) = \int e^{is(\sigma-\zeta')} a(x,s,\zeta,\sigma)\,ds\,d\sigma = \int (\mathcal{F}'a)(x,\zeta'-\sigma,\zeta,\sigma)\,d\sigma,$$

where $\mathcal{F}'$ is Fourier transform in the second slot (so $\mathcal{F}'a$ is Schwartz in this variable!) and directly checking the stability estimates. For the converse, if b is supported in $|\zeta'| < C'|\zeta''|$, as one may assume, one can take

$$a(x,s,\zeta,\sigma) = (2\pi)^{-k} b(x,\sigma,\zeta'')\chi(\langle\zeta'\rangle/\langle\zeta''\rangle)\chi_0(s),$$

where $\chi \in C_c^\infty(\mathbb{R})$ is identically 1 on $[0, 2C']$, while $\chi_0 \in C_c^\infty(\mathbb{R}^k)$ is such that if b is supported in $|x| < R$ then $\chi_0(s)$ is identically 1 on $|s| < 2R$. Here the localizer χ makes a into a symbol of the desired product type in (ζ,σ), while χ_0 localizes the support in s. With this definition of a,

$$\int (\mathcal{F}'a)(x,\zeta'-\sigma,\zeta,\sigma)\,d\sigma = \chi(\langle\zeta'\rangle/\langle\zeta''\rangle)\int b(x,\sigma,\zeta'')(2\pi)^{-k}\hat{\chi}_0(\zeta'-\sigma)\,d\sigma;$$

by the support conditions on χ and b and as $\hat{\chi}_0$ is Schwartz, dropping the factor χ only causes a Schwartz error to obtain $\tilde{b}(x,\zeta) = \int b(x,\sigma,\zeta'')(2\pi)^{-k}\hat{\chi}_0(\zeta'-\sigma)\,d\sigma$. Now,

$$\int e^{ix'\cdot\zeta'}\tilde{b}(x,\zeta')\,d\zeta' = \chi_0(x)\int e^{ix'\cdot\zeta'} b(x,\zeta')\,d\zeta' = \int e^{ix'\cdot\zeta'} b(x,\zeta')\,d\zeta',$$

so the distributions defined by a and b differ by an element of $C^\infty(\mathbb{R}^n)$ as claimed.

We remark that, although we do not use this point of view here, the regularity statement (5.2)-(5.3) for b amount to the statement that b is a conormal function on the blow up of $\mathbb{R}^n \times \overline{\mathbb{R}^n}$, with the second factor radially compactified, at $\mathbb{R}^n \times \partial\overline{\mathbb{R}^{n-k}_{\zeta''}}$, i.e. at infinity in ζ where $\zeta' = 0$, with order M on the front face, and order $M + M'$ on the lift of $\mathbb{R}^n \times \partial\overline{\mathbb{R}^n}$, where $M = p - n/4 + k/2$, $M' = l - k/2$ as before.

Indeed, a further argument shows that first, modulo $C^\infty(\mathbb{R}^n)$, the x'' dependence of b can be eliminated via expanding b in Taylor series around $x'' = 0$ and noting that $(x'')^\alpha$ becomes $(-1)^{|\alpha|}D^\alpha_{\zeta''}$ after an integration by parts, so in view of the symbolic estimates in ζ'' corresponds to reduced p, with an asymptotic summation argument completing the argument. Next, modulo $I^p(\tilde{\Lambda}_1)$, the x' dependence of b can be eliminated by a similar argument, expanding in Taylor series in x', which via integration by parts gives $(-1)^\alpha D^\alpha_{\zeta'}$, thus reducing l, which via an asymptotic summation argument completes the claim. Hence, it may be assumed that, modulo a term in $I^p(\tilde{\Lambda}_1)$, a paired Lagrangian distribution is the inverse Fourier transform of a conormal function on the blow up of $\overline{\mathbb{R}^n}$ at $\partial\overline{\mathbb{R}^{n-k}_{\zeta''}}$, i.e. at infinity in ζ where $\zeta' = 0$, with order M on the front face, and order $M + M'$ on the lift of $\partial\overline{\mathbb{R}^n}$, where $M = p - n/4 + k/2$, $M' = l - k/2$ as before.

One immediate consequence is:

Lemma 5.1. *If* $p_1 \le p_2$ *and* $p_1 + l_1 \le p_2 + l_2$ *then* $I^{p_1,l_1}(\Lambda_0, \Lambda_1) \subset I^{p_2,l_2}(\Lambda_0, \Lambda_1)$.

Proof. It suffices to consider the model pair, $(\tilde\Lambda_0, \tilde\Lambda_1)$. Since the class of differential operators under which one has stability in the two cases is the same, one just has to remark that for $p_1' \le p_2'$, $p_1' + l_1' \le p_2' + l_2'$,

$$\langle\zeta\rangle^{p_1'}\langle\zeta'\rangle^{l_1'} \le \langle\zeta\rangle^{p_1'}\langle\zeta'\rangle^{l_2'}\langle\zeta'\rangle^{p_2'-p_1'} \le \langle\zeta\rangle^{p_1'}\langle\zeta'\rangle^{l_2'}\langle\zeta\rangle^{p_2'-p_1'} = \langle\zeta\rangle^{p_2'}\langle\zeta'\rangle^{l_2'}.$$

□

Another immediate consequence is:

Lemma 5.2.

$$I^p(\Lambda_0) \subset I^{p-k/2,k/2}(\Lambda_0, \Lambda_1).$$

Proof. Again, it suffices to consider the model pair, $(\tilde\Lambda_0, \tilde\Lambda_1)$. An element of $I^p(\tilde\Lambda_0)$ can be written, modulo $C^\infty(\mathbb{R}^n)$, as the inverse Fourier transform of a symbol in $S^{p-\frac{n}{4}}(\mathbb{R}^n)$. But $S^{p-\frac{n}{4}}(\mathbb{R}^n)$ is conormal on $\overline{\mathbb{R}^n}$, of order $p - \frac{n}{4}$, hence on its blow up at $\partial\overline{\mathbb{R}^{n-k}_{\zeta''}}$, with order $M = M + M' = p - n/4$ *both* on the front face, and on the lift of $\partial\overline{\mathbb{R}^n}$. In terms of $I^{\tilde p,\tilde l}(\tilde\Lambda_0, \tilde\Lambda_1)$ this corresponds to orders $\tilde p = p - k/2$, $\tilde l = k/2$, proving the lemma. □

Note from the proof that one *cannot* lower $\tilde p = p - k/2$ even by increasing $\tilde l = k/2$. In fact, on the one hand, for an element of $S^{\tilde p',\tilde l'}$ the growth rate at the front face is determined by $\tilde p'$ alone, and on the other hand for $u \in I^p(\tilde\Lambda_0)$, the growth rate at this place is determined by p in general (i.e. there is no extra decay at the front face compared to other directions).

One can now easily describe the principal symbol on Λ_1 in general (without homogeneity discussions as in [6]). For this purpose it is useful to work with half-densities to avoid having to tensor with bundles that vary with the particular problem we want to study (such as half-density bundles from the base space X, or a factor of the base space on product spaces $X = X_L \times X_R$). Since the half-density bundles are trivial, from now on, without further comments, we trivialize them on the base manifold, as well as its factors, so as to *regard distributions (e.g. elements of* $I^{p,l}(\Lambda_0, \Lambda_1)$*) as distributional half-densities, and distributions with values in densities on the right factor* X_R *(which are the Schwartz kernels of operators acting on functions) also as distributional half-densities.*

Lemma 5.3. *Suppose* $u \in I^{p,l}(\Lambda_0, \Lambda_1)$ *given by an inverse Fourier transform* $\mathcal{F}^{-1}b$*, b conormal on the blow up of* $[\overline{\mathbb{R}^n_\zeta}; \partial\overline{\mathbb{R}^{n-k}_{\zeta''}}]$ *supported in* $\langle\zeta'\rangle \le C\langle\zeta''\rangle$*, with order M on the front face, and order* $M + M'$ *on the lift of* $\partial\overline{\mathbb{R}^n}$*, where* $M = p - n/4 + k/2$*,* $M' = l - k/2$ *as before. Let* $a = (\mathcal{F}')^{-1}b$*, where* $\mathcal{F}'$ *is partial Fourier transform in the primed variables. Then*

$$a \in S^{p-n/4+k/2}(\mathbb{R}^{n-k}_{\zeta''}; I^{M'+\frac{k}{4}}(\mathbb{R}^k_{x'}; N^*\{0\})), \tag{5.4}$$

and the equivalence class of a modulo $S^{p-n/4+k/2-1}(\mathbb{R}^k_{x'} \setminus 0; \mathbb{R}^{n-k}_{\zeta''})$ *satisfies*

$$[(2\pi)^{\frac{(n-2k)}{4}} a|_{x'\neq 0}\, |dx'|^{1/2}|d\zeta''|^{1/2}] = \sigma_{\Lambda_1\setminus\Lambda_0,p}(u), \tag{5.5}$$

with the right hand side being the standard principal symbol of a (microlocal) element of $I^p(\Lambda_1)$. The equivalence class of

$$(2\pi)^{\frac{(n-2k)}{4}} a\,|dx'|^{1/2}\,|d\zeta''|^{1/2} \text{ modulo } S^{p-1-n/4+k/2}(\mathbb{R}^{n-k}_{\zeta''}; I^{M'+1+\frac{k}{4}}(\mathbb{R}^k_{x'}; N^*\{0\}))$$

is the principal symbol of u on Λ_1, which is well-defined.

Furthermore,

$$a \in S^{p-1-n/4+k/2}(\mathbb{R}^{n-k}_{\zeta''}; I^{M'+1+\frac{k}{4}}(\mathbb{R}^k_{x'}; N^*\{0\})) \Longrightarrow u \in I^{p-1,l+1}(\Lambda_0, \Lambda_1),$$

while if

$$\tilde{a} \in S^{p-n/4+k/2}(\mathbb{R}^{n-k}_{\zeta''}; I^{M'+\frac{k}{4}}(\mathbb{R}^k_{x'}; N^*\{0\}))$$

then there is $u \in I^{p,l}(\Lambda_0, \Lambda_1)$ such that the principal symbol of u on Λ_1 is $\tilde{a}$.

Proof. Note that elements of $S^{M,M'}$ with the stated support condition are exactly the functions on $\mathbb{R}^n$ with a bound $|b| \leq C\langle\zeta''\rangle^M\langle\zeta'\rangle^{M'}$ which is stable upon iteratively applying finite products of $\zeta'_j D_{\zeta'_m}, D_{\zeta'_m}, \zeta''_j D_{\zeta''_m}, D_{\zeta''_m}$ to b, so it consists exactly of elements of $S^M(\mathbb{R}^{n-k}_{\zeta''}; S^{M'}(\mathbb{R}^k))$ with the stated support. Since the partial inverse Fourier transform in the primed variables maps $S^{M'}(\mathbb{R}^k_{x'})$ contiuously to $I^{M'+\frac{k}{4}}(\mathbb{R}^k_{x'}; N^*\{0\})$, (5.4) follows immediately. As the standard parameterization of a conormal distribution in $I^p(\Lambda_1)$ is

$$(2\pi)^{-(n+2(n-k))/4} \int e^{ix''\cdot\zeta''}\tilde{a}(x', x'', \zeta'')\,d\zeta'',$$

with $\tilde{a} \in S^{p+(n-2(n-k))/4}(\mathbb{R}^n_x; \mathbb{R}^{n-k}_{\zeta''})$ with principal symbol given by the equivalence class of the restriction of $\tilde{a}$ to $x' = 0$, while

$$u = (2\pi)^{-n+k} \int e^{ix''\cdot\zeta''}(\mathcal{F}')^{-1}b(x', \zeta'')\,d\zeta'',$$

with $(\mathcal{F}')^{-1}b(x', \zeta'')$ in $S^M(\mathbb{R}^{n-k}_{\zeta''}; C^\infty(\mathbb{R}^k \setminus 0))$, (5.5) follows.

Since conversely we have that the partial Fourier transform in the primed variables maps $S^M(\mathbb{R}^{n-k}_{\zeta''}; I^{M'+\frac{k}{4}}(\mathbb{R}^k_{x'}; N^*\{0\}))$ to $S^M(\mathbb{R}^{n-k}_{\zeta''}; S^{M'}(\mathbb{R}^k))$, if $u = \mathcal{F}^{-1}b$, and $b \in S^{M,M'}$ satisfies

$$(\mathcal{F}')^{-1}b \in S^{M-1}(\mathbb{R}^{n-k}_{\zeta''}; I^{M'+1+\frac{k}{4}}(\mathbb{R}^k_{x'}; N^*\{0\})),$$

then $b \in S^{M-1,M'+1}$ and thus $u \in I^{p-1,l+1}$. Further, if

$$\tilde{a} \in S^{p-n/4+k/2}(\mathbb{R}^{n-k}_{\zeta''}; I^{M'+\frac{k}{4}}(\mathbb{R}^k_{x'}; N^*\{0\}))$$

then defining b to be $(2\pi)^{-\frac{(n-2k)}{4}}(\mathcal{F}'\tilde{a})\chi$, where χ is a symbol on $\mathbb{R}^n$, with support in $\langle\zeta'\rangle < 2\langle\zeta''\rangle$, identically 1 on $\langle\zeta'\rangle < \langle\zeta''\rangle$, then $b - (2\pi)^{-\frac{(n-2k)}{4}}(\mathcal{F}'\tilde{a}) \in S^{M-N,M'+N}$ for every $N \geq 0$, and thus

$$(2\pi)^{\frac{(n-2k)}{4}}(\mathcal{F}')^{-1}b - \tilde{a} \in S^{M-1}(\mathbb{R}^{n-k}_{\zeta''}; I^{M'+1+\frac{k}{4}}(\mathbb{R}^k_{x'}; N^*\{0\}))$$

as claimed. $\square$

This description of paired Lagrangians is rather convenient for describing what happens when Λ_0 and Λ_1 are interchanged.

Proposition 5.4. *For $l < -k/2$ and $N \in \mathbb{N}$ such that $l + N < -k/2$ one has*

$$I^{p,l}(\Lambda_0, \Lambda_1) \subset I^p(\Lambda_1) + I^{p-N-\frac{k}{2}, N+\frac{k}{2}}(\Lambda_1, \Lambda_0).$$

On the other hand, for $l > -k/2$,

$$I^{p,l}(\Lambda_0, \Lambda_1) \subset I^{p+l, \frac{k}{2}}(\Lambda_1, \Lambda_0).$$

In both cases the inclusion maps are continuous, i.e. in the first case, when restricted to distributions with support in a fixed compact set, for any M there is M' and $C > 0$ such that for $u \in I^{p,l}(\Lambda_0, \Lambda_1)$ there are $I^p(\Lambda_1)$ and $u_2 \in I^{p-N-\frac{k}{2}, N+\frac{k}{2}}(\Lambda_1, \Lambda_0)$ with

$$\|u_1\|_{I^p(\Lambda_1);M} + \|u_2\|_{I^{p-N-\frac{k}{2}, N+\frac{k}{2}}(\Lambda_1,\Lambda_0);M} \leq C\|u\|_{I^{p,l}(\Lambda_0,\Lambda_1);M'}, \tag{5.6}$$

where $\|.\|_{I^p(\Lambda_1);M}$, etc., denotes the Mth seminorm giving the topology on $I^p(\Lambda_1)$, etc.

Note that when $l > -k/2$, $I^p(\Lambda_1) \subset I^{p,l}(\Lambda_0, \Lambda_1)$ is included in $I^{p+l,\frac{k}{2}}(\Lambda_1, \Lambda_0)$ by Lemma 5.2 and Lemma 5.1, while the same conclusion does not hold when $l < -k/2$ necessitating the addition of $I^p(\Lambda_1)$ explicitly to the right hand side.

Proof. As usual, it suffices to consider the model Lagrangians. It is straightforward to write down an explicit homogeneous symplectomorphism, and quantize it as a Fourier integral operator, microlocally near $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1$. Explicitly, where $C|\xi''_q| > \langle \xi \rangle$, as one may always arrange microlocally near a point in the intersection by suitably picking the index q, letting e_q be the corresponding coordinate unit vector, one can take the symplectomorphism

$$(x', x'', \xi', \xi'') \mapsto (-\frac{\xi'}{\xi''_q}, x'' + \frac{x' \cdot \xi'}{\xi''_q}\, e_q, \xi''_k x', \xi''),$$

and quantize it as

$$\begin{aligned} Fu(y) &= \int e^{i(y''-x''+(x'\cdot y')e_q)\cdot\xi''} |\xi''_q|^{k/2}\, u(x)\, dx\, d\xi'' \\ &= \int e^{iy''\cdot\xi''} |\xi''_q|^{k/2} (\mathcal{F}u)(-\xi''_q y', \xi'')\, d\xi'', \end{aligned}$$

where the symbol $|\xi''_q|^{k/2}$ is chosen to make F elliptic of order 0. Thus, for $u \in I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$, assuming as we may that u is the inverse Fourier transform of an element b of $S^{p',l'}$ with $p' = p - n/4 + k/2$, $l' = l - k/2$, and with support in $|\xi| \leq C|\xi''_q|$, $|\xi''_q| \geq 1$,

(5.7)
$$\begin{aligned} Fu(y) = \int e^{iy''\cdot\xi''} |\xi''_q|^{k/2} b(-\xi''_q y', \xi'')\, d\xi'' &= \int e^{i(y'\cdot\xi' + y''\cdot\xi'')} |\xi''_q|^{k/2} (\mathcal{F}'\tilde{b})(\xi', \xi'')\, d\xi, \\ \tilde{b}(\zeta', \zeta'') = b(\zeta''_q \zeta', \zeta''), \end{aligned}$$

where

$$\mathcal{F}'\tilde{b}(\xi', \xi'') = (2\pi)^{-k} \int e^{i\xi'\cdot\zeta'} \tilde{b}(\zeta', \xi'')\, d\zeta'$$

is the partial inverse Fourier transform of $\tilde{b}$. Thus, Fu is (up to a constant factor) the inverse Fourier transform of

$$a(\xi', \xi'') = |\xi''_q|^{k/2} (\mathcal{F}'\tilde{b})(\xi', \xi'') = |\xi''_q|^{-k/2} (\mathcal{F}'b)((\xi''_q)^{-1}\xi', \xi'') = (\tilde{\mathcal{F}}'b)(\xi', \xi''),$$

with $\tilde{\mathcal{F}}'$ defined by the last equation, and in order to prove the proposition, we only need to show that (with $p' = p - n/4 + k/2$, $l' = l - k/2$)

$$
\begin{aligned}
&l' > -k \Rightarrow \tilde{\mathcal{F}}' S^{p',l'} \subset S^{p'+l'+k/2,0}\\
&l' < -k \Rightarrow \tilde{\mathcal{F}}' S^{p',l'} \subset S^{p'-k/2}(\mathbb{R}^n) + S^{p'-N-k/2,N},
\end{aligned}
\tag{5.8}
$$

with continuous inclusions. We first prove the first implication as well as the second in the special case $N = 0$, when the first term on the right hand side can be absorbed in the second. Since it is straightforward to check that the differential operators under which we require iterative regularity transform properly, the main issue is to obtain sup bounds. But

(5.9)

$$
\begin{aligned}
|(\tilde{\mathcal{F}}'b)(\xi',\xi'')| &\le |\xi''_q|^{-k/2} \int |b(\zeta',\xi'')|\,d\zeta' \lesssim |\xi''_q|^{-k/2} \int \langle \zeta' \rangle^{l'} |\xi''_q|^{p'}\,d\zeta'\\
&\le |\xi''_q|^{p'-k/2} \Big(\int_{|\zeta'|\le 1} d\zeta' + \int_{1\le|\zeta'|\le C|\xi''_q|} |\zeta'|^{l'}\,d\zeta' \Big) \lesssim |\xi''_q|^{p'-k/2}(1 + |\xi''_q|^{l'+k}),
\end{aligned}
$$

so the conclusion immediately follows. (We remark that if $l' = -k$, a logarithmic term in $|\xi''_q|$ would appear on the right hand side, so in terms of spaces with polynomial weights, we would have to lose $\epsilon > 0$ to end up in $S^{p'-k/2+\epsilon,0}$, which is the result one obtains if one simply replaces l' by $l' + \epsilon$ and applies the statement in that case, hence not stating the case $l = -k/2$ separately.) Now, for general $N \ge 1$, we expand $(\tilde{\mathcal{F}}'b)(\xi',\xi'')$ in Taylor series around $\xi' = 0$ to order $N - 1$,

$$
\begin{aligned}
(\tilde{\mathcal{F}}'b)(\xi',\xi'') = &\sum_{|\alpha|\le N-1} \frac{1}{\alpha!} (\xi')^\alpha \partial^\alpha_{\xi'} (\tilde{\mathcal{F}}'b)(0,\xi'')\\
&+ \sum_{|\alpha|=N} \frac{N}{\alpha!} \int_0^1 (1-t)^{N-1} (\xi')^\alpha \partial^\alpha_{\xi'} (\tilde{\mathcal{F}}'b)(t\xi',\xi'')\,dt,
\end{aligned}
\tag{5.10}
$$

and check that the two terms are respectively in $S^{p'-k/2}(\mathbb{R}^n)$ and $S^{p'-N-k/2,N}$ when microlocalized to $|\xi'| \le \tilde{C}|\xi''_q|$. Here the key point is that

$$
\xi'^\alpha D^\alpha_{\xi'} (\mathcal{F}'b)((\xi''_q)^{-1}\xi',\xi'') = (-1)^{|\alpha|} \xi'^\alpha |\xi''_q|^{-|\alpha|} (\mathcal{F}'((M')^\alpha b))((\xi''_q)^{-1}\xi',\xi''),
$$

where M'_j is multiplication by the jth primed coordinate function, with $(M')^\alpha$ then defined by the standard multiindex notation, so $(M')^\alpha b \in S^{p',l'+|\alpha|}$, and thus, in view of (5.8) with the already proved case, $N = 0$, the αth term in (5.10) is in $S^{p'+l'+k/2,|\alpha|}$ if $l' + |\alpha| > -k$, and in $S^{p'-|\alpha|-k/2,|\alpha|}$ if $l' + |\alpha| < -k$, with the additional information (in view of the evaluation at $\xi' = 0$) that if $|\alpha| < N$ then in fact the αth term is in $S^{p'+l'+|\alpha|+k/2}(\mathbb{R}^n)$ if $l' + |\alpha| > -k$, and in $S^{p'-k/2}(\mathbb{R}^n)$ if $l' + |\alpha| < -k$. This proves (5.8) when $N \ge 0$ is an integer with $l' + N < -k$, and thus the proposition. □

Corollary 5.5. *For $l < -k/2$ and for $\epsilon > 0$,*

$$
I^{p,l}(\Lambda_0,\Lambda_1) \subset I^p(\Lambda_1) + I^{p+l+\epsilon,-l-\epsilon}(\Lambda_1,\Lambda_0),
$$

with continuous inclusions.

Note that the second summand has order p on Λ_1, i.e. it did not increase when reversing the order of the two Lagrangians, while it has order $p + l + \epsilon$ on Λ_0, so it only increased by ϵ as compared to the left hand side. This is an affordable loss when Λ_0 is thought of as carrying a 'small singularity', while any loss on Λ_1 is unaffordable.

Also note that in view of Lemma 5.1, the corollary indeed becomes stronger if one decreases ϵ.

Proof. If $l = -k/2 - \epsilon - N$ for some $N \in \mathbb{N}$ and $\epsilon > 0$, then this is just Proposition 5.4. Below we assume that seminorms are actually norms, as we may (by including the weighted sup norm without derivatives on the symbol in all of them), and that they get stronger with increasing index M. Let $I^{p-k/2-\epsilon}_{M'}(\Lambda_0, \Lambda_1)$ denote the completion of $I^{p-k/2-\epsilon}(\Lambda_0, \Lambda_1)$ with respect to the M'th norm, so the result is a Banach space; thus, for $M'' \geq M'$, the identity map on $I^{p-k/2-\epsilon}(\Lambda_0, \Lambda_1)$ extends to a continuous map

$$I^{p-k/2-\epsilon}_{M''}(\Lambda_0, \Lambda_1) \to I^{p-k/2-\epsilon}_{M'}(\Lambda_0, \Lambda_1),$$

and the completeness of $I^{p-k/2-\epsilon}(\Lambda_0, \Lambda_1)$ as a Fréchet space means that

$$I^{p-k/2-\epsilon}(\Lambda_0, \Lambda_1) = \cap_{M'} I^{p-k/2-\epsilon}_{M'}(\Lambda_0, \Lambda_1).$$

Indeed, if $u \in \cap_M I^{p-k/2-\epsilon}_M(\Lambda_0, \Lambda_1)$ then for each M there is a Cauchy sequence in $I^{p-k/2-\epsilon}(\Lambda_0, \Lambda_1)$ converging to u; we may assume that this Cauchy sequence is of the form $\{u_{M,j}\}_{j=1}^\infty$ with $\|u_{M,j} - u\|_M \leq 2^{-j}$. Then the diagonal sequence $u_j = u_{j,j}$ is Cauchy with respect to all norms M, and it converges to u in all of these, so by the completeness of $I^{p-k/2-\epsilon}(\Lambda_0, \Lambda_1)$, $u \in I^{p-k/2-\epsilon}(\Lambda_0, \Lambda_1)$. We use similar notation for completions of other spaces with respect to various norms below.

The complex interpolation spaces for $I^{p,-k/2-\epsilon}_{M'}(\Lambda_0, \Lambda_1)$ and $I^{p,-k/2-\epsilon-N}_{M'}(\Lambda_0, \Lambda_1)$ are $I^{p,-k/2-\epsilon-N\theta}_{M'}(\Lambda_0, \Lambda_1)$, $\theta \in [0, 1]$, since in the interpolation only the weight corresponding to Λ_0 is changed, and the seminorms are weighted L^∞ bounds, i.e. the interpolation is actually for a family of multiplication operators. Similarly, the complex interpolation spaces between $I^{p-k/2,k/2}_M(\Lambda_1, \Lambda_0)$, and $I^{p-N-k/2,N+k/2}_M(\Lambda_1, \Lambda_0)$, are $I^{p-N\theta-k/2,N\theta+k/2}_M(\Lambda_1, \Lambda_0)$, $\theta \in [0, 1]$; now both weights are interpolated, but this still is interpolation for a family of multiplication operators. In view of the continuity of the inclusion map

$$I^{p,-k/2-\epsilon-N}(\Lambda_0, \Lambda_1) \hookrightarrow I^p(\Lambda_1) + I^{p-N-k/2,N+k/2}(\Lambda_1, \Lambda_0)$$

for $N \in \mathbb{N}$, for all M there is M' such that the inclusion map extends to a map from the M'th completion of the left hand side to the Mth completion of the right hand side. Thus, complex interpolation is applicable, and yields that

$$I^{p,-k/2-\epsilon-N\theta}_{M'}(\Lambda_0, \Lambda_1) \hookrightarrow I^p_M(\Lambda_1) + I^{p-N\theta-k/2,N\theta+k/2}_M(\Lambda_1, \Lambda_0),\ \theta \in [0, 1].$$

In particular, as

$$I^{p,-k/2-\epsilon-N\theta}(\Lambda_0, \Lambda_1) \subset I^{p,-k/2-\epsilon-N\theta}_{M'}(\Lambda_0, \Lambda_1),$$

the inclusion map extends to

$$I^{p,-k/2-\epsilon-N\theta}(\Lambda_0, \Lambda_1) \hookrightarrow I^p_M(\Lambda_1) + I^{p-N\theta-k/2,N\theta+k/2}_M(\Lambda_1, \Lambda_0),\ \theta \in [0, 1],$$

for all M, with the spaces on the right becoming stronger with M. Since the intersections of these spaces is $I^p(\Lambda_1) + I^{p-N\theta-k/2,N\theta+k/2}(\Lambda_1, \Lambda_0)$, we deduce that

$$I^{p,-k/2-\epsilon-N\theta}(\Lambda_0, \Lambda_1) \hookrightarrow I^p(\Lambda_1) + I^{p-N\theta-k/2,N\theta+k/2}(\Lambda_1, \Lambda_0),\ \theta \in [0, 1].$$

As $N \in \mathbb{N}$ is arbitrary,

$$I^{p,-k/2-\epsilon-m}(\Lambda_0,\Lambda_1) \subset I^p(\Lambda_1) + I^{p-m-k/2,m+k/2}(\Lambda_1,\Lambda_0).$$

when $m \geq 0$ real, which is just a rewriting of the statement of the corollary. □

We also recall the composition rule of Antoniano and Uhlmann [1] for flow-outs, with Λ_1 the flow-out of $\Lambda_0 = N^*\mathrm{diag}$, as referred to in [3, Proposition 1.39], namely (with k the codimension of the intersection)

$$I^{p,l}(\Lambda_0,\Lambda_1) \circ I^{p',l'}(\Lambda_0,\Lambda_1) \subset I^{p+p'+k/2,l+l'-k/2}(\Lambda_0,\Lambda_1). \tag{5.11}$$

We recall the set-up of flow-outs here, phrased in the general codimension case as in Greenleaf and Uhlmann [3]. Thus, one of the Lagrangians is the conormal bundle $N^*\mathrm{diag}$ of the diagonal, and the other is a flow-out $\Lambda = \Lambda_\Gamma$ corresponding to a conic, codimension k, involutive (i.e. coisotropic) $\Gamma \subset T^*\mathbb{R}^n$. Such a Γ is defined by the vanishing of k functions p_i which Poisson commute on Γ; Λ_Γ is then the set of points $((x,\xi),(y,-\eta)) \in T^*\mathbb{R}^{2n}$ such that $(y,\eta) = \exp(\sum t_j \mathsf{H}_{p_j})(x,\xi)$ for some $t \in \mathbb{R}^k$. We give a concrete example: if $\Gamma = T^*_Y X$ with Y defined by $x' = 0$, $x' \in \mathbb{R}^k$, then one can take $x'_1,\ldots,x'_k$ as the Poisson commuting functions, and then Λ_Γ consists of points $((x,\xi),(y,\eta))$ such that $x = y \in Y$ (i.e. $x' = 0 = y'$, $x'' = y''$) and $\xi + \eta \in N^*_x Y$ (i.e. $\xi'' = -\eta''$), i.e.

$$\Lambda_\Gamma = N^*\{x' = 0 = y',\ x'' = y''\}.$$

Another example, considered in [3], is with $\tilde\Gamma$ given by $\xi' = 0$, so

$$\tilde\Lambda = \Lambda_{\tilde\Gamma} = \{((x,\xi),(y,\eta)) :\ \xi' = 0 = \eta',\ \xi'' = -\eta'',\ x'' = y''\}. \tag{5.12}$$

For purposes of considering elements of $I^{p,l}(\Lambda_0,\Lambda_1)$ as operators on functions or distributions on $\mathbb{R}^n$, it is important whether the Lagrangians intersect $T^*\mathbb{R}^n \times o_{\mathbb{R}^n}$ or $o_{\mathbb{R}^n} \times T^*\mathbb{R}^n$, with $o_{\mathbb{R}^n}$ denoting the zero section of $T^*\mathbb{R}^n$. Our first example, with $\Gamma = T^*_Y X$, $\Lambda_0 = N^*\mathrm{diag}$ and $\Lambda_1 = \Lambda_\Gamma$ contains covectors of both types, namely points like

$$\{x' = 0 = y',\ x'' = y'',\ \xi'' = -\eta'' = 0,\ \xi' = 0,\ \eta' \neq 0\}$$

and

$$\{x' = 0 = y',\ x'' = y'',\ \xi'' = -\eta'' = 0,\ \eta' = 0,\ \xi' \neq 0\};$$

these are in the intersections of $N^*\{x' = 0 = y',\ x'' = y''\}$ with $N^*\{y' = 0\}$ resp. $N^*\{x' = 0\}$. The behavior at these intersections is best considered in terms of another Lagrangian pair, discussed below after (5.31), and for now we assume that the wave front set of the elements of $I^{p,l}(\Lambda_0,\Lambda_1)$ we consider is disjoint from $T^*\mathbb{R}^n \times o_{\mathbb{R}^n}$ and $o_{\mathbb{R}^n} \times T^*\mathbb{R}^n$. We write

$$\begin{aligned} &\tilde I^{*,*}(\Lambda_0,\Lambda_1) \\ &\quad = \{K \in I^{*,*}(\Lambda_0,\Lambda_1) :\ \mathrm{WF}(K) \cap (T^*\mathbb{R}^n \times o_{\mathbb{R}^n}) = \emptyset, \\ &\qquad\qquad \mathrm{WF}(K) \cap (o_{\mathbb{R}^n} \times T^*\mathbb{R}^n) = \emptyset\}. \end{aligned} \tag{5.13}$$

If one reverses the order of the Lagrangians, i.e. Λ_0 is the flow-out of $\Lambda_1 = N^*\mathrm{diag}$, then for $l, l' < -k/2$, one has

$$I^{p,l}(\Lambda_0,\Lambda_1) \subset I^p(\Lambda_1) + I^{p+l+\epsilon,-l-\epsilon}(\Lambda_1,\Lambda_0),$$

with a similar decomposition for $I^{p',l'}(\Lambda_0,\Lambda_1)$. Now, by (5.11) in the last case (and in fact all the other, simpler, statements can be reduced to this using Lemma 5.2),

$$\begin{aligned}
&I^p(\Lambda_1)\circ I^{p'}(\Lambda_1)\subset I^{p+p'}(\Lambda_1),\\
&I^p(\Lambda_1)\circ I^{p'+l'+\epsilon,-l'-\epsilon}(\Lambda_1,\Lambda_0)\subset I^{p+p'+l'+\epsilon,-l'-\epsilon}(\Lambda_1,\Lambda_0),\\
&I^{p+l+\epsilon,-l-\epsilon}(\Lambda_1,\Lambda_0)\circ I^{p'}(\Lambda_1)\subset I^{p+p'+l+\epsilon,-l-\epsilon}(\Lambda_1,\Lambda_0)\\
&I^{p+l+\epsilon,-l-\epsilon}(\Lambda_1,\Lambda_0)\circ I^{p'+l'+\epsilon,-l'-\epsilon}(\Lambda_1,\Lambda_0)\\
&\qquad\qquad\subset I^{p+p'+l+l'+2\epsilon+k/2,-l-l'-2\epsilon-k/2}(\Lambda_1,\Lambda_0).
\end{aligned}$$

Thus,

$$\begin{aligned}
&I^{p,l}(\Lambda_0,\Lambda_1)\circ I^{p',l'}(\Lambda_0,\Lambda_1)\\
&\qquad\subset I^{p+p'}(\Lambda_1)+I^{p+p'+l'+\epsilon,-l'-\epsilon}(\Lambda_1,\Lambda_0)+I^{p+p'+l+\epsilon,-l-\epsilon}(\Lambda_1,\Lambda_0),
\end{aligned}$$

which suffices for our purposes. (Note that the order of the two Lagrangians is reversed on the two sides!)

We now recall a result of Greenleaf and Uhlmann:

Proposition 5.6. *(See* [3, Theorem 3.3]*) An operator $A\in I^{p,l}(\Lambda_1,\Lambda_0)$ (with, say, compactly supported Schwartz kernel) is continuous $H^{m'}\to H^m$ if*

$$p+\frac{k}{2}\leq m'-m \text{ and } p+l\leq m'-m.$$

Note that the first condition is exactly the boundedness condition for elements of $I^p(\Lambda_0)$, while the second one is that of elements of $I^{p+l}(\Lambda_1)$.

There is actually an error in the proof of [3, Theorem 3.3]. Recall that the proposition is reduced to the case of $m=m'=0$ and equality holding in one of the two inequalities. The $p+l=0$ (and then $l\geq k/2$, so $p\leq -k/2$) case is the problematic one in the proof; note that this means that the order on the flow-out, Λ_0, which is regarded as the main Lagrangian, is small compared to that on Λ_1, the conormal bundle of the diagonal. This is a problem since $\mathrm{Id}\in I^0(\Lambda_1)$ is assumed to be to be $I^{p,l}(\Lambda_1,\Lambda_0)$, but as we remarked after this only holds for $p=-k/2$, $l=k/2$, and *not for smaller values of* p. However, this can be fixed: by Lemma 5.1, if $p+l=0$, $p<-k/2$, then

$$I^{p,l}(\Lambda_1,\Lambda_0)\subset I^{-k/2,k/2}(\Lambda_1,\Lambda_0),$$

so one may assume that $p=-k/2$, $l=k/2$, in which case the rest of the argument goes through.

In view of Corollary 5.5, we deduce:

Proposition 5.7. *With $\Lambda_1=N^*\mathrm{diag}$, Λ_0 its flow out, $\tilde I^{*,*}(\Lambda_0,\Lambda_1)$ as in* (5.13), *$K\in\tilde I^{p,l}(\Lambda_0,\Lambda_1)$ is bounded from $H^{m'}$ to H^m if*

$$p\leq m'-m,\ p+l<m'-m-\frac{k}{2}.$$

Note that the assumptions are the criterion (except that equality is also allowed in the criterion) for elements of $I^p(\Lambda_1)$, resp. $I^{p+l}(\Lambda_0)$, to be bounded in the stated manner.

Proof. If $l < -k/2$ then Corollary 5.5 gives that for all $\epsilon > 0$,

$$I^{p,l}(\Lambda_0, \Lambda_1) \subset I^p(\Lambda_1) + I^{p+l+\epsilon,-l-\epsilon}(\Lambda_1, \Lambda_0).$$

Now, elements of $I^p(\Lambda_1)$ are bounded from $H^{m'}$ to H^m when $p \leq m' - m$, while those of $I^{p+l+\epsilon,-l-\epsilon}(\Lambda_1, \Lambda_0)$ are bounded from $H^{m'}$ to H^m when

$$p + l + \epsilon + \frac{k}{2} \leq m' - m \text{ and } p \leq m' - m,$$

taking $\epsilon > 0$ sufficiently small (so that $\epsilon \leq m' - m - \frac{k}{2} - p - l$, note that the right hand side is positive), the proposition follows.

If $l \geq -k/2$ then for $\epsilon > 0$, using $l + k/2 + \epsilon > 0$, and thus in view of Lemma 5.1:

$$I^{p,l}(\Lambda_0, \Lambda_1) \subset I^{p+l+k/2+\epsilon,-k/2-\epsilon}(\Lambda_0, \Lambda_1),$$

so by the first part of the proof $I^{p,l}(\Lambda_0, \Lambda_1)$ is bounded from $H^{m'}$ to H^m when

$$p + l + k/2 + \epsilon \leq m' - m, \ p + l < m' - m - \frac{k}{2}.$$

Taking $0 < \epsilon < m' - m - \frac{k}{2} - (p+l)$, the inequalities are satisfied, and the proposition follows. □

However, while boundedness is important for our purposes, we also need to show that the classes $I^{p,l}(\Lambda_0, \Lambda_1)$ satisfy a composition law. For this, as well as other, purposes, we consider another model of cleanly intersecting Lagrangians, related to the $\Gamma = T^*_Y X$ case considered above.

This other model of a cleanly intersecting Lagrangian pair is, in $T^*\mathbb{R}^n \setminus o$, where $\mathbb{R}^n = \mathbb{R}^k_{x'} \times \mathbb{R}^{n-k-d}_{x''} \times \mathbb{R}^d_{x'''}$,

$$\Lambda_0 = N^*\{x' = 0, \ x'' = 0\}, \ \Lambda_1 = N^*\{x'' = 0\}. \tag{5.14}$$

One may assume (via localization in the double primed dual variables, and using that one is near the intersection $\Lambda_0 \cap \Lambda_1$) that one is working in the region where $|\xi''_q| > C\langle\xi\rangle$, and then this pair is reduced to the standard Lagrangian pair $(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ considered above via the homogeneous symplectomorphism

$$(x', x'', x''', \xi', \xi'', \xi''') \mapsto (x', x'' + \frac{x''' \cdot \xi'''}{\xi''_q} e_q, -\frac{\xi'''}{\xi''_q}, \xi', \xi'', \xi''_q x'''),$$

which is quantized by the elliptic 0th order FIO

$$Fu(y) = \int e^{i[(y'-x')\cdot\xi' + (y''-x'')\cdot\xi'' + (x'''\cdot y''')e_q\cdot\xi'']} |\xi''|^{d/2} u(x) \, dx.$$

The characterization of $I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ as inverse Fourier transforms modulo $I^p(\tilde{\Lambda}_1)$ of elements of $S^{p,l}$ gives that they can also be described, modulo $I^p(\Lambda_1)$, by oscillatory integrals

$$\int e^{i(y'\cdot\xi' + y''\cdot\xi'')} b(y''', \xi', \xi'') \, d\xi' \, d\xi'', \tag{5.15}$$

where $b \in S^{p-\frac{n}{4}+\frac{k}{2}+\frac{d}{2}, l-\frac{k}{2}}(\mathbb{R}^d_{x'''}; \mathbb{R}^{n-k-d}_{\xi''}; \mathbb{R}^k_{\xi'})$. Thus, one has the inverse Fourier transform in the primed and double primed variables, with the triple primed parameters serving as parameters, i.e. one can add parametric variables to the above parameterization using $N^*\{x' = 0, x'' = 0\}$ and $N^*\{x'' = 0\}$ at the cost of shifting the orders appropriately. The principal symbol of (5.15) on Λ_1 is then, with $\mathcal{F}'$ the inverse Fourier transform in the primed variables,

$$(2\pi)^{\frac{(3n+2k-2d)}{4}}(\mathcal{F}')^{-1}b\,|dy'|^{1/2}\,|d\xi''|^{1/2}\,|dy'''|^{1/2} \tag{5.16}$$

in

$$\begin{aligned} &S^{p-n/4+k/2+d/2}(\mathbb{R}^d_{y'''};\mathbb{R}^{n-k-d}_{\xi''};I^{l-\frac{k}{4}}(\mathbb{R}^k_{y'};N^*\{0\})) \\ &= S^{p-n/4+k/2+d/2}(\mathbb{R}^{n-k-d}_{\xi''};I^{l-\frac{k+d}{4}}(\mathbb{R}^{k+d}_{y',y'''};N^*\{y'=0\})) \end{aligned} \tag{5.17}$$

modulo

$$\begin{aligned} &S^{p-n/4+k/2+d/2-1}(\mathbb{R}^d_{y'''};\mathbb{R}^{n-k-d}_{\xi''};I^{l+1-\frac{k}{4}}(\mathbb{R}^k_{y'};N^*\{0\})) \\ &= S^{p-n/4+k/2+d/2-1}(\mathbb{R}^{n-k-d}_{\xi''};I^{l+1-\frac{k+d}{4}}(\mathbb{R}^{k+d}_{y',y'''};N^*\{y'=0\})). \end{aligned}$$

With this parameterization it is straightforward to see, as was shown by Greenleaf and Uhlmann in [4, Lemma 1.1], that if Y and Z are transversal manifolds of codimension d_1, resp. d_2, in $\mathbb{R}^n$, then the product of distributions conormal to Y and Z, respectively, is a sum of paired Lagrangian distributions associated to the pairs $(N^*(Y\cap Z),N^*Y)$ and $(N^*(Y\cap Z),N^*Z)$. More precisely,

$$I^{[\mu]}(Y)I^{[\mu']}(Z)\subset I^{[\mu,\mu']}(Y\cap Z,Y)+I^{[\mu',\mu]}(Y\cap Z,Z),$$

where

$$I^{[\mu,\mu']}(Y\cap Z,Y)=I^{\mu+\frac{d_1}{2}-\frac{n}{4},\mu'+\frac{d_2}{2}}(N^*(Y\cap Z),N^*Y). \tag{5.18}$$

(Here the left hand side is denoted by $I^{\mu,\mu'}(Y,Y\cap Z)$, $Y\cap Z=Y_2\subset Y_1=Y$ in [4] just after Equation (1.4). Then the equality in (5.18) is the extreme left hand side of the first displayed equation after Equation (1.4) being equal to the extreme right hand side. The middle expression in this equation is *not* equal to the extreme right hand side.)

Note that here the codimension of the intersection of the two Lagrangians N^*Y and $N^*(Y\cap Z)$ is d_2, and thus using

$$I^{[\mu]}(Y)=I^{\mu+\frac{d_1}{2}-\frac{n}{4}}(N^*Y),\ I^{[\mu']}(Z)=I^{\mu'+\frac{d_2}{2}-\frac{n}{4}}(N^*Z),$$

one has

$$I^{\mu}(N^*Y)I^{\mu'}(N^*Z)\subset I^{\mu,\mu'+\frac{n}{4}}(N^*(Y\cap Z),N^*Y)+I^{\mu',\mu+\frac{n}{4}}(N^*(Y\cap Z),N^*Z), \tag{5.19}$$

We remark here that one must be careful in ordering the Lagrangians, as mentioned above; this *is* the correct ordering. Thus, the 'main' Lagrangians are the original ones, N^*Y and N^*Z; $N^*(Y\cap Z)$ carries a relative singularity only.

A special case of the model of (5.14) in $\mathbb{R}^{2n}=\mathbb{R}^n_x\times\mathbb{R}^n_y$ (note the change of dimension!), with $\mathbb{R}^n=\mathbb{R}^k_{x'}\times\mathbb{R}^{n-k}_{x''}$ is, with (ξ,η) the dual variables of (x,y),

$$\begin{aligned} &\Lambda_1=\{x'=y',\ x''=y'',\ \xi'=-\eta',\ \xi''=-\eta''\}=N^*\mathrm{diag}, \\ &\Lambda_0=\{x'=0=y',\ x''=y'',\ \xi''=-\eta''\}=N^*\{x'=0=y',\ x''=y''\}, \end{aligned} \tag{5.20}$$

with codimension k intersection; this corresponds to the flowout with $\Gamma=T^*_YX$, $Y=\{x'=0\}$, discussed above. Then the parameterization of $I^{p,l}(\Lambda_0,\Lambda_1)$, modulo $I^p(\Lambda_1)$, is

$$\int e^{i[(x'-y')\cdot\xi'+(x''-y'')\cdot\xi''+x'\cdot\eta']}a(x'',\xi',\xi'',\eta')\,d\xi\,d\eta',\ a\in S^{p,l-\frac{k}{2}}(\mathbb{R}^{n-k}_{x''};\mathbb{R}^n_\xi;\mathbb{R}^k_{\eta'}),$$

with a conic neighborhood of $\eta'=0$ in $(\mathbb{R}^n_\xi\times\mathbb{R}^k_{\eta'})\setminus 0$ corresponding to a neighborhood of the intersection $\Lambda_0\cap\Lambda_1$ (so ξ is the 'large' variable on the parameter space, note that it is indeed the variable in the parameterization of the conormal bundle of the diagonal), and the x'' dependence can be replaced by y'' dependence. (To see this form of parameterization, write $z''=x-y$, $z'=x'$, $z'''=x''$ then $\Lambda_1=N^*\{z''=0\}$, $\Lambda_0=N^*\{z'=0,\ z''=0\}$. Replacing x' and x'' by y' and y'' in the definition of z' and z''' gives the other parameterization.) Here the principal symbol is, with $\mathcal{F}'$ the Fourier transform in the last variable, η',

$$(2\pi)^{n+k}(\mathcal{F}')^{-1}a\,|dy'|^{1/2}\,|d\xi|^{1/2}\,|dy'''|^{1/2} \tag{5.21}$$

in

$$S^p(\mathbb{R}^{n-k}_{x''};\mathbb{R}^n_\xi;I^{l-\frac{k}{4}}(\mathbb{R}^k_{x'};N^*\{0\}))=S^p(\mathbb{R}^n_{\xi''};I^{l-\frac{n}{4}}(\mathbb{R}^n_x;N^*\{x'=0\})) \tag{5.22}$$

modulo

$$S^{p-1}(\mathbb{R}^{n-k}_{x''};\mathbb{R}^n_\xi;I^{l+1-\frac{k}{4}}(\mathbb{R}^k_{x'};N^*\{0\}))=S^{p-1}(\mathbb{R}^n_{\xi''};I^{l+1-\frac{n}{4}}(\mathbb{R}^n_x;N^*\{x'=0\})).$$

Writing out the composition we have:

Proposition 5.8. *With $\Lambda_1=N^*\mathrm{diag}$, Λ_0 its flow out, the subset $\tilde I^{*,*}(\Lambda_0,\Lambda_1)$ of $I^{*,*}(\Lambda_0,\Lambda_1)$ defined in (5.13), satisfies that if $l+l'<0$ and $L=\max(l,l',l+l'+k/2)$, then*

$$\tilde I^{p,l}(\Lambda_0,\Lambda_1)\circ\tilde I^{p',l'}(\Lambda_0,\Lambda_1)\subset\tilde I^{p+p',L}(\Lambda_0,\Lambda_1). \tag{5.23}$$

Furthermore, with $-(l+l')>\delta>0$, modulo

$$\begin{aligned}&S^{p+p'-\min(1,\delta)}(\mathbb{R}^{n-k}_{x''};\mathbb{R}^n_\xi;I^{L+\delta-\frac{k}{4}}(\mathbb{R}^k_{x'};N^*\{0\}))\\&=S^{p+p'-\min(1,\delta)}(\mathbb{R}^n_{\xi''};I^{L+\delta-\frac{n}{4}}(\mathbb{R}^n_x;N^*\{x'=0\}))\end{aligned}$$

the principal symbol on $\Lambda_1=N^*\mathrm{diag}$ in

$$S^{p+p'}(\mathbb{R}^{n-k}_{x''};\mathbb{R}^n_\xi;I^{L-\frac{k}{4}}(\mathbb{R}^k_{x'};N^*\{0\}))=S^{p+p'}(\mathbb{R}^n_{\xi''};I^{L-\frac{n}{4}}(\mathbb{R}^n_x;N^*\{x'=0\}))$$

of the composition of two operators is the product of their principal symbols.

Remark 5.9. As one can always decrease the second order l at the cost of increasing the first order p, see Lemma 5.1, this result also gives that if $l+l'\geq 0$ then for any $\ell>l+l'$, with $L=\max(l-\ell,l',l-\ell+l'+k/2)$,

$$\tilde I^{p,l}(\Lambda_0,\Lambda_1)\circ\tilde I^{p',l'}(\Lambda_0,\Lambda_1)\subset\tilde I^{p+p'+\ell,L}(\Lambda_0,\Lambda_1). \tag{5.24}$$

However, the increase of the order on Λ_1 relative to the $l+l'<0$ case makes this a much less useful result.

Remark 5.10. The constraint $l+l'<0$ is exactly the constraint under which elements of $S^p(\mathbb{R}^n_{\xi''};I^{l-\frac{n}{4}}(\mathbb{R}^n_x;N^*\{x'=0\}))$ and $S^{p'}(\mathbb{R}^n_{\xi''};I^{l'-\frac{n}{4}}(\mathbb{R}^n_x;N^*\{x'=0\}))$ can be multiplied in view of the lack of smoothness of these symbols in x'. Namely, the issue is multiplication for elements of $I^{l-\frac{n}{4}}(\mathbb{R}^n_x;N^*\{x'=0\})$ and $I^{l'-\frac{n}{4}}(\mathbb{R}^n_x;N^*\{x'=0\})$ which come from partial inverse Fourier transforms in x' of symbols of order $l-k/2$, resp. $l'-k/2$. Typical members of these classes are asymptotically homogeneous of degree $l-k/2$, resp. $l'-k/2$, so their partial inverse Fourier transforms in x' are, modulo smooth functions, homogeneous of degree $-k/2-l$, resp. $-k/2-l'$. The restriction $l+l'<0$ means that the total homogeneity is $>-k$, i.e. is strictly greater than that of a delta distribution on $x'=0$. Marginally disallowed products are thus, in the case $k = 1$, a delta distribution and a step function at a hypersurface; any more smoothness than that of the step function (in terms of conormal order) means that the functions is continuous and may be multiplied by the δ distribution. Thus, in this sense, this proposition is *sharp*.

Proof. Let

$$A \in \tilde{I}^{p,l}(\Lambda_0, \Lambda_1),\ B \in \tilde{I}^{p',l'}(\Lambda_0, \Lambda_1);$$

we may assume that A and B both have wave front set near the intersection of the two Lagrangians. Write A resp. B as an oscillatory integral with the amplitude independent of the right, resp. left, base variable, i.e.

$$(Av)(x) = \int e^{i[(x'-y')\cdot\xi' + (x''-y'')\cdot\xi'' + x'\cdot\eta']} a(x'', \xi', \xi'', \eta')\, d\xi\, d\eta'\, v(y)\, dy,$$
$$a \in S^{p, l-\frac{k}{2}}(\mathbb{R}^{n-k}_{x''}; \mathbb{R}^n_\xi; \mathbb{R}^k_{\eta'}),$$

resp.

$$(Bu)(y) = \int e^{i[(y'-z')\cdot\zeta' + (y''-z'')\cdot\zeta'' + z'\cdot\mu']} b(z'', \zeta', \zeta'', \mu')\, d\zeta\, d\mu'\, u(z)\, dz,$$
$$b \in S^{p', l'-\frac{k}{2}}(\mathbb{R}^{n-k}_{z''}; \mathbb{R}^n_\zeta; \mathbb{R}^k_{\mu'}),$$

with

$$|\xi| \geq 1, |\eta'| \leq \epsilon|\xi| \text{ on } \operatorname{supp} a, \text{ and } |\zeta| \geq 1, |\mu'| \leq \epsilon|\zeta| \text{ on } \operatorname{supp} b,$$

for $\epsilon < 1/2$. Note that the wave front set of the Schwartz kernel of A (over $x' = y' = 0$, $x'' = y''$) is contained in the set of covectors of the form $(\xi' + \eta', \xi'', -\xi', -\xi'')$ such that a is not Schwartz in the direction (ξ', ξ'', η'), i.e. (ξ', ξ'', η') is not in the microsupport of a. Since we do not want covectors of the kind $o \times T^*\mathbb{R}^n$ in the wave front set, we need $(\xi' + \eta', \xi'')$ bounded away from 0 on the microsupport of a when $(\xi', \xi'') \neq 0$, which is accomplished by our requirement that $\epsilon < 1/2$.

Thus, with $\mathcal{F}$ denoting the Fourier transform on $\mathbb{R}^n$,

$$(Av)(x) = \int e^{i[x'\cdot\xi' + x''\cdot\xi'' + x'\cdot\eta']} a(x'', \xi', \xi'', \eta') (\mathcal{F}v)(\xi)\, d\xi\, d\eta',$$

while Bu is the inverse Fourier transform in ζ of

$$\int e^{i[-z'\cdot\zeta' - z''\cdot\zeta'' + z'\cdot\mu']} (2\pi)^n b(z'', \zeta', \zeta'', \mu')\, d\mu'\, u(z)\, dz.$$

Therefore,

$$(ABu)(x) = \int e^{i[(x'-z')\cdot\xi' + (x''-z'')\cdot\xi'' + x'\cdot\eta' + z'\cdot\mu']}$$
$$(2\pi)^n a(x'', \xi', \xi'', \eta') b(z'', \xi', \xi'', \mu')\, d\xi\, d\eta'\, d\mu'\, u(z)\, dz,$$

i.e. the Schwartz kernel of AB is given by the oscillatory integral

$$\int e^{i[(x'-z')\cdot\xi' + (x''-z'')\cdot\xi'' + x'\cdot\eta' + z'\cdot\mu']}$$
$$(2\pi)^n a(x'', \xi', \xi'', \eta') b(z'', \xi', \xi'', \mu')\, d\xi\, d\eta'\, d\mu'.$$

We rewrite the phase as

$$(x' - z') \cdot (\xi' - \mu') + (x'' - z'') \cdot \xi'' + x' \cdot (\eta' + \mu').$$

Letting $\nu' = \eta' + \mu'$, $\zeta' = \xi' - \mu'$, we deduce that the Schwartz kernel of AB is

$$\int e^{i[(x'-z')\cdot\zeta' + (x''-z'')\cdot\xi'' + x'\cdot\nu']} c(x'', z'', \zeta', \xi'', \nu')\, d\zeta'\, d\xi''\, d\nu',$$

$$c(x'', z'', \zeta', \xi'', \nu') = (2\pi)^n \int a(x'', \zeta' + \mu', \xi'', \nu' - \mu') b(z'', \zeta' + \mu', \xi'', \mu')\, d\mu'.$$

Thus, to show (5.23), we merely need to show that

$$c \in S^{p+p', L-k/2}(\mathbb{R}^{n-k}_{x''} \times \mathbb{R}^{n-k}_{z''}; \mathbb{R}^n_\xi; \mathbb{R}^k_{\nu'}), \tag{5.25}$$

and then the composition result follows. Note that in view of the support conditions on a and b, on the support of the integrand of c, $|\nu' - \mu'|, |\mu'| \leq \epsilon|\zeta + \mu'|$ (here $\zeta \in \mathbb{R}^n$), thus $|\mu'| \leq \frac{\epsilon}{1-\epsilon}|\zeta|$, $|\nu'| \leq 2\frac{\epsilon}{1-\epsilon}|\zeta|$, and thus the integral is certainly convergent, *without restrictions on* l, l', with c supported in $|\nu'| \leq 2|\zeta|$, and moreover $|\zeta + \mu'|$ is bounded from above and below by positive multiples of $|\zeta|$. For $l + l' < 0$, one gets, for an absolute constant $C > 0$, and with $\|a\|_{S^{p, l-\frac{k}{2}}, 0}$, etc., denoting 0th symbol norms (sup norms),

$$|c| \leq C \|a\|_{S^{p, l-\frac{k}{2}}, 0} \|b\|_{S^{p', l'-\frac{k}{2}}, 0} \langle\zeta\rangle^{p+p'} \int_{\mathbb{R}^k} \langle\nu' - \mu'\rangle^{l-k/2} \langle\mu'\rangle^{l'-k/2}\, d\mu'; \tag{5.26}$$

here for ν' in a compact set, one gets uniform bounds for the integral as the integrand is then bounded by $\tilde{C}\langle\mu'\rangle^{l+l'-k}$; $l + l' < 0$ is used here strongly. (If one does not assume $l + l' < 0$, one needs to use that $|\mu'| \lesssim |\zeta|$ on the support of the integrand, so $\mathbb{R}^k$ can be replaced by the ball $B_{|\zeta|}(0)$, and one obtains a positive power of $|\zeta|$ as a result when integrating, which allows one to obtain a paired Lagrangian symbolic estimate but with the rather undesirable increase of the order $p + p'$ on Λ_1. See also Remark 5.9.) Further, for $l + l' < 0$, the integral on the right hand side can be estimated, uniformly as $|\nu'| \to \infty$, by

$$C'(\langle\nu'\rangle^{l+l'} + \langle\nu'\rangle^{l-k/2} + \langle\nu'\rangle^{l'-k/2}) \leq C''\langle\nu'\rangle^{L-k/2}. \tag{5.27}$$

Indeed, for $|\nu'| \leq 1$, say, we already explained this estimate. Otherwise we break up the region of integration into $|\mu'| \leq |\nu'|/2$, resp. $|\nu' - \mu'| \leq |\nu'|/2$, resp. $|\nu'|/2 \leq |\mu'|, |\nu' - \mu'| \leq 2|\nu'|$, resp. $2|\nu'| \leq |\mu'|$, resp. $2|\nu'| \leq |\nu' - \mu'|$. Note that the last two regions are not disjoint, but the union of the five regions is $\mathbb{R}^k$. On the first, resp. second of these, $\langle\nu' - \mu'\rangle$, resp. $\langle\mu'\rangle$ is bounded from above and below by a positive multiple of $\langle\nu'\rangle$, so the corresponding weight can be pulled outside the integral, so in the first case one is reduced to the estimate

$$\int_{B_{|\nu'|/2}(0)} \langle\mu'\rangle^{l-k/2}\, d\mu' \lesssim (1 + |\nu'|^{l+k/2}),$$

resulting in an overall bound $|\nu'|^{l'-k/2}(1 + |\nu'|^{l+k/2})$, yielding that (5.27) is satisfied in this case, with a similar estimate in the second case. In the third case, both $\langle\nu' - \mu'\rangle$ and $\langle\mu'\rangle$ is bounded from above and below by a positive multiple of $\langle\nu'\rangle$, and one obtains a bound $\lesssim |\nu'|^{l+l'}$. In the fourth, resp. fifth case, $\langle\nu' - \mu'\rangle$, resp. $\langle\mu'\rangle$ is bounded from above and below by a positive multiple of $\langle\mu'\rangle$, resp. $\langle\nu' - \mu'\rangle$, so in the fourth case one is reduced to the estimate

$$\int_{|\mu'| \geq 2|\nu'|} \langle\mu'\rangle^{l+'l-k}\, d\mu' \lesssim \langle\nu'\rangle^{l+l'},$$

with a similar bound in the fifth case; these use $l+l'<0$. This proves (5.27), and thus gives the 0th seminorm estimate of the claimed $S^{p+p',L}(\mathbb{R}^{n-k}_{x''}\times\mathbb{R}^{n-k}_{z''};\mathbb{R}^n_\xi;\mathbb{R}^k_{\nu'})$ statement, (5.25), for c.

The derivatives can be handled easily, with this being immediate for ζ, x'' and z'' derivatives, while for $\nu'_j\partial_{\nu'_k}$ derivatives one writes $\nu'_j\partial_{\nu'_k}=(\nu'_j-\mu'_j)\partial_{\nu'_k}+\mu'_j\partial_{\nu'_k}$ under the integral, then the first term is handled by the symbol bounds for a, while for the second one rewrites $\mu'_j\partial_{\nu'_k}a$ as $-\mu'_j\partial_{\mu'_k}a+\mu'_j\partial_{\zeta'_k}a$, integrates by parts for the first term to use the symbol estimates of b, while the symbol estimates for a plus the bounds for μ' in terms of $\zeta+\mu'$ handle the second term. Proceeding inductively, one deduces that (5.25) holds.

To prove the principal symbol property, take $N\geq 1$ integer. (Here $N=1$ suffices; taking N larger one can obtain further terms in the Λ_1-symbolic expansion of the composition.) We expand a,b in Taylor series in their second argument, $\zeta'+\mu'$, around ζ' with the integral remainder formula involving Nth derivatives. In case of a, this gives terms

$$\frac{1}{\alpha!}(\mu')^\alpha(\partial^\alpha_{\zeta'}a)(x'',\zeta',\xi'',\nu'-\mu')$$

with $|\alpha|<N$ in the expansion, and the remainder is a sum of integrals with $|\alpha|=N$:

$$\int_0^1\frac{N}{\alpha!}(1-t)^N(\mu')^\alpha(\partial^\alpha_{\zeta'}a)(x'',\zeta'+t\mu',\xi'',\nu'-\mu')\,dt;$$

similar expressions hold for b, with $(\mu')^\beta\partial^\beta_{\zeta'}$ being the relevant derivatives. The $(\alpha\beta)$th term (with $|\alpha|\leq N$, $|\beta|\leq N$) in c inside the integral has bounds

$$\lesssim\langle\zeta'+\mu'\rangle^{p+p'-|\alpha|-|\beta|}\langle\nu'-\mu'\rangle^{l-k/2}\langle\mu'\rangle^{l'+|\alpha|+|\beta|-k/2},$$

and thus if $l+l'+|\alpha|+|\beta|<0$, the contribution to c is in

$$\begin{aligned}&S^{p+p'-|\alpha|-|\beta|,L_{\alpha\beta}},\ \text{with}\\ &L_{\alpha\beta}=\max(l,l'+|\alpha|+|\beta|,l+l'+|\alpha|+|\beta|+k/2)\leq L+|\alpha|+|\beta|.\end{aligned}$$

If $l+l'+|\alpha|+|\beta|\geq 0$, then, letting

$$M=-\delta+|\alpha|+|\beta|>l+l'+|\alpha|+|\beta|\geq 0,$$

so $M<|\alpha|+|\beta|$, and using

$$\langle\mu'\rangle^{l'+|\alpha|+|\beta|-k/2}\lesssim\langle\zeta\rangle^M\langle\mu'\rangle^{l'+|\alpha|+|\beta|-k/2-M}$$

(by the support conditions), we obtain that the contribution of the $(\alpha\beta)$th term to c is in

$$\begin{aligned}&S^{p+p'-|\alpha|-|\beta|+M,\tilde L_{\alpha\beta}},\ \text{with}\\ &\tilde L_{\alpha\beta}=\max(l,l'+|\alpha|+|\beta|-M,l+l'+|\alpha|+|\beta|+k/2-M)\\ &\qquad\leq L+|\alpha|+|\beta|-M.\end{aligned}$$

This gives that modulo $S^{p+p'-\min(1,\delta),L+\min(1,\delta)}$, c is given by the convolution

$$(2\pi)^n\int a(x'',\zeta',\xi'',\nu'-\mu')b(z'',\zeta',\xi'',\mu')\,d\mu'.$$

Taylor expanding b in z'' around x'' and integrating by parts in ξ'' gives that further this can be replaced by

$$\tilde c(x'',\zeta',\xi'',\nu')=(2\pi)^n\int a(x'',\zeta',\xi'',\nu'-\mu')b(x'',\zeta',\xi'',\mu')\,d\mu'$$

modulo $S^{p+p'-1,L+1}$. The Λ_1-principal symbol of the distribution corresponding to $\tilde{c}$ is $(2\pi)^{n-k}$ times the partial inverse Fourier transform in ν' of $\tilde{c}$. Since the inverse Fourier transform of a convolution in $\mathbb{R}^k$ is $(2\pi)^k$ times the product of the inverse Fourier transforms of the factors, we deduce that this principal symbol is

$$\begin{aligned}(2\pi)^{n+k}&((\mathcal{F}')^{-1}\tilde{c})\,|d\zeta'|^{1/2}\,|d\xi''|^{1/2}\\ &= (2\pi)^{n+k}((\mathcal{F}')^{-1}a)(2\pi)^{n+k}((\mathcal{F}')^{-1}b)\,|d\zeta'|^{1/2}\,|d\xi''|^{1/2},\end{aligned}$$

i.e. it is the product of the principal symbols of a and b, as claimed. □

Remark 5.11. Note that the proof we just gave also shows that if

$$\tilde{a} \in S^p(\mathbb{R}^n_\xi; I^{l-\frac{n}{4}}(\mathbb{R}^n_x; N^*\{x'=0\})),\ \tilde{b} \in S^{p'}(\mathbb{R}^n_\xi; I^{l'-\frac{n}{4}}(\mathbb{R}^n_x; N^*\{x'=0\})),$$

with $l+l'<0$, then with $L=\max(l,l',l+l'+k/2)$,

$$\tilde{a}\tilde{b} \in S^{p+p'}(\mathbb{R}^n_\xi; I^{L-\frac{n}{4}}(\mathbb{R}^n_x; N^*\{x'=0\})).$$

This does not require a conic support condition on the partial (x'-)Fourier transforms a, resp. b, of $\tilde{a}$, resp. $\tilde{b}$ like one we did above; one is estimating a partial convolution c of a and b in the dual variable μ' of x', and the estimates boil down to (5.27) being satisfied for the integral on the right hand side of (5.26). Further, this shows that

$$\|\tilde{a}\tilde{b}\|_{S^{p+p'}(I^{L-\frac{n}{4}});0} \leq C\|\tilde{a}\|_{S^p(I^{l-\frac{n}{4}});0}\|\tilde{b}\|_{S^{p'}(I^{l'-\frac{n}{4}});0}, \tag{5.28}$$

where we used a short hand notation for the symbol spaces discussed above to simplify the notation. Now, the higher order product-type symbol norms for the partial Fourier transform, of the partial convolution c are equivalent to a product of ∂_{ξ_j}, $\xi_k\partial_{\xi_j}$, ∂_{x_j}, $\mu'_j\partial_{\mu'_k}$, $\partial_{\mu'_k}$ being applied iteratively to c and the zeroth $S^{p+p',L-\frac{k}{2}}$ norm being evaluated. As c is the partial Fourier transform of $\tilde{a}\tilde{b}$ in x', this means ∂_{ξ_j}, $\xi_k\partial_{\xi_j}$, ∂_{x_j}, $\partial_{x'_j}x'_k$, x'_k being applied iteratively to $\tilde{a}\tilde{b}$, and the zeroth $S^{p+p',L-\frac{k}{2}}$ norm of the partial Fourier transform of the result being evaluated. (Here x'_k can be dropped if one assumes compact support for $\tilde{a}$ or $\tilde{b}$; one can also replace $\partial_{x'_j}x'_k$ by $x'_k\partial_{x'_j}$.) Using Leibniz' rule, which is valid by the density of order $-\infty$ symbols in μ', resp. order $-\infty$ conormal distributions in x', and using (5.28), the seminorms of $\tilde{a}\tilde{b}$ in $S^{p+p'}(I^{L-\frac{n}{4}})$ are bounded by

$$\|\tilde{a}\tilde{b}\|_{S^{p+p'}(I^{L-\frac{n}{4}});k} \leq C_k\|\tilde{a}\|_{S^p(I^{l-\frac{n}{4}});k}\|\tilde{b}\|_{S^{p'}(I^{l'-\frac{n}{4}});k}. \tag{5.29}$$

We record here a statement regarding square roots of conormal distributions that will be useful later; it allows us to construct square root of the principal symbols of paired Lagrangian distributions.

Lemma 5.12. *Suppose that* $a \in S^p(\mathbb{R}^n_\xi; I^{l-\frac{n}{4}}(\mathbb{R}^n_x; N^*\{x'=0\}))$ *with* $l<-k/2$, *and with* $a \geq c|\xi|^p$, $c>0$, *for* $|\xi|\geq R$, *on a conic open set* $\Gamma \subset \mathbb{R}^n_\xi$. *Let* $l' \in (l,-k/2)$. *Then*

$$b=\sqrt{a} \in S^{p/2}(\Gamma_\xi; I^{l'-\frac{n}{4}}(\mathbb{R}^n_x; N^*\{x'=0\})).$$

Note that under the assumptions, a is the inverse Fourier transform in μ', the dual variable of x', of a symbol of order $l-k/2<-k$, so a is actually continuous, and indeed Hölder α for $0<\alpha<-(l+k/2)$. Thus, the pointwise statement $a \geq c|\xi|^p$ actually makes sense.

Proof. Note that the statement is a consequence of the positive ellipticity of a away from $x'=0$, so we may work in an arbitrarily small neighborhood of $x'=0$ as is convenient. Given $\epsilon>0$, we first decompose $a=a_1+a_2$ with

$$a_1 \in S^p(\mathbb{R}^n_\xi; C^\infty(\mathbb{R}^n_x)) = S^p(\mathbb{R}^n_x, \mathbb{R}^n_\xi),$$
$$a_2 \in S^p(\mathbb{R}^n_\xi; I^{l'-\frac{n}{4}}(\mathbb{R}^n_x; N^*\{x'=0\})),$$

and with $a_1 \geq (c/2)|\xi|^p$, for $|\xi|\geq R$, on $\Gamma\subset\mathbb{R}^n_\xi$, while $\|a_2\|_{S^p(\mathbb{R}^n_\xi; I^{l'-\frac{n}{4}});0} < \epsilon$ (here we use shorthand notation as in the above remark). To do so, we note that

$$a = a_0 + \mathcal{F}^{-1}_{\mu'} b,\ b\in S^{p,l-k/2}(\mathbb{R}^{n-k}_{x''};\mathbb{R}^n_\xi;\mathbb{R}^k_{\mu'}),\ a_0\in S^p(\mathbb{R}^n_\xi; C^\infty(\mathbb{R}^n_x)).$$

Now, given $\epsilon'>0$, the standard approximation argument, using $b_R = b\phi(\mu'/R)$, where $\phi\equiv 1$ near 0, has compact support, letting $R\to\infty$ gives $b_1'\in S^{p,-\infty}$ such that $\|b-b_1'\|_{S^{p,l'-k/2};0} < \epsilon'$. Then, as $l'-k/2 < -k$, with $b_2 = b-b_1'$,

$$\sup|\langle\xi\rangle^{-p}\mathcal{F}^{-1}_{\mu'} b_2| \leq C_0\|b_2\|_{S^{p,l'-k/2};0} < C_0\epsilon'.$$

Thus, with $a_2 = \mathcal{F}^{-1}_{\mu'} b_2$, $a_1 = a - a_2 = a_0 + \mathcal{F}^{-1}_{\mu'} b_1'$, $a > (c - C_0\epsilon')|\xi|^p$. Now let $\epsilon' = \min(\epsilon, c/(2C_0))$; then a_1 and a_2 satisfy all conditions.

We note that as a_1 is elliptic on Γ, with a positive elliptic lower bound,

$$\sqrt{a_1}\in S^{p/2}(\Gamma_\xi; C^\infty(\mathbb{R}^n)),$$
$$\tilde a = a_1^{-1}a_2 \in S^0(\Gamma_\xi; I^{l'-\frac{n}{4}}(\mathbb{R}^n_x; N^*\{x'=0\})),$$

and $\tilde a$ vanishes at $x'=0$. We write

$$b = \sqrt{a_1}\sqrt{1+(a_1^{-1}a_2)},$$

and we are reduced to showing that

$$\sqrt{1+\tilde a}\in S^0(\Gamma_\xi; I^{l'-\frac{n}{4}}(\mathbb{R}^n_x; N^*\{x'=0\})). \tag{5.30}$$

We expand $f=\sqrt{1+.}$ in Taylor series, whose radius of convergece 1. By Remark 5.11,

$$\tilde a^N \in S^0(\Gamma_\xi; I^{l'-\frac{n}{4}}(\mathbb{R}^n_x; N^*\{x'=0\})),$$

with

$$\|\tilde a^N\|_{S^0(I^{l'-\frac{n}{4}});0} \leq C^{N-1}\|\tilde a\|^N_{S^0(I^{l'-\frac{n}{4}});0},$$

which follows from (5.28) by induction. This shows that, provided $\|\tilde a\|_{S^0(I^{l'-\frac{n}{4}});0} < C^{-1}$ (which holds if $\epsilon < C^{-1}$), the Taylor series converges in the 0th $S^0(I^{l'-\frac{n}{4}})$-norm. Then differentiating the Taylor series with respect to operators giving rise to the symbol topology, as discussed in Remark 5.11, preserves the $S^0(I^{l'-\frac{n}{4}})$-estimates in view of the chain rule for derivatives, which gives $(f'\circ\tilde a)(V\tilde a)$, where V is one of ∂_{ξ_j}, $\xi_k\partial_{\xi_j}$, ∂_{x_j}, $x_k'\partial_{x_j'}$, x_k', and the fact that Va satisfies $S^0(I^{l-\frac{n}{4}})$-estimates as well, plus the fact that f' also has Taylor series with radius of convergence 1. Iterating this argument proves (5.30), and thus the lemma. □

As we already mentioned, a different model for the Lagrangians in $\mathbb{R}^{2n}$, used by Greenleaf and Uhlmann [3], is the pair $(N^*\mathrm{diag}, \Lambda_{\tilde\Gamma})$ when $\tilde\Gamma$ given by $\xi'=0$, so

$$\tilde\Lambda = \Lambda_{\tilde\Gamma} = \{((x,\xi),(y,\eta)):\ \xi'=0=\eta',\ \xi''=-\eta'',\ x''=y''\}.$$

With this model, paired Lagrangian distributions in $I^{p,l}(N^*\mathrm{diag},\tilde\Lambda)$ are given by oscillatory integrals

$$\int e^{i[(x'-y'-s)\cdot\zeta'+(x''-y'')\cdot\zeta+s\sigma]}a(x,y,s,\zeta,\sigma)\,ds\,d\zeta\,d\sigma,$$

with $a\in S^{M,M'}(\mathbb{R}^{2n+k},\mathbb{R}^n,\mathbb{R}^k)$, $M=p+k/2$, $M'=l-k/2$ (there is a typo in [3] in their definition of the first order after (1.31)). Note here the flow-out is the second Lagrangian, reversed as compared to Proposition 5.8, which is convenient to apply the results of Antoniano and Uhlmann, but is not convenient in our case.

Since the structure of the projection maps of the left and the right factors matters for composition purposes (i.e. just because all Lagrangian pairs can be put to a model form via a symplectomorphism on $\mathbb{R}^{2n}$, it does not follow that they all have the same composition properties!), we also need another special case of the model of (5.14) in $\mathbb{R}^{2n}=\mathbb{R}^n_x\times\mathbb{R}^n_y$, with $\mathbb{R}^n=\mathbb{R}^k_{x'}\times\mathbb{R}^{n-k}_{x''}$ and with (ξ,η) the dual variables of (x,y), as before. This is

$$\begin{aligned}&\Lambda_1=\{x'=0,\ \xi''=0,\ \eta'=0,\ \eta''=0\}=N^*\{x'=0\},\\&\Lambda_0=\{x'=0=y',\ x''=y'',\ \xi''=-\eta''\}=N^*\{x'=0=y',\ x''=y''\},\end{aligned}\tag{5.31}$$

this time with codimension n intersection. Note that here Λ_0 is the same 'flow-out' Lagrangian as in (5.20), but Λ_1 a Lagrangian of the form $\Lambda_1^\sharp\times o_{\mathbb{R}^n}$, with $\Lambda_1^\sharp$ Lagrangian in $T^*\mathbb{R}^n\setminus o_{\mathbb{R}^n}$, which means that if an operator with Schwartz kernel in $I^p(\Lambda_1)$ is applied to even a $C_c^\infty(\mathbb{R}^n_y)$ function, the result is not C^∞, merely Lagrangian on $\Lambda_1^\sharp$. (There is a dual phenomenon if one reverses the x and the y factors, namely then the operator cannot be applied to all distributions.) For this pair, the parameterization, modulo $I^p(\Lambda_1)$, is

$$\int e^{i[x'\cdot\xi'-y'\cdot\eta'+(x''-y'')\cdot\eta'']}a(x'',\xi',\eta',\eta'')\,d\xi'\,d\eta,\ a\in S^{p+\frac{n-k}{2},l-\frac{n}{2}}(\mathbb{R}^{n-k}_{x''};\mathbb{R}^k_{\xi'};\mathbb{R}^n_\eta),\tag{5.32}$$

where a conic neighborhood of $\Lambda_0\cap\Lambda_1$ corresponds to a conic neighborhood of $\eta=0$ in $\mathbb{R}^k_{\xi'}\times\mathbb{R}^n_\eta$ (so now ξ' is the 'large variable' on the parameter space), and the x'' dependence can again be replaced by y'' dependence. (To see this form of parameterization, write $z''=x'$, $z'=(-y',x''-y'')$, $z'''=x''$ then $\Lambda_1=N^*\{z''=0\}$, $\Lambda_0=N^*\{z'=0,\ z''=0\}$. Replacing x'' by y'' in the definition of z''' gives the other parameterization.)

We first note the action of pseudodifferential operators applied from either factor to this pair:

Lemma 5.13. *Let Λ_0,Λ_1 as in (5.31), with x's being the left variables. Then for $Q\in\Psi^s(\mathbb{R}^n)$ (of proper support), and for $K\in I^{p,l}(\Lambda_0,\Lambda_1)$, $QK\in I^{p+s,l}(\Lambda_0,\Lambda_1)$ while $KQ\in I^{p,l+s}(\Lambda_0,\Lambda_1)$.*

Proof. As before, it suffices to consider kernels K of the form (5.32), or its y''-dependent analogue, for kernels in $I^p(\Lambda_1)$, as well as those in $I^{p+l}(\Lambda_0)$ with wave front set disjoint from $\Lambda_0\cap\Lambda_1$ (thus away from covectors with vanishing dual-to-y components), can easily be treated by standard results.

In order to find QK, write K in the form (5.32), but with x'' dependence replaced by y'' dependence. Writing Q as left quantization,

$$Qv(z)=(2\pi)^{-n}\int e^{i(z'\cdot\zeta'+z''\cdot\zeta'')}q(z',z'',\zeta',\zeta'')(\mathcal{F}v)(\zeta',\zeta'')\,d\zeta'\,d\zeta'',$$

and using that (5.32) with x''-dependence replaced by y''-dependence gives, when applied to a C_c^∞ function u,

$$\left(\mathcal{F}_{\xi',\eta''}\left(\int e^{i[-y'\cdot\eta'-y''\cdot\eta'']}a(y'',\xi',\eta',\eta'')\,u(y',y'')\,d\eta'\,dy'\,dy''\right)\right)(x',x''),$$

we conclude that

$$\begin{aligned}QKu(z) =&(2\pi)^{-n}\int e^{i(z'\cdot\zeta'+z''\cdot\zeta''-y'\cdot\eta'-y''\cdot\zeta'')}\\ &\qquad q(z',z'',\zeta',\zeta'')a(y'',\zeta',\eta',\zeta'')\,u(y',y'')\,d\eta'\,dy'\,dy''\,d\zeta'\,d\zeta'',\end{aligned}$$

so the Schwartz kernel of QK is given by the oscillator integral

$$QK = (2\pi)^{-n}\int e^{i(z'\cdot\zeta'-y'\cdot\eta'+(z''-y'')\cdot\zeta'')}q(z',z'',\zeta',\zeta'')a(y'',\zeta',\eta',\zeta'')\,d\eta'\,d\zeta'\,d\zeta'',$$

which is of the desired form.

Composition from the right can be checked similarly, using (5.32) as stated, with x''-dependence. □

Most crucially we need mapping properties of these operators on Sobolev spaces.

Proposition 5.14. *Let Λ_0,Λ_1 as in (5.31), with x's being the left variables. Then for $K\in I^{p,l}(\Lambda_0,\Lambda_1)$ with wave front set disjoint from $o_{\mathbb{R}^n}\times T^*\mathbb{R}^n$, and for $m,m'\in\mathbb{R}$,*

$$p+l<m+m'-\frac{k}{2}\ \text{and}\ p<m-\frac{n}{2}\Rightarrow K\in\mathcal{L}(H^{m'},H^{-m}). \tag{5.33}$$

Remark 5.15. Note that the first condition of (5.33) is almost exactly the statement that a distribution in $I^{p+l}(\Lambda_0)$ with wave front set away from $\Lambda_0\cap\Lambda_1$ is bounded from $H^{m'}$ to H^{-m}, with 'almost' referring to the loss of the normally allowed equality, cf. Proposition 5.7 and the remarks afterwards. On the other hand, the second condition in (5.33) is exactly the condition that an element of $I^p(\Lambda_1)$ maps distributions (or even just C^∞, for that matter) into H^{-m}.

Proof. We first remark that (5.33) is equivalent to the combination of the two conditions: either

$$l\geq m'-\frac{k}{2}+\frac{n}{2}\ \text{and}\ p+l<m+m'-\frac{k}{2}, \tag{5.34}$$

or

$$l<m'-\frac{k}{2}+\frac{n}{2}\ \text{and}\ p<m-\frac{n}{2}. \tag{5.35}$$

Indeed, (5.33) automatically implies these two, and conversely, if $l\geq m'-\frac{k}{2}+\frac{n}{2}$ then subtracting the first inequality from the second yields $p<m-\frac{n}{2}$ while if $l<m'-\frac{k}{2}+\frac{n}{2}$ then adding the inequalities yields $p+l<m+m'-\frac{k}{2}$.

Now, in view of Lemma 5.13, at the cost of replacing p by $p-m$ and l by $l-m'$, as we now do, it suffices to consider L^2-boundedness. Further, one may assume that K is of the form (5.32), with a supported in the region $\langle\eta\rangle\leq\langle\xi'\rangle$. We claim that if we let $A(x'',\eta'')$ be the operator on $\mathbb{R}^k_{x'}$ given by

$$(A(x'',\eta'')u)(x')=\int e^{i[x'\cdot\xi'-y'\cdot\eta']}a(x'',\xi',\eta',\eta'')u(y')\,d\xi'\,d\eta'\,dy',\ u\in C_c^\infty(\mathbb{R}^k),$$

and if either set of conditions (5.34), resp. (5.35), is satisfied then

$$A \in S^0(\mathbb{R}^{n-k}_{x''}; \mathbb{R}^{n-k}_{\eta''}; \mathcal{L}(L^2(\mathbb{R}^k), L^2(\mathbb{R}^k))), \tag{5.36}$$

i.e. it is an operator-valued symbol of order 0, which thus by the operator-valued version of the standard calculus, see [8, Section 18.1, Remark 2], gives a bounded operator

$$\mathcal{L}(L^2(\mathbb{R}^{n-k}; L^2(\mathbb{R}^k)); L^2(\mathbb{R}^{n-k}; L^2(\mathbb{R}^k))) = \mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)),$$

proving the proposition.

But with $\mathcal{F}'$ denoting the Fourier transform in the primed variables,

$$(A(x'', \eta'')u)(x') = (2\pi)^k \left((\mathcal{F}')^{-1}(\int a(x'', ., \eta', \eta'')(\mathcal{F}u)(\eta')\, d\eta') \right)(x'),$$

i.e. $\mathcal{F}' A (\mathcal{F}')^{-1}$ has Schwartz kernel $(2\pi)^k a(x'', \xi', \eta', \eta'')$, with the action in the primed variables. First we check that $A(.,.)$ is a uniformly bounded family of bounded operators (and indeed, a uniformly bounded family of Hilbert-Schmidt operators). This follows if we show that

$$a(x'', \xi', \eta', \eta'') \in L^\infty(\mathbb{R}^{2(n-k)}_{x'',\eta''}; L^2(\mathbb{R}^{2k}_{\xi',\eta'})),$$

which in turn follows if for some $\delta > 0$,

$$\langle \xi' \rangle^{\frac{k}{2}+\delta} \langle \eta' \rangle^{\frac{k}{2}+\delta} a \in L^\infty(\mathbb{R}^{2n}).$$

But

$$\langle \xi' \rangle^{\frac{k}{2}+\delta} \langle \eta' \rangle^{\frac{k}{2}+\delta} |a| \leq C \langle \xi' \rangle^{p-m+\frac{n}{2}+\delta} \langle \eta \rangle^{l-m'-\frac{n-k}{2}+\delta} \tag{5.37}$$

Now, if $l - m' - \frac{n-k}{2} \geq 0$ then $\langle \eta \rangle^{l-m'-\frac{n-k}{2}+\delta} \leq \langle \xi' \rangle^{l-m'-\frac{n-k}{2}+\delta}$, and thus

$$\langle \xi' \rangle^{\frac{k}{2}+\delta} \langle \eta' \rangle^{\frac{k}{2}+\delta} |a| \leq C \langle \xi' \rangle^{p+l-m-m'+\frac{k}{2}+2\delta}, \tag{5.38}$$

and thus is bounded since $p + l - m - m' + \frac{k}{2} < 0$ means that one can take sufficiently small $\delta > 0$ to still have $p + l - m - m' + \frac{k}{2} + 2\delta \leq 0$. On the other hand, if $l - m' - \frac{n-k}{2} < 0$ then $p - m + \frac{n}{2} < 0$ as well, so one may choose $\delta > 0$ sufficiently small so that the right hand side of (5.37) is bounded.

Since $D_{\eta''_j}$, $\eta''_j D_{\eta''_i}$, $D_{x''_j}$ preserve the symbolic order of a, analogous properties follow when these differential operators are applied to $A(.,.)$ iteratively, implying that (5.36) holds, which in turn completes the proof of the proposition. □

If the role of the x and y variables is reversed one has

$$\begin{aligned} \hat{\Lambda}_1 &= \{y' = 0,\ \eta'' = 0,\ \xi' = 0,\ \xi'' = 0\} = N^*\{y' = 0\}, \\ \Lambda_0 &= \{x' = 0 = y',\ x'' = y'',\ \xi'' = -\eta''\} = N^*\{x' = 0 = y',\ x'' = y''\}, \end{aligned} \tag{5.39}$$

as the modified model. Either essentially repeating the arguments given above, or noting that if $K \in I^{p,l}(\Lambda_0, \hat{\Lambda}_1)$ then its adjoint is in $I^{p,l}(\Lambda_0, \Lambda_1)$, and thus via dualization one obtains mapping properties of K from Proposition 5.14, one has

Proposition 5.16. *Let $\Lambda_0, \hat{\Lambda}_1$ as in (5.39), with x's being the left variables. Then for $K \in I^{p,l}(\Lambda_0, \Lambda_1)$ with wave front set disjoint from $o_{\mathbb{R}^n} \times T^*\mathbb{R}^n$, and for $m, m' \in \mathbb{R}$,*

$$p + l < m + m' - \frac{k}{2} \text{ and } p < m' - \frac{n}{2} \Rightarrow K \in \mathcal{L}(H^{m'}, H^{-m}). \tag{5.40}$$

Even if a distribution is Lagrangian associated to Λ_0 (i.e. has no singularity at Λ_1), the fact that Λ_0 intersects $T^*\mathbb{R}^n \times o_{\mathbb{R}^n}$ means that the standard results on mapping properties do not apply. However, one *can* regard this distribution as a paired Lagrangian associated to (Λ_0, Λ_1) and apply the previous propositions:

Corollary 5.17. *Let Λ_0 be as in* (5.31)*, with x's being the left variables. Then for $K \in I^p(\Lambda_0)$ with wave front set disjoint from $o_{\mathbb{R}^n} \times T^*\mathbb{R}^n$ and for $m, m' \in \mathbb{R}$,*

$$p < m + m' - \frac{k}{2} \text{ and } p < m \Rightarrow K \in \mathcal{L}(H^{m'}, H^{-m}). \tag{5.41}$$

In case $K \in I^p(\Lambda_0)$ with wave front set disjoint from $T^*\mathbb{R}^n \times o_{\mathbb{R}^n}$, then the conditions become

$$p < m + m' - \frac{k}{2} \text{ and } p < m' \Rightarrow K \in \mathcal{L}(H^{m'}, H^{-m}). \tag{5.42}$$

We remark that mapping properties of certain Lagrangian distributions with canonical relations intersecting the zero section were studied in a different setting by Greenleaf and Uhlmann [5].

Proof. With Λ_1 as in (5.31), $\Lambda_0 \cap \Lambda_1$ has codimension n in either of these two Lagrangians, and thus by Lemma 5.2, $I^p(\Lambda_0) \subset I^{p-\frac{n}{2},\frac{n}{2}}(\Lambda_0, \Lambda_1)$. Thus by Proposition 5.14, K is bounded as claimed provided $p < m + m' - \frac{k}{2}$ and $p < m$, which completes the proof. ☐

As an example, with $\operatorname{codim} Y = k$, $\dim X = n$, consider

$$f \in I^{[-s_0]}(Y) = I^{-s_0-(\dim X - 2k)/4}(N^*Y).$$

Then the pullback $\pi_L^* f$ of f to $X \times X$, via the left projection to X, is in $I^{[-s_0]} = I^{-s_0-\dim Y/2}(N^*(Y \times X))$. Since for $A \in \Psi^s(X)$ one has $K_A \in I^s(N^*\text{diag})$ (with K_A denoting the Schwartz kernel of A), the Schwartz kernel K_{fA} of fA is $(\pi_L^* f)K_A$, and by (5.19) one has

$$\begin{aligned} K_{fA} \in & I^{s,-s_0+k/2}(N^*(\text{diag} \cap (Y \times X)), N^*\text{diag}) \\ & + I^{-s_0-\dim Y/2, s+n/2}(N^*(\text{diag} \cap (Y \times X)), N^*(Y \times X)). \end{aligned} \tag{5.43}$$

Similar results apply to Af, with the left and the right factors interchanged.

In the special case $A = \text{Id}$ we get

$$\begin{aligned} K_{f\ \text{Id}} \in & I^{0,-s_0+k/2}(N^*(\text{diag} \cap (Y \times X)), N^*\text{diag}) \\ & + I^{-s_0-\dim Y/2, n/2}(N^*(\text{diag} \cap (Y \times X)), N^*(Y \times X)). \end{aligned} \tag{5.44}$$

In this case one could write the multiplication also as a multiplication from the right factor, and thus deduce that the second summand can be dropped. (This also follows directly from [3].) However, this has no impact on the following consequence:

Proposition 5.18. *Multiplication by $f \in I^{[-s_0]}(Y) = I^{-s_0-(\dim X-2k)/4}(N^*Y)$ is bounded $H^s \to H^s$ provided $s_0 > \operatorname{codim} Y$ and $-s_0 + k/2 < s < s_0 - k/2$.*

Proof. In view of (5.44) and Propositions 5.7 and 5.14 and Corollary 5.17, multiplication by f is bounded $H^s \to H^s$ provided

$$\begin{aligned} & -s_0 + k/2 < -k/2, \\ & -s_0 - \dim Y/2 < s - n/2, \\ & -s_0 + k/2 < -s, \end{aligned}$$

which gives exactly the constraints in the proposition. □

6. Elliptic estimates

In this section we discuss microlocal elliptic estimates, which help take care of the regions of phase space one would like to think of as 'irrelevant' for wave propagation purposes. Here, and in the next section, we denote the position (base) variable by x, the dual variable by ξ, and use local coordinates in which Y is given by $\{x'=0\}$.

So suppose that $g \in I^{[-s_0]}(Y)$, $\operatorname{codim} Y = k$, with G the dual metric. For simplicity, we reduce the problem from $\Box$ to

$$P = (\det g)^{1/2}\Box = \sum_{ij} D_i (\det g)^{1/2} G_{ij} D_j.$$

If $\Box u = f \in H^s$, then by Proposition 5.18 multiplication by $(\det g)^{1/2} \in I^{[-s_0]}(Y)$ preserves H^s if

$$\begin{aligned} &s_0 > k,\\ &-s_0 + k/2 < s < s_0 - k/2. \end{aligned} \tag{6.1}$$

Thus,

$$Pu = (\det g)^{1/2} f \in H^s;$$

so under these constraints, we may instead study the equation $Pu = \tilde{f}$. We write

$$g_{ij} = (\det g)^{1/2} G_{ij},\ P = \sum_{ij} D_i g_{ij} D_j, \tag{6.2}$$

and note that P is formally self-adjoint with respect to the Euclidean inner product.

For $A \in \Psi^{2s-2}(X)$; we need to compute the Schwartz kernel of PA (or AP) as a (sum of) paired Lagrangian distribution(s). The Schwarz kernel K_{D_iA} of D_iA is $D_{i,L}K_A$ (where the subscript L denotes the derivative acting on the left factor of $X \times X$), while the Schwartz kernel K_{AD_i} of AD_i is $-D_{i,R}K_A$, we deduce that the Schwartz kernel of PA, resp. AP, is

$$K_{PA} = \sum D_{i,L} g_{ij,L} D_{j,L} K_A, \qquad K_{AP} = \sum D_{j,R} g_{ij,R} D_{i,R} K_A.$$

Here $g_{ij,L}$, resp. $g_{ij,R}$, is the pullback of g_{ij} from the left, resp. right, factor. Now, $K_A \in I^{2s-2}(N^*\mathrm{diag})$, so $D_{i,L}K_A, D_{i,R}K_A \in I^{2s-1}(N^*\mathrm{diag})$. Now as $g_{ij} \in I^{[-s_0]}(Y)$, by (5.43) (with the left and right factors interchanged in the first case),

$$\begin{aligned} g_{ij,R} D_{i,R} K_A \in & I^{2s-1,-s_0+k/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*\mathrm{diag})\\ &+ I^{-s_0-\dim Y/2, 2s-1+n/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*(X \times Y)), \end{aligned}$$

and

$$\begin{aligned} g_{ij,L} D_{i,L} K_A \in & I^{2s-1,-s_0+k/2}(N^*(\mathrm{diag} \cap (Y \times X)), N^*\mathrm{diag})\\ &+ I^{-s_0-\dim Y/2, 2s-1+n/2}(N^*(\mathrm{diag} \cap (Y \times X)), N^*(Y \times X)), \end{aligned}$$

so in particular away from the intersections, these are Lagrangian associated to the conormal bundles of $X \times Y$ (or $Y \times X$), diag, as well as their intersection, $(Y \times Y) \cap \mathrm{diag}$, with orders $I^{[-s_0]} = I^{-s_0-2\dim Y/4}$, $I^{[2s-1]} = I^{2s-1}$ and $I^{[-s_0+2s-1]} = I^{-s_0+2s-1+k/2}$. Applying $D_{j,R}$, resp. $D_{j,L}$ increases the orders on all Lagrangians, i.e. in terms of paired Lagrangians it increases the first order (corresponding to the main Lagrangian, i.e. the second in the pair, dictating the singular behavior) by 1, see in particular Lemma 5.13. Thus, we conclude:

Lemma 6.1. *For* $g \in I^{[-s_0]}(Y)$*,* $A \in \Psi^{2s-2}(X)$ *with compactly supported Schwartz kernel,*

$$K_{AP} \in I^{2s,-s_0+k/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*\mathrm{diag}) + I^{-s_0+1-\dim Y/2, 2s-1+n/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*(X \times Y)), \tag{6.3}$$

and

$$K_{PA} \in I^{2s,-s_0+k/2}(N^*(\mathrm{diag} \cap (Y \times X)), N^*\mathrm{diag}) + I^{-s_0+1-\dim Y/2, 2s-1+n/2}(N^*(\mathrm{diag} \cap (Y \times X)), N^*(Y \times X)). \tag{6.4}$$

Now consider K_{AP}. Note that microlocally away from the intersection of the two Lagrangians, microlocally near $N^*(\mathrm{diag} \cap (X \times Y))$,

$$I^{2s,-s_0+k/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*\mathrm{diag})$$

is just

$$I^{2s-s_0+k/2}(N^*(\mathrm{diag} \cap (X \times Y))),$$

and $N^*(\mathrm{diag} \cap (X \times Y)) = N^*(\mathrm{diag} \cap (Y \times Y))$ intersects $T^*\mathbb{R}^n \times o_{\mathbb{R}^n}$ at $N^*(Y \times X)$. Thus, we need to use Corollary 5.17 as well when discussing boundedness between Sobolev spaces. In view of Propositions 5.7 and 5.16 and Corollary 5.17, microlocally away from $N^*\mathrm{diag}$, AP is bounded from $H^{s-\epsilon_0}$ to $H^{-s+\epsilon_0}$ provided

$$\begin{aligned} & -s_0 + 2s + k/2 < 2s - 2\epsilon_0 - k/2, \\ & -s_0 + 1 - \dim Y/2 < s - \epsilon_0 - \frac{n}{2} \text{ and} \\ & -s_0 + 2s + k/2 < s - \epsilon_0, \end{aligned} \tag{6.5}$$

i.e.

$$\begin{aligned} & k + 2\epsilon_0 < s_0, \\ & s > -s_0 + \epsilon_0 + 1 + k/2 \text{ and} \\ & s < s_0 - \epsilon_0 - k/2. \end{aligned} \tag{6.6}$$

Notice that these inequalities imply (6.1). Note that if the first inequality holds then

$$-s_0 + \epsilon_0 + 1 + k/2 < -k/2 - \epsilon_0 + 1 < 1 - k/2,$$

so when $s \geq 1-k/2$, the second inequality in (6.6) is automatic when the first holds. Moreover, if the stronger inequality $1 + k + 2\epsilon_0 < s_0$ is assumed in place of the first in (6.6) (we need the stronger inequality below in the hyperbolic setting), then for $s \geq -k/2$ it assures that the second one holds. An analogous (in some sense, dual) computation applies to PA, using Proposition 5.14 in place of Proposition 5.16, and yielding the same constraints, (6.6). We state these results as a lemma:

Lemma 6.2. *For* $g \in I^{[-s_0]}(Y)$*,* $A \in \Psi^{2s-2}(X)$ *with compactly supported Schwartz kernel,* PA, AP *are, microlocally away from* $N^*\mathrm{diag}$*, bounded from* $H^{s-\epsilon_0}$ *to* $H^{-s+\epsilon_0}$ *provided* (6.6) *is satisfied.*

Microlocal elliptic regularity is now a straightforward consequence. Consider $q_0 \notin \Sigma$. We shall assume that $q_0 \notin \mathrm{WF}^{s-1/2}(u)$, thus there is a conic neighborhood O of q_0 on which u is microlocally in $H^{s-1/2}$; we may take O disjoint from Σ. With p the principal symbol of P, $p(q_0) \neq 0$, and we may assume that $\mathrm{sign}\, p$ is constant on O. We take $A \in \Psi^{2s-2}(X)$ with principal symbol a_0^2 elliptic at q_0, supported close to q_0, in the region where $\operatorname{sign} p$ is constant, with $\mathrm{WF}'(A) \subset O$ and $A = A^*$. Then the principal symbol of AP on $N^*\mathrm{diag}$ is

$$a_0^2 p \in S^{2s}(\mathbb{R}^n_\xi; I^{-s_0-n/4+k/2}(\mathbb{R}^n_x; N^*\{x'=0\})) = S^{2s}(\mathbb{R}^n_\xi; I^{[-s_0]}(\mathbb{R}^n_x; N^*\{x'=0\})).$$

By assumption,

$$p \in S^2(\mathbb{R}^n_\xi; I^{-s_0-n/4+k/2}(\mathbb{R}^n_x; N^*\{x'=0\})).$$

has a fixed (non-zero) sign, $\operatorname{sign} p(q_0)$, on $\operatorname{supp} a_0$, so by Lemma 5.12, for $\epsilon_1 > 0$ (which we take as small as convenient),

$$a_0^2 p = (\operatorname{sign} p(q_0)) b^2,\ b = a_0\sqrt{|p|} \in S^s(\mathbb{R}^n_\xi; I^{-s_0-n/4+k/2+\epsilon_1}(\mathbb{R}^n_x; N^*\{x'=0\})).$$

Let

$$B \in I^{s,-s_0+k/2+\epsilon_1}(N^*(\mathrm{diag}\cap(X\times Y)), N^*\mathrm{diag})$$

with principal symbol b; then by Proposition 5.8, taking into account that $2(-s_0 + k/2) < -k - 4\epsilon_0 < -1$ so there is a full order gain in the symbolic calculation,

$$AP = (\operatorname{sign} p(q_0)) B^* B + F$$

with

$$\begin{aligned} F \in & I^{2s-1,1-s_0+k/2+\epsilon_1}(N^*(\mathrm{diag}\cap(X\times Y)), N^*\mathrm{diag}) \\ & + I^{-s_0+1-\dim Y/2, 2s-1+n/2}(N^*(\mathrm{diag}\cap(X\times Y)), N^*(X\times Y)), \end{aligned} \tag{6.7}$$

so F has order corresponding to $\Psi^{2s-1}(X)$ on the conormal bundle of the diagonal, and elsewhere it has the same orders as AP had, apart from the $\epsilon_1 > 0$ loss from the symbolic construction of B. In view of Propositions 5.7 and 5.16 and Corollary 5.17, for $\epsilon_0' = \min(1/2, \epsilon_0)$ and $\epsilon_1 > 0$ sufficiently small (since we have strict inequalities in (6.6)), F is bounded from $H^{s-\epsilon_0'}$ to $H^{-s+\epsilon_0'}$ (here we possibly reduced ϵ_0 to ϵ_0' in order to deal with the diagonal singularity, which we thus far ignored), if (6.6) holds. Thus, subject to these limitations on ϵ_0', s_0 and s, $\langle Fu, u\rangle$ is bounded by the a priori assumptions. Since the constraint on ϵ_0' is purely due to the diagonal singularity, it is convenient to write

$$\begin{aligned} & F = F' + F'', \\ & F' \in I^{2s-1,1-s_0+k/2+\epsilon_1}(N^*(\mathrm{diag}\cap(X\times Y)), N^*\mathrm{diag}), \\ & F'' \in I^{-s_0+1-\dim Y/2, 2s-1+n/2+\epsilon_1}(N^*(\mathrm{diag}\cap(X\times Y)), N^*(X\times Y)), \end{aligned} \tag{6.8}$$

with the wave front set of F' in a prescribed arbitrary conic neighborhood of $N^*\mathrm{diag}$ – note that away from $N^*\mathrm{diag}$, elements of

$$I^{2s-1,1-s_0+k/2+\epsilon_1}(N^*(\mathrm{diag}\cap(X\times Y)), N^*\mathrm{diag})$$

are in $I^{-s_0+1-\dim Y/2, 2s-1+n/2+\epsilon_1}(N^*(\mathrm{diag}\cap(X\times Y)), N^*(X\times Y))$, so can always be regarded as part of F''. Concretely, as $\mathrm{WF}'(A) \subset O$, so the wave front set of K_{AP} intersects $N^*\mathrm{diag}$ only in $O\times O'$, we demand, as we may, that

$$\mathrm{WF}(K_B),\ \mathrm{WF}(K_{F'}) \subset O \times O',$$

where the prime on O denotes the usual twisting, i.e. the switch of the sign of the second covector. (Note that this means in particular that $\mathrm{WF}(K_{F'})$ does not contain covectors in $(T^*X\setminus o)\times o$ and $o\times(T^*X\setminus o)$.) With such a decomposition, for $\epsilon_1 > 0$ sufficiently small, F'' is bounded from $H^{s-\epsilon_0}$ to $H^{-s+\epsilon_0}$ so $\langle F''u, u\rangle$ is bounded, while u being in $H^{s-1/2}$ on O, $\langle F'u, u\rangle$ is bounded by the a priori assumptions. Further, with $Q \in \Psi^{s-2}(X)$ elliptic with positive principal symbol q, with parametrix $G \in \Psi^{2-s}(X)$ with positive principal symbol g, such that $GQ = \mathrm{Id} + R$, $R \in \Psi^{-\infty}(X)$, and for $\delta > 0$,

$$
\begin{aligned}
|\langle Au, Pu\rangle| &\leq |\langle G^* Au, QPu\rangle| + |\langle Au, RPu\rangle| \\
&\leq \delta \|G^* Au\|^2 + \delta^{-1}\|QPu\|^2 + |\langle Au, RPu\rangle|,
\end{aligned}
\tag{6.9}
$$

where the last two terms are bounded by the a priori assumptions. In order to absorb the $G^* A \in \Psi^s(X)$ term, to deal with the regularizer, as well as to facilitate the direct translation to a wave front set statement, it is convenient to replace $B^* B$ by

$$
B_1^* B_1 + B_2^* B_2 + c^2 (G^* A)^* (G^* A)
\tag{6.10}
$$

where $c > 0$ is a small constant,

$$
\begin{aligned}
&B_1 \in I^s(N^* \mathrm{diag}), \\
&B_2 \in I^{s, -s_0 + k/2 + \epsilon_1}(N^*(\mathrm{diag} \cap (Y \times Y)), N^* \mathrm{diag}).
\end{aligned}
$$

This is achieved as follows. Let ρ be a positive elliptic homogeneous degree 1 function on $T^* X \setminus o$. Since $\operatorname{supp} a_0$ is compact, disjoint from Σ, $|p| \geq c_0^2 \rho^2$ on it for some $c_0 > 0$. Further, the principal symbol $\tilde{g}$ of G^* satisfies $|\tilde{g}| \leq C' \rho^{2-s}$, and that of a_0 satisfies $|a_0| \leq C'' \rho^{s-1}$, so the principal symbol $a_0^4 g^2$ of $(G^* A)^* (G^* A)$ is then bounded by $C^2 \rho^2 a_0^2$. Then let $c = \frac{c_0}{2C}$, so the symbol of $c^2 (G^* A)^* (G^* A)$ is bounded by $\frac{c_0^2}{4} \rho^2 a_0^2$. Now let

$$
b_1 = \frac{c_0}{2} a_0 \rho, \ b_2 = \left(|p| \rho^{-2} - \frac{c_0^2}{4} - c^2 a_0^4 g^2 \rho^{-2} \right)^{1/2} a_0 \rho.
$$

Then on $\operatorname{supp} a_0$, the factor inside the parentheses is a homogeneous degree zero C^∞ function bounded below by a positive constant, thus the square root is C^∞. Taking B_j with principal symbols b_j, (6.10) has principal symbol $|p| a_0^2$, hence

$$
AP = (\operatorname{sign} p(q_0)) \left(B_1^* B_1 + B_2^* B_2 + c^2 (G^* A)^* (G^* A) \right) + F,
$$

with F satisfying (6.8) (but possibly different from the F given by $B^* B$). Then

$$
\langle Pu, Au\rangle = \langle APu, u\rangle = (\operatorname{sign} p(q_0)) \Big(\|B_1 u\|^2 + \|B_2 u\|^2 + c\|G^* Au\|^2 \Big) + \langle Fu, u\rangle,
$$

which we justify via a standard regularization argument, recalled below, so using (6.9) to estimate the left hand side from above, and taking $\delta > 0$ sufficiently small, $\delta \|G^* Au\|^2$ can be absorbed in the right hand side. This gives the conclusion that $B_j u \in L^2$ for $j = 1, 2$, which allows us to conclude that $\mathrm{WF}^s(u)$ is disjoint from the elliptic set of B_1.

Finally, the regularization argument is to replace A by $A_r = \Lambda_r A \Lambda_r$, $r \in [0, 1]$, where $\Lambda_r \in \Psi^{-1}$ for $r > 0$, Λ_r is uniformly bounded in Ψ^0, and $\Lambda_r \to \mathrm{Id}$ in Ψ^ϵ for $\epsilon > 0$, and thus strongly in L^2; one may take Λ_r formally self-adjoint for convenience. (One can for instance take Λ_r to be a quantization of $(1 + r\rho)^{-1}$; and then replace it by its self-adjoint part which does not affect the principal symbol or the boundedness and convergence properties, as in Section 2.) Then $\Lambda_r A \Lambda_r P$ has the same principal symbol, uniformly in Ψ^{2s}, as

$$
\Lambda_r (\operatorname{sign} p(q_0)) \Big(B_1^* B_1 + B_2^* B_2 + c (G^* A)^* (G^* A) \Big) \Lambda_r,
$$

and correspondingly

$$\begin{aligned}\langle Pu, A_r u\rangle &= \langle A_r P u, u\rangle \\ &= (\operatorname{sign} p(q_0))\Big(\|B_1\Lambda_r u\|^2 + \|B_2\Lambda_r u\|^2 + c\|G^* A\Lambda_r u\|^2\Big) + \langle F_r u, u\rangle,\end{aligned}$$

where F_r is uniformly bounded in Ψ^{2s-1}, and is in Ψ^{2s-3} for $r > 0$. Here the calculations such as the first equality and $\|\Lambda_r B_1 u\|^2 = \langle \Lambda_r B_1^* B_1 \Lambda_r u, u\rangle$ follow since for $r > 0$ on O, which contains the (conic or essential) support of a_0, u is in $H^{s-1/2}$ by the a priori assumptions, and the sum of the diagonal orders of the operators involved is $\leq 2s - 1$. Now letting $r \to 0$ gives uniform bounds for $\|B_j\Lambda_r u\|_{L^2}$, and thus proves $B_j u \in L^2$ in view of the weak compactness of the unit ball in L^2 and since $B_j\Lambda_r u \to B_j u$ in distributions. As $q_0 \in \Sigma$ was arbitrary, we conclude that

Lemma 6.3. *Suppose that* (6.6) *holds. If* $u \in H^{s-\epsilon_0}_{\mathrm{loc}}$, $Pu \in H^{s-2}_{\mathrm{loc}}$, *then* $\mathrm{WF}^s(u) \subset \Sigma \cup \mathrm{WF}^{s-1/2}(u)$.

Now one can iterate this, gradually increasing s by $\leq 1/2$; here we also return to $\Box$ instead of P:

Proposition 6.4. *Suppose that* $k + 1 + 2\epsilon_0 < s_0$ *and* $-k/2 < s < s_0 - \epsilon_0 - k/2$. *If* $u \in H^{s-\epsilon_0}_{\mathrm{loc}}$, $\Box u \in H^{s-2}_{\mathrm{loc}}$, *then* $\mathrm{WF}^s(u) \subset \Sigma$.

Proof. First apply Lemma 6.3 with $s' = \min(s - \epsilon_0 + 1/2, s) \leq s$ in place of s (and ϵ_0 unchanged); then

$$s' \geq s - \epsilon_0 + 1/2 > -k/2 - \epsilon_0 + 1/2 - (s_0 - k - 1 - 2\epsilon_0) > -s_0 + k/2 + \epsilon_0 + 1,$$

so the second inequality in (6.6) holds, and all others hold because $s' \leq s$. Since $u \in H^{s-\epsilon_0}_{\mathrm{loc}}$ implies that $\mathrm{WF}^{s'-1/2}(u) = \emptyset$, the conclusion of the lemma gives $\mathrm{WF}^{s'}(u) \subset \Sigma$. Now repeat this argument with $s'' = \min(s' + 1/2, s) \in [s', s]$, so (6.6) holds for s'' in place of s, to conclude $\mathrm{WF}^{s''}(u) \subset \Sigma$. An inductive argument gives $\mathrm{WF}^s(u) \subset \Sigma$ in a finite number of steps, as desired. □

7. Propagation estimate

We now return to the positive commutator propagation estimates, but unlike the smooth coefficients in Section 2, we consider $g \in I^{[-s_0]}(Y)$, $\operatorname{codim} Y = k$. We again work with the reduced operator $P = \sum_{ij} D_i g_{ij} D_j$ given by (6.2), replacing $\Box u = f \in H^{s-1}$ by $Pu = (\det g)^{1/2} f \in H^{s-1}$ provided (in view of Proposition 5.18)

$$\begin{aligned} &s_0 > k, \\ &-s_0 + k/2 < s - 1 < s_0 - k/2. \end{aligned} \tag{7.1}$$

So suppose that $A \in \Psi^{2s-1}(X)$; we need to compute the Schwartz kernel of $[P, A]$ as a (sum of) paired Lagrangian distribution(s). By the remarks at the beginning of Section 6, and writing

$$[P, A] = \sum [D_i, A] g_{ij} D_j + \sum D_i [g_{ij}, A] D_j + \sum D_i g_{ij} [D_j, A],$$

the Schwartz kernel of $[P, A]$ is

$$\begin{aligned} K_{[P,A]} = &- \sum D_{j,R} g_{ij,R} (D_{i,L} + D_{i,R}) K_A + \sum D_{i,L} g_{ij,L} (D_{j,L} + D_{j,R}) K_A \\ &- \sum D_{i,L} D_{j,R} (g_{ij,L} - g_{ij,R}) K_A. \end{aligned}$$

As before, $g_{ij,L}$, resp. $g_{ij,R}$, is the pullback of g_{ij} from the left, resp. right, factor. Now, $K_A \in I^{2s-1}(N^*\mathrm{diag})$, and $D_{i,L} + D_{i,R}$ is tangent to the diagonal, so $(D_{i,L} + D_{i,R})K_A \in I^{2s-1}(N^*\mathrm{diag})$ still. Now as $g_{ij} \in I^{[-s_0]}(Y)$, by (5.43) (with the left and right factors interchanged),

$$\begin{aligned}&g_{ij,R}(D_{i,L} + D_{i,R})K_A\\ &\in I^{2s-1,-s_0+k/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*\mathrm{diag})\\ &\qquad + I^{-s_0-\dim Y/2,2s-1+n/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*(X \times Y)),\end{aligned}$$

and applying $D_{j,R}$ increases the orders on all Lagrangians by 1, see in particular Lemma 5.13. Thus,

$$\begin{aligned}&\sum D_{j,R}g_{ij,R}(D_{i,L} + D_{i,R})K_A\\ &\in I^{2s,-s_0+k/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*\mathrm{diag})\\ &\qquad + I^{-s_0+1-\dim Y/2,2s-1+n/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*(X \times Y)).\end{aligned} \tag{7.2}$$

The right hand side is exactly the same space as what we obtained in (6.3) and (6.4). As in Lemma 6.2, we deduce that microlocally away from $N^*\mathrm{diag}$, (7.2) is bounded from $H^{s-\epsilon_0}$ to $H^{-s+\epsilon_0}$ provided (6.6) holds.

An analogous computation applies to $\sum D_{i,L}g_{ij,L}(D_{j,L} + D_{j,R})K_A$, yielding the same constraints, (6.6).

A similar computation applies to $D_{i,L}D_{j,R}(g_{ij,L} - g_{ij,R})K_A$, i.e. when g_{ij} is commuted through A. However, while the order on $N^*\mathrm{diag}$ is the same as in the above cases, the order on the *other* Lagrangians is just that of $D_{i,L}D_{j,R}g_{ij,L}K_A$ and $D_{i,L}D_{j,R}g_{ij,R}K_A$, i.e. the commutator does not proved additional help as compared to the product. This means a loss of 1 order on $N^*(\mathrm{diag} \cap (Y \times Y))$ as compared to (7.2), but no extra loss on $N^*(Y \times X)$ since $D_{i,R}$ is characteristic there. Concretely, as above,

$$\begin{aligned}&g_{ij,R}K_A\\ &\in I^{2s-1,-s_0+k/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*\mathrm{diag})\\ &\qquad + I^{-s_0-\dim Y/2,2s-1+n/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*(X \times Y)),\end{aligned}$$

and so, using Lemma 5.13 for the second summand on the right hand side,

$$\begin{aligned}&D_{i,L}D_{j,R}g_{ij,R}K_A\\ &\in I^{2s+1,-s_0+k/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*\mathrm{diag})\\ &\qquad + I^{-s_0+1-\dim Y/2,2s+n/2}(N^*(\mathrm{diag} \cap (X \times Y)), N^*(X \times Y)),\end{aligned}$$

Similarly,

$$\begin{aligned}&D_{i,L}D_{j,R}g_{ij,L}K_A\\ &\in I^{2s+1,-s_0+k/2}(N^*(\mathrm{diag} \cap (Y \times X)), N^*\mathrm{diag})\\ &\qquad + I^{-s_0+1-\dim Y/2,2s+n/2}(N^*(\mathrm{diag} \cap (Y \times X)), N^*(Y \times X)),\end{aligned}$$

and so, in principle,

$$\begin{aligned}&D_{i,L}D_{j,R}(g_{ij,L}-g_{ij,R})K_A\\&\in I^{2s+1,-s_0+k/2}(N^*(\mathrm{diag}\cap(Y\times Y)),N^*\mathrm{diag})\\&\qquad+I^{-s_0+1-\dim Y/2,2s+n/2}(N^*(\mathrm{diag}\cap(X\times Y)),N^*(X\times Y))\\&\qquad+I^{-s_0+1-\dim Y/2,2s+n/2}(N^*(\mathrm{diag}\cap(Y\times X)),N^*(Y\times X)).\end{aligned}$$

However, by the standard pseudodifferential calculus, the principal symbol on $N^*\mathrm{diag}$ in

$$S^{2s+1}(I^{-s_0+k/2-n/4})/S^{2s}(I^{-s_0+k/2+1-n/4}),$$

where we used short hand notation so that e.g.

$$S^{2s+1}(I^{-s_0+k/2-n/4})=S^{2s+1}(\mathbb{R}^n_\xi;I^{-s_0+k/2-n/4}(\mathbb{R}^n_x;N^*\{x'=0\})),$$

vanishes since it is given by (the equivalence class of) $\xi_i\xi_j g_{ij}(x)a(x,\xi)$ for both $D_{i,L}D_{j,R}g_{ij,L}K_A$ and $D_{i,L}D_{j,R}g_{ij,R}K_A$, so by Lemma 5.3,

$$\begin{aligned}&D_{i,L}D_{j,R}(g_{ij,L}-g_{ij,R})K_A\\&\in I^{2s,-s_0+1+k/2}(N^*(\mathrm{diag}\cap(Y\times Y)),N^*\mathrm{diag})\\&\qquad+I^{-s_0+1-\dim Y/2,2s+n/2}(N^*(\mathrm{diag}\cap(X\times Y)),N^*(X\times Y))\\&\qquad+I^{-s_0+1-\dim Y/2,2s+n/2}(N^*(\mathrm{diag}\cap(Y\times X)),N^*(Y\times X)).\end{aligned}\tag{7.3}$$

Thus, the only change compared to the previous calculations for boundedness $H^{s-\epsilon_0}\to H^{-s+\epsilon_0}$ away from $N^*\mathrm{diag}$ is that (6.5) is replaced by

$$\begin{aligned}&-s_0+2s+1+k/2<2s-2\epsilon_0-k/2\\&-s_0+1-\dim Y/2<s-\epsilon_0-\frac{n}{2}\ \text{and}\\&-s_0+2s+1+k/2<s-\epsilon_0,\end{aligned}\tag{7.4}$$

thus (6.6) is replaced by

$$\begin{aligned}&k+1+2\epsilon_0<s_0\\&s>-s_0+\epsilon_0+1+k/2\ \text{and}\\&s<s_0-\epsilon_0-1-k/2.\end{aligned}\tag{7.5}$$

Note that these inequalities imply (7.1). The first of these inequalities implies

$$-s_0+\epsilon_0+1+k/2<-k/2-\epsilon_0,$$

so again, when $s>-k/2$, the second inequality automatically holds if the first holds.

Now, the actual argument proceeds as follows. We want to take $A\in\Psi^{2s-1}$ satisfying (2.6); this requires $\tilde\eta$ and σ_j to be fixed; as in Section 2 we also use a positive elliptic order 1 symbol ρ. As in Section 2, we then actually arrange that A is of the form $\check A^2$, with $\check A\in\Psi^{s-1/2}$ formally self-adjoint; and in fact we take

$$A_r=\Lambda_r A\Lambda_r,\ \check A_r=\check A\Lambda_r,$$

with Λ_r as before with symbol ϕ_r. The functions $\tilde\eta$ and σ_j depend on p only via (2.3) and (2.4), both of which are purely conditions at $\bar q$. Now, if g_{ij} are conormal to Y and $\bar q\in T^*_Y X$, then for $s_0>1+k$, p is still C^1, and thus H_p is continuous, and is indeed C^{α_0}, $\alpha_0<s_0-1-k$. Correspondingly, $\mathsf{H}_p(\bar q)$ is well-defined, and

one can use it in the definition of the C^∞ functions $\tilde\eta$ and σ_j on S^*X. However, as $\mathsf{H}_p\sigma$ is now C^{α_0}, instead of (2.12) one has for $\alpha=\min(1,\alpha_0)>0$,

$$|\mathsf{H}_p\sigma_j|\leq C_0(\omega^{1/2}+|\tilde\eta|)^\alpha, \tag{7.6}$$

so $|\mathsf{H}_p\omega|\leq C\omega^{1/2}(\omega^{1/2}+|\tilde\eta|)^\alpha$. Using (2.7), we now deduce that $|\mathsf{H}_p\omega|\leq\frac{c_0}{2}\epsilon^2\delta$ provided that $\frac{c_0}{2}\epsilon^2\delta\geq C''(\epsilon\delta)\delta^\alpha$, i.e. that $\epsilon\geq C'\delta^\alpha$ for some constant C' independent of ϵ, δ. Taking $\epsilon\sim\delta^\alpha$, the size of the parabola at $\tilde\eta=-\delta$ is roughly $\omega^{1/2}\sim\delta^{1+\alpha}$, which still suffices for the proof of propagation of singularities in view of $\alpha>0$, as we have localized along a single direction, namely the direction of H_p at $\bar q$.

We also assume at first (an assumption that will be eliminated by an iterative procedure) that

$$\mathrm{WF}'(A)\subset O,\ O=\mathrm{WF}^{s-1/2}(u)^c; \tag{7.7}$$

note that given O the δ-localization of a makes this achievable.

By construction, the principal symbol of the commutator along the conormal bundle of the diagonal, which is in $S^{2s}(\mathbb{R}^n_\xi;I^{-s_0+k/2+1-n/4}(\mathbb{R}^n_x;N^*\{x'=0\}))$, is still of the form (2.2), though now the symbols have a conormal singularity at Y as well. More precisely, with

$$e_0=\rho^{2s}e\in S^{2s}(\mathbb{R}^n_\xi;I^{-s_0+k/2+1-n/4}(\mathbb{R}^n_x;N^*\{x'=0\})),$$

as in (2.8) times the weight ρ^{2s}, cf. (2.6), and

$$b_0=\rho^s b\in S^s(\mathbb{R}^n_\xi;I^{-s_0+k/2+1-n/4+\epsilon_1}(\mathbb{R}^n_x;N^*\{x'=0\})),$$

as in (2.10) times the weight ρ^s (getting b_0 to lie in the indicated space uses Lemma 5.12, applied to $\mathsf{H}_p\phi$, which is bounded away from 0; this gives the loss of $\epsilon_1>0$ which one can take as small as convenient, as we did in the elliptic setting), one takes B,E paired Lagrangian associated to $N^*\mathrm{diag}$ and $N^*(\mathrm{diag}\cap(Y\times Y))$ with principal symbols given by b_0,e_0 on $N^*\mathrm{diag}$, more precisely

$$B\in I^{s,-s_0+1+k/2+\epsilon_1}(N^*(\mathrm{diag}\cap(Y\times Y)),N^*\mathrm{diag}),$$

and

$$E\in I^{2s,-s_0+1+k/2}(N^*(\mathrm{diag}\cap(Y\times Y)),N^*\mathrm{diag})$$

so they are in Ψ^s and Ψ^{2s} on $N^*\mathrm{diag}\setminus N^*(\mathrm{diag}\cap(Y\times Y))$, and the orders on $N^*(\mathrm{diag}\cap(Y\times Y))$ given by $I^{[-s_0+s+1]}=I^{-s_0+s+1+k/2+\epsilon_1}$ for B and $I^{[-s_0+s+1]}=I^{-s_0+2s+1+k/2}$ for E; one can also arrange (by applying a pseudodifferential operator microlocally the identity near $N^*\mathrm{diag}$ but with wave front set in $O\times O'$) that the Schwartz kernels of B,E satisfy

$$\mathrm{WF}'(K_B),\mathrm{WF}'(K_E)\subset O\times O',$$

where O' is the usual twisted version of O (sign of the covector switched). Then by Proposition 5.8, taking into account that $2(-s_0+1+k/2)<-k-4\epsilon_0<-1$ so there is a full order gain in the symbolic calculation (if we take $\epsilon_1>0$ sufficiently small),

$$i[P,A]=B^*B+E+F,$$

where away from $N^*(Y\times X)\cup N^*(X\times Y)$, at which F has the same orders as the commutator, as given in (7.2) and (7.3) (i.e. is dictated by the second of these, as these are greater),

$$F\in I^{2s-1,-s_0+2+k/2+\epsilon_1}(N^*(\mathrm{diag}\cap(Y\times Y)),N^*\mathrm{diag}).$$

As in the elliptic setting, we break up F:

$$
\begin{aligned}
&F = F' + F'',\\
&F' \in I^{2s-1,-s_0+2+k/2+\epsilon_1}(N^*(\mathrm{diag}\cap(X\times Y)),N^*\mathrm{diag}),\\
&F'' \in I^{-s_0+1-\dim Y/2,2s+n/2+\epsilon_1}(N^*(\mathrm{diag}\cap(X\times Y)),N^*(X\times Y))\\
&\qquad + I^{-s_0+1-\dim Y/2,2s+n/2+\epsilon_1}(N^*(\mathrm{diag}\cap(Y\times X)),N^*(Y\times X)),
\end{aligned}
\tag{7.8}
$$

with the wave front set of the Schwartz kernel of F' in

$$\mathrm{WF}(K_{F'})\subset O\times O';$$

note that away from $N^*\mathrm{diag}$, elements of

$$I^{2s-1,-s_0+2+k/2+\epsilon_1}(N^*(\mathrm{diag}\cap(X\times Y)),N^*\mathrm{diag})$$

are in $I^{-s_0+1-\dim Y/2,2s+n/2+\epsilon_1}(N^*(\mathrm{diag}\cap(X\times Y)),N^*(X\times Y))$, so can always be regarded as part of F''. With such a decomposition, for $\epsilon_1>0$ sufficiently small, in view of Propositions 5.14 and 5.16 and Corollary 5.17, F'' is bounded from $H^{s-\epsilon_0}$ to $H^{-s+\epsilon_0}$ so $\langle F''u,u\rangle$ is bounded. On the other hand F' is bounded $H^{s-1/2}$ to $H^{1/2-s}$ by Proposition 5.7, and has wave front set in $O\times O'$, so u being in $H^{s-1/2}$ on O, $\langle F'u,u\rangle$ is bounded by the a priori assumptions as well. Thus, subject to (7.5), $\langle Eu,u\rangle$ and $\langle Fu,u\rangle$ are bounded by the a priori assumptions.

Further, as in Section 2, if $Pu\in H^{s-1}_{\mathrm{loc}}(X)$, then with $Q\in\Psi^{1/2}(X)$ elliptic with positive principal symbol $\rho^{1/2}$, with parametrix $G\in\Psi^{-1/2}(X)$, such that $GQ=\mathrm{Id}+R$, $R\in\Psi^{-\infty}(X)$, we use (2.16) to control $|\langle A_r u,Pu\rangle|$. In order to absorb the $Q\check{A}_r\in\Psi^s(X)$ term in (2.16), and to deal with the regularizer and the weight as in Section 2, as well as to facilitate the direct translation to a wave front set statement, we replace B^*B by $B_{1,r}^*B_{1,r}+B_{2,r}^*B_{2,r}+M^2(Q\check{A}_r)^*(Q\check{A}_r)$ where $M>0$ is a large constant,

$$
\begin{aligned}
&B_{1,r}=B_1\Lambda_r,\ B_1\in I^s(N^*\mathrm{diag})=\Psi^s,\\
&B_{2,r}\in I^{s,-s_0+1+k/2+\epsilon_1}(N^*(\mathrm{diag}\cap(Y\times Y)),N^*\mathrm{diag}),
\end{aligned}
$$

$B_{2,r}$ uniformly bounded in $I^{s,-s_0+1+k/2+\epsilon_1}(N^*(\mathrm{diag}\cap(Y\times Y)),N^*\mathrm{diag})$ and with the Schwartz kernel of $B_{j,r}$ having (uniform) wave front set in $O\times O'$. To achieve this, we proceed as in (2.13)-(2.14), and we recall that we arranged that $\mathsf{H}_p\phi\geq c_0/2$, and thus writing

$$\mathsf{H}_p\phi=\psi_1+\psi_2,\ \psi_1\equiv c_0/4,\ \psi_2\geq c_0/4,$$

we let

$$b_{1,r}=\rho^s\phi_r\digamma^{-1/2}\delta^{-1/2}\sqrt{\psi_1}\sqrt{\chi_0'\left(\digamma^{-1}\Big(2\beta-\frac{\phi}{\delta}\Big)\right)}\sqrt{\chi_1\left(\frac{\tilde\eta+\delta}{\epsilon\delta}+1\right)},\tag{7.9}$$

and

$$
\begin{aligned}
b_{2,r}&=\rho^s\phi_r\digamma^{-1/2}\delta^{-1/2}c_{2,r}\sqrt{\chi_0'\left(\digamma^{-1}\Big(2\beta-\frac{\phi}{\delta}\Big)\right)}\sqrt{\chi_1\left(\frac{\tilde\eta+\delta}{\epsilon\delta}+1\right)},\\
c_{2,r}&=\left(\psi_2-\Big(\big((2s-1)-r\rho\phi_r\big)(\rho^{-1}\mathsf{H}_p\rho)+M^2\Big)\digamma^{-1}\delta\Big(2\beta-\frac{\phi}{\delta}\Big)^2\right)^{1/2}
\end{aligned}
\tag{7.10}
$$

and let $B_{j,r}$ have principal symbol $b_{j,r}$, noting that $b_{1,r}$ is C^∞ (i.e. does not have a conormal singularity). As in Section 2, the expression in the large parentheses defining $c_{2,r}$ is bounded below by a positive constant (uniformly in r) for $F > 0$ sufficiently large as $|2\beta - \frac{\phi}{\delta}| \leq 4$ on $\operatorname{supp} a$. Then the analogue of (2.15) is

$$\|B_{1,r}u\|^2 + \|B_{2,r}u\|^2 + M^2\|Q\breve{A}_r u\|^2 \leq 2|\langle A_r u, Pu\rangle| + |\langle E_r u, u\rangle| + |\langle F_r u, u\rangle|.$$

Using (2.16) to estimate the first term on the right hand side from above, $\|Q\breve{A}_r u\|^2$ can be absorbed in the $M^2\|Q\breve{A}_r u\|^2$ term on the left hand side (for $M > 1$). This gives the conclusion that $B_{j,0}u \in L^2$ for $j = 1, 2$, which allows us to conclude that $\mathrm{WF}^s(u)$ is disjoint from the elliptic set of $B_{1,0}$.

What we have proved is the following:

Lemma 7.1. *Suppose that* (7.5) *holds. Let* $\alpha = \min(1, \alpha_0) \in (0, 1]$, $\alpha_0 < s_0 - 1 - k$, *and let* $U \subset X$ *be coordinate chart (identified with a subset of* $\mathbb{R}^n$*). For any* $K \subset \Sigma \cap T^*_U X$ *compact there exists* $\delta_0 > 0$ *and* $C_0 > 0$ *such that the following holds. If* $u \in H^{s-\epsilon_0}_{\mathrm{loc}}$, $Pu \in H^{s-1}_{\mathrm{loc}}$, $\delta \in (0, \delta_0)$ *and* $q_0 \in K$ *and if the Euclidean metric ball around* $q_0 + \delta\mathsf{H}_p(q_0)$ *of radius* $C_0\delta^{1+\alpha}$ *is disjoint from* $\mathrm{WF}^s(u)$, *and the Euclidean metric tube (union of metric balls) around the straight line segment connecting* q_0 *and* $q_0 + \delta\mathsf{H}_p(q_0)$ *of radius* $C_0\delta^{1+\alpha}$ *is disjoint from* $\mathrm{WF}^{s-1/2}(u)$ *then* $q_0 \notin \mathrm{WF}^s(u)$.

The analogous conclusion also holds with $q_0 + \delta\mathsf{H}_p(q_0)$ *replaced by* $q_0 - \delta\mathsf{H}_p(q_0)$.

As in the elliptic case, one can eliminate the background regularity assumption on the metric tube; here one needs to proceed more directly, shrink the supports of the cutoffs defining a slightly in each step of the iteration, as is standard, see [8, Section 24.5], last paragraph of the proof of Proposition 24.5.1, and the end of Section 2. The key point in starting the iteration is that with $s' = \min(s - \epsilon_0 + 1/2, s) \leq s$, if $k + 1 + 2\epsilon_0 < s_0$ and $-k/2 < s$ then

$$s' \geq s - \epsilon_0 + 1/2 > -k/2 - \epsilon_0 + 1/2 - (s_0 - k - 1 - 2\epsilon_0) > -s_0 + k/2 + \epsilon_0 + 1,$$

so the second inequality in (7.5) holds; all others follow at once from those of s using $s' \leq s$.

Proposition 7.2. *Suppose that* $k + 1 + 2\epsilon_0 < s_0$ *and* $-k/2 < s < s_0 - \epsilon_0 - 1 - k/2$. *Let* $\alpha = \min(1, \alpha_0) \in (0, 1]$, $\alpha_0 < s_0 - 1 - k$, *and let* $U \subset X$ *be coordinate chart (identified with a subset of* $\mathbb{R}^n$*). For any* $K \subset \Sigma \cap T^*_U X$ *compact there exists* $\delta_0 > 0$ *and* $C_0 > 0$ *such that the following holds. If* $u \in H^{s-\epsilon_0}_{\mathrm{loc}}$, $\Box u \in H^{s-1}_{\mathrm{loc}}$, $\delta \in (0, \delta_0)$ *and* $q_0 \in K$ *and if the metric ball around* $q_0 + \delta\mathsf{H}_p(q_0)$ *of radius* $C_0\delta^{1+\alpha}$ *is disjoint from* $\mathrm{WF}^s(u)$ *then* $q_0 \notin \mathrm{WF}^s(u)$.

The analogous conclusion also holds with $q_0 + \delta\mathsf{H}_p(q_0)$ *replaced by* $q_0 - \delta\mathsf{H}_p(q_0)$.

8. Propagation of singularities

In order to convert Proposition 7.2 into a propagation of singularities along bicharacteristics statement, we need a more precise analysis of the bicharacteristics. One has the following lemma, which is just a version of the argument of Melrose and Sjöstrand [11, 12], see also [8, Chapter XXIV] and [10].

Lemma 8.1. *(Version of* [8, Lemma 24.3.15]*.) Suppose that* $\alpha \in (0, 1]$ *and* H_p *is in* C^α. *Suppose that* F *is a closed subset of* Σ *with the property that for every* $U \subset X$ *coordinate chart and for every* $K \subset \Sigma \cap T^*_U X$ *compact there exists* $\delta_0 > 0$ *and* $C_0 > 0$ *such that for all* $t \in (-\delta_0, \delta_0) \setminus \{0\}$ *and* $q_0 \in K \cap F$ *there exists* $q = q(t, q_0) \in F$ *in the metric ball* $B(q_0 + t\mathsf{H}_p(q_0), C_0|t|^{1+\alpha})$ *around* $q_0 + t\mathsf{H}_p(q_0)$ *of radius* $C_0|t|^{1+\alpha}$. *Then for every* $q_0 \in F$ *there is a bicharacteristic* $\gamma : (t_-, t_+) \to F$ *with* $\gamma(0) = q_0$ *and such that* γ *leaves every compact subset of* F *when* $t \to t_\pm$.

Proof. One can follow the proof of [8, Lemma 24.3.15] quite closely, ignoring case (i). Here we present a slightly different version of the argument, following [10], see also [19, Proof of Theorem 8.1].

A standard argument based on Zorn's lemma shows that it suffices to prove the local assertion that for every $q_0 \in F$ there exists a bicharacteristic $\gamma : [-\epsilon, \epsilon] \to \Sigma$, $\epsilon > 0$, with $\gamma(0) = q_0$ and such that $\gamma(t) \in F$ for $t \in [-\epsilon, \epsilon]$. Indeed, it suffices to do a one-sided version, i.e. that if $q_0 \in F$ then

$$\begin{gathered}\text{there exists a bicharacteristic } \gamma : [-\epsilon, 0] \to \Sigma,\ \epsilon > 0,\\ \gamma(0) = q_0,\ \gamma(t) \in F,\ t \in [-\epsilon, 0],\end{gathered} \tag{8.1}$$

for the existence of a bicharacteristic on $[0, \epsilon]$ can be demonstrated similarly by replacing the forward propagation estimates by backward ones, and piecing together the two bicharacteristics $\gamma_\pm$ gives one defined on $[-\epsilon, \epsilon]$ since at 0 they both satisfy $\frac{d}{dt}\gamma_\pm(0) = \mathsf{H}_p(q_0)$, so the curve defined on $[-\epsilon, \epsilon]$ is C^1 with the correct derivative everywhere.

Let $\mathcal{U}$ be a neighborhood of q_0 with $\overline{\mathcal{U}} \subset T^*_U X$ so H_p is Hölder-α in $\overline{\mathcal{U}}$, and is in particular bounded; $\sup \|\mathsf{H}_p\| \leq C'$. Let $\mathcal{U}_0$ be a smaller neighborhood with closure in $\mathcal{U}$ and (with δ_0 as in Proposition 7.2) $\epsilon \in (0, \delta_0)$ such that for any $q \in \mathcal{U}_0$, $\|q' - q\| \leq (C' + C_0\epsilon^\alpha)\epsilon$ implies $q' \in \mathcal{U}$. Suppose that $0 < \delta < \epsilon$, $q \in \mathcal{U}_0$. For $q \in T^*X$, let

$$D(q, \delta) = B(q - \delta\mathsf{H}_p(q), C_0\delta^{1+\alpha}) \cap F. \tag{8.2}$$

For each integer $N \geq 1$ now we define a sequence of $2^N + 1$ points $q_{j,N}$, $0 \leq j \leq 2^N$ integer, which will be used to construct points $\gamma(-j2^{-N}\epsilon)$ on the desired bicharacteristic $\gamma : [-\epsilon, 0] \to F$ through q_0. Namely, let $\delta = 2^{-N}\epsilon$, $q_{0,N} = q_0$, and choose $q_{j+1,N} \in D(q_{j,N}, \delta)$; such $q_{j+1,N}$ exists by assumption. Here one needs to check that $q_{j,N} \in \mathcal{U}$ inductively for $0 \leq j \leq 2^N$, but this follows as

$$\begin{aligned}\|q_{j,N} - q_0\| &\leq \sum_{i=0}^{j-1} \|q_{i+1,N} - q_i\|\\ &\leq j(C'2^{-N}\epsilon + C_0(2^{-N}\epsilon)^{1+\alpha}) \leq C'\epsilon + C_0 2^{-\alpha N}\epsilon^{1+\alpha}.\end{aligned} \tag{8.3}$$

Let $\gamma_N : [-\epsilon, 0]$ be the curve defined by $\gamma_N(t) = q_{j,N}$ for $t = -j2^{-N}\epsilon$, with γ given by the straight line between successive dyadic points. Thus, by an estimate similar to (8.3), γ_N is a uniformly Lipschitz family with

$$\|\gamma_N(t) - \gamma_N(t')\| \leq (C' + C_0\epsilon^\alpha)|t - t'|,$$

and thus there is a subsequence γ_{N_k} converging uniformly to some γ; as F is closed, γ takes values in F. It remains to check the differentiability of γ, and that $\frac{d}{dt}\gamma(t) = \mathsf{H}_p(\gamma(t))$. For this it suffices to show that there is $\tilde{C}_0 > 0$ such that for all relevant t and δ,

$$\gamma(t + \delta) \in B(\gamma(t) + \delta\mathsf{H}_p(\gamma(t)), \tilde{C}_0|\delta|^{1+\alpha}),$$

which follows if we show the analogous statement for γ_N (with constant $\tilde{C}_0$ independent of N) when t and $t + \delta$ are both dyadic points (so $\delta = -k\epsilon 2^{-N}$ is such as well). This is straightforward to check from the definition of γ_N since, with C_α the

Hölder-α constant of H_p on $\overline{\mathcal{U}}$, so $\|\mathsf{H}_p(q) - \mathsf{H}_p(q')\| \leq C_\alpha \|q - q'\|^\alpha$,

$$\begin{aligned}
&\|\gamma_N(t - k\epsilon 2^{-N}) - \gamma_N(t) + k\epsilon 2^{-N}\mathsf{H}_p(\gamma_N(t))\| \\
&\leq \sum_{j=0}^{k-1} \|\gamma_N(t-(j+1)\epsilon 2^{-N}) - \gamma_N(t - j\epsilon 2^{-N}) + \epsilon 2^{-N}\mathsf{H}_p(\gamma_N(t - j\epsilon 2^{-N}))\| \\
&\quad + \sum_{j=0}^{k-1} \epsilon 2^{-N}\|\mathsf{H}_p(\gamma_N(t - j\epsilon 2^{-N})) - \mathsf{H}_p(\gamma_N(t))\| \\
&\leq \sum_{j=0}^{k-1} C_0(\epsilon 2^{-N})^{1+\alpha} + \sum_{j=0}^{k-1} C_\alpha \epsilon 2^{-N}(j\epsilon 2^{-N})^\alpha \\
&\leq (kC_0 + \frac{C_\alpha}{1+\alpha}k^{1+\alpha})(\epsilon 2^{-N})^{1+\alpha} \leq (C_0 + \frac{C_\alpha}{1+\alpha})(k\epsilon 2^{-N})^{1+\alpha},
\end{aligned}$$

which gives the desired estimate with $\tilde{C}_0 = C_0 + \frac{C_\alpha}{1+\alpha}$. □

Applying the lemma with $F = \mathrm{WF}^s(u)$, Proposition 7.2 implies Theorem 1.4, which we restate as a corollary:

Corollary 8.2. *Suppose that* $k + 1 + 2\epsilon_0 < s_0$ *and* $-k/2 < s < s_0 - \epsilon_0 - 1 - k/2$*. Then for* $u \in H^{s-\epsilon_0}_{\mathrm{loc}}$*,* $\Box u \in H^{s-1}_{\mathrm{loc}}$*,* $\mathrm{WF}^s(u)$ *is a union of maximally extended bicharacteristics in* Σ*.*

A corollary of Theorem 1.4 is the following global regularity result:

Corollary 8.3. *If* $s_0 > 1+k/2$*,* $-k/2 < s' < s < s_0-1-k/2$*,* $u \in H^{s'}_{\mathrm{loc}}$*,* $\Box u \in H^{s-1}_{\mathrm{loc}}$ *and for each* $q \in \Sigma$ *the bicharacteristic through* q *has a point* q' *on it which is not in* $\mathrm{WF}^s(u)$*, then* $u \in H^s_{\mathrm{loc}}$*.*

Proof. By microlocal elliptic regularity which is valid with this s, $\mathrm{WF}^s(u) \subset \Sigma$. Now let $\epsilon_0 = \min((s_0 - k - 1)/2, s_0 - 1 - k/2 - s)/2 > 0$. Then for $s' \leq \tilde{s} \leq s$, the hypotheses of Corollary 8.2, apart from possibly $u \in H^{\tilde{s}-\epsilon_0}$, are satisfied with s replaced by $\tilde{s}$ and with this ϵ_0. Thus, taking $\tilde{s} = \min(s, s' + \epsilon_0)$, all hypotheses are satisfied, so as a point on any bicharacteristic is not in $\mathrm{WF}^s(u)$ and thus not in $\mathrm{WF}^{\tilde{s}}(u)$, one concludes that $\mathrm{WF}^{\tilde{s}}(u) = \emptyset$, i.e. $u \in H^{\tilde{s}}_{\mathrm{loc}}$. If $\tilde{s} = s$, we are done, otherwise we have $u \in H^{s'+\epsilon_0}_{\mathrm{loc}}$ repeat the argument, with $\tilde{s} = \min(s, s' + 2\epsilon_0)$; in finite number of steps we conclude that $u \in H^s_{\mathrm{loc}}$. □

A further consequence is:

Corollary 8.4. *Suppose* $s_0 > 1+k/2$*,* $0 \leq s < s_0-1-k/2$*. Let* $\Box_+^{-1} f \in H^{1,-\infty}_{\mathrm{b,loc}}(X)$ *denote the forward solution for* $\Box u = f$*, i.e. for* $f \in H^{-1,-\infty}_{\mathrm{b,loc}}(X)$ *supported in* $t > t_0$*,* $u = \Box_+^{-1} f$ *is supported in* $t > t_0$*.*

If $f \in H^{s-1}_{\mathrm{loc}}$ *is supported in* $t > t_0$*, then* $u = \Box_+^{-1} f \in H^s_{\mathrm{loc}}$*.*

An analogous result holds with $\Box_+^{-1}$ *replaced by the backward solution operator* $\Box_-^{-1}$ *and* $t > t_0$ *replaced by* $t < t_0$*.*

Proof. First we note $f \in H^{s-1}_{\mathrm{loc}}(X)$ implies $f \in H^{-1,s}_{\mathrm{b,loc}}(X)$, and thus $u = \Box_+^{-1} f \in H^{1,s-1}_{\mathrm{b,loc}}(X) \subset L^2(X)$. Then we merely need to observe that every bicharacteristic reaches $t < t_0$, where u vanishes, thus is in H^s_{loc}, so Corollary 8.3 is applicable with $s' = 0$ and yields the conclusion. □

References

[1] José L. Antoniano and Gunther A. Uhlmann. A functional calculus for a class of pseudo-differential operators with singular symbols. In *Pseudodifferential operators and applications (Notre Dame, Ind., 1984)*, volume 43 of *Proc. Sympos. Pure Math.*, pages 5–16. Amer. Math. Soc., Providence, RI, 1985.

[2] Dan-Andrei Geba and Daniel Tataru. A phase space transform adapted to the wave equation. *Comm. Partial Differential Equations*, 32(7-9):1065–1101, 2007.

[3] Allan Greenleaf and Gunther Uhlmann. Estimates for singular Radon transforms and pseudodifferential operators with singular symbols. *J. Funct. Anal.*, 89(1):202–232, 1990.

[4] Allan Greenleaf and Gunther Uhlmann. Recovering singularities of a potential from singularities of scattering data. *Comm. Math. Phys.*, 157(3):549–572, 1993.

[5] Allan Greenleaf and Gunther Uhlmann. Characteristic space-time estimates for the wave equation. *Math. Z.*, 236(1):113–131, 2001.

[6] V. Guillemin and G. Uhlmann. Oscillatory integrals with singular symbols. *Duke Math. J.*, 48(1):251–267, 1981.

[7] L. Hörmander. Fourier integral operators, I. *Acta Mathematica*, 127:79–183, 1971.

[8] L. Hörmander. *The analysis of linear partial differential operators,* vol. 1-4. Springer-Verlag, 1983.

[9] Lars Hörmander. On the existence and the regularity of solutions of linear pseudo-differential equations. *Enseignement Math. (2)*, 17:99–163, 1971.

[10] G. Lebeau. Propagation des ondes dans les variétés à coins. *Ann. Scient. Éc. Norm. Sup.*, 30:429–497, 1997.

[11] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math*, 31:593–617, 1978.

[12] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. II. *Comm. Pure Appl. Math*, 35:129–168, 1982.

[13] R. B. Melrose and G. A. Uhlmann. Lagrangian intersection and the Cauchy problem. *Comm. Pure and Appl. Math.*, 32:483–519, 1979.

[14] Richard B. Melrose. Transformation of boundary problems. *Acta Math.*, 147(3-4):149–236, 1981.

[15] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters Ltd., Wellesley, MA, 1993.

[16] Richard B. Melrose and Paolo Piazza. Analytic K-theory on manifolds with corners. *Adv. Math.*, 92(1):1–26, 1992.

[17] Hart F. Smith. A parametrix construction for wave equations with $C^{1,1}$ coefficients. *Ann. Inst. Fourier (Grenoble)*, 48(3):797–835, 1998.

[18] Daniel Tataru. Strichartz estimates for operators with nonsmooth coefficients and the non-linear wave equation. *Amer. J. Math.*, 122(2):349–376, 2000.

[19] A. Vasy. Propagation of singularities for the wave equation on manifolds with corners. *Annals of Mathematics*, 168:749–812, 2008.

[20] A. Vasy. Diffraction at corners for the wave equation on differential forms. *Commun. in PDEs*, 35:1236–1275, 2010.

[21] András Vasy. Geometric optics and the wave equation on manifolds with corners. In *Recent advances in differential equations and mathematical physics*, volume 412 of *Contemp. Math.*, pages 315–333. Amer. Math. Soc., Providence, RI, 2006.

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