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Characterization and 'Source-Receiver' Continuation of Seismic Reflection Data*

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Abstract: In reflection seismology one places sources and receivers on the Earth's surface. The source generates elastic waves in the subsurface, that are reflected where the medium properties, stiffness and density, vary discontinuously. In the field, often, there are obstructions to collect seismic data for all source-receiver pairs desirable or needed for data processing and application of inverse scattering methods. Typically, data are measured on the Earth's surface; the location of the receiver relative to the source can be coordinated by offset and azimuth. We employ the term *data continuation* to describe the act of computing data that have not been collected in the field. Seismic data are commonly modeled by a scattering operator developed in a high-frequency, single scattering approximation. We initially focus on the determination of the range of the forward scattering operator that models the singular part of the data in the mentioned approximation. This encompasses the analysis of the properties of, and the construction of, a minimal elliptic projector that projects a space of distributions on the data acquisition manifold to the range of the mentioned scattering operator. This projector can be directly used for the purpose of seismic data continuation, and is derived from the global parametrix of a homogeneous pseudodifferential equation the solution of which coincides with the range of the scattering operator. We illustrate the data continuation by a numerical example.

1. Introduction

In reflection seismology one places sources and receivers on the Earth's surface. The source generates elastic waves in the subsurface, that are reflected where the medium properties, stiffness and density, vary discontinuously. Seismic data collected in the field are often not ideal for data processing and application of inverse scattering methods. Typically, data are measured on the Earth's (two-dimensional) surface; the location of the

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receiver relative to the source can be coordinated by offset and azimuth. We employ the term *data continuation* to describe the act of computing data that have not been collected in the field. Special cases of data continuation are the so-called 'transformation to zero offset' (derived from what seismologists call Dip MoveOut [7] to generate data at zero offsets, and 'transformation to common azimuth' (derived from what seismologists call Azimuth MoveOut [2]) to generate data at a fixed, prescribed azimuth. Data continuation can also play the role of 'forward extrapolation' [9] in a data regularization scheme.

Seismic data are commonly modeled by a scattering operator developed in a high-frequency single scattering approximation. In this approximation one assumes that the medium is described by a singular contrast superimposed on a smooth background. Under geological constraints, often, the contrast is a conormal distribution. Initially, we focus on the determination of the range of the forward scattering operator that models the singular part of the data in the single scattering approximation. This encompasses the analysis of the properties of, and the construction of, a minimal elliptic projector that projects the space of distributions on the acquisition manifold to the range of the mentioned scattering operator. This projector can be directly used for the purpose of seismic data continuation, and is derived from the global parametrix of a homogeneous pseudo-differential equation the solution of which coincides with the range of the scattering operator.

Through characterization of features in the data, applications of data continuation extend to survey design (i.e. the design of the acquisition geometry describing the locations of the source-receiver pairs). The range of the scattering operator can also be used as a criterion for muting the data for features that are undesirable for the purpose of imaging the data (such as multiple scattered waves).

The notion of data continuation has been introduced in exploration seismology quite some time ago. As early as in 1982, Bolondi *et al.* [3] came up with the idea of describing data *offset continuation* and Dip MoveOut in the form of solving a partial differential equation. Their approach, built on the approach of Deregowski and Rocca [7], is valid in homogeneous media for acoustic waves while their partial differential operator is approximate only. An 'exact' partial differential equation for space dimension $n = 2$ that addresses mentioned offset continuation was later derived by Goldin [11]. In this application it is implicitly used that the kernel of the associated partial differential operator determines the range of the operator that models the singular part of seismic data – in the single scattering approximation. The operator can be written in the form of a generalized Radon transform.

Heuristically, the procedure and analysis presented in this paper can be thought of as a generalization from two to higher dimensions, from acoustic to elastic, and from homogeneous to heterogeneous media, of Goldin's 'offset continuation' equation. Let the data be denoted by $d = d(s, r, t)$, where s denotes source position, r receiver position, t the time, while $(s, r, t) \in Y$ and Y denotes the acquisition manifold. Let $Y \subset \mathbb{R}^{2n-1}$. We introduce the map

$$\kappa\colon\ (s, r, t) \mapsto (z, t_n, h),\ z = \tfrac{1}{2}(r + s),\ h = \tfrac{1}{2}(r - s),\ t_n = t_n(h, t) = \sqrt{t^2 - \frac{4h^2}{v^2}},$$

where, v is the acoustic wave speed. Let r be the pull back of d by the inverse of this map, $r = (\kappa^{-1})^* d$. The singular support of r can be parametrized by (z, h) according to $(z, T_n(z, h), h)$ with $T_n(z, h) = t_n(h, T(z, h))$, in which the function $T(z, h)$ denotes the traveltime of a particular reflection in the data; in seismological terms, the function

T_n is the traveltime 'after Normal MoveOut correction'. Goldin's equation is of the form ($n = 2$)

$$P'r = 0, \quad P' := t_n \frac{\partial^2}{\partial t_n \partial h} + h \left(\frac{\partial^2}{\partial h^2} - \frac{\partial^2}{\partial z^2} \right). \tag{1}$$

This equation is supplemented with the initial conditions

$$r(z, t_n, h)|_{h=h_0}, \quad \frac{\partial r}{\partial h}(z, t_n, h)|_{h=h_0}.$$

The first initial condition represents what seismologists call a post-normal -moveout constant-offset section at half offset h_0; the second initial condition is the first-order derivative of post-normal-moveout section at half offset h_0. Goldin's equation is not exact in the sense that it does not account for the symbols of the reflection operators associated with the reflectors in the subsurface.

The notion of data continuation has also been introduced and exploited in helical x-ray transmission tomography (CT) [22]. Consider a flat area detector, which is contained in the plane described by Cartesian coordinates (u, v). Let R denote the radius of the helix and $2\pi h$ its pitch. Let λ denote the angle describing rotation of the cone vertex. The axial shift of the assembly of the x-ray source and the detector is denoted by ζ. The data are denoted by $g = g(u, v, \lambda, \zeta)$. In this case, John's equation [19] describing the range of the Radon transform is used. John's equation for the x-ray transform in dimension 3 is given by

$$R^2 \frac{\partial^2 g}{\partial u \partial \zeta} - 2u \frac{\partial g}{\partial v} + (R^2 + u^2) \frac{\partial^2 g}{\partial u \partial v} = R \frac{\partial^2 g}{\partial \lambda \partial v} - Rh \frac{\partial^2 g}{\partial v \partial \zeta} + uv \frac{\partial^2 g}{\partial v^2}. \tag{2}$$

This equation is supplemented with the initial conditions

$$g(u, v, \lambda, \zeta)|_{\zeta=0}$$

that are measured. (The standard form of John's equation is much simpler than (2).) Gel'fand and Graev [10] have generalized John's result to k-planes in $\mathbb{R}^n$.

John's (and Goldin's) partial differential equation in higher space dimension is second order and of ultrahyperbolic type.

The seismic forward scattering operator is a Fourier integral operator and can be identified with a generalized Radon transform [1, 5, 23]. We characterize seismic data by analyzing the range of the forward scattering operator. This range coincides with the kernel of a self-adjoint, second-order pseudodifferential operator, P, derived from annihilators, P_i, of the data, d,

$$P_i d = 0, \quad P = \sum_i P_i^2. \tag{3}$$

Let Q denote the global parametrix of P. The mentioned elliptic minimal projector then follows to be

$$\pi = I - QP + \text{smoothing operator} \tag{4}$$

and provides the Fourier integral operator for continuing the singular part of the data. The annihilators are functionally dependent on the background medium, and hence can be used to form a criterion to estimate it. This estimation is known to seismologists as 'velocity analysis' and can be formulated as a reflection tomography problem. Thus, data continuation and reflection tomography, and imaging, are intimately connected.

The results presented in this paper are based on the work by Guillemin and Uhlmann [15]. Here, we speak of 'data' continuation rather than 'offset' continuation, because our approach continues data in sources and receivers and not only in offset due to the heterogeneity of the subsurface we can allow.

2. Modeling of Seismic Data in the Single Scattering Approximation

The propagation and scattering of seismic waves is governed by the elastic wave equation, which is written in the form

$$W_{il} u_l = f_i, \tag{5}$$

where

$$u_l = \sqrt{\rho(x)}(\text{displacement})_l, \quad f_i = \frac{1}{\sqrt{\rho(x)}}(\text{volume force density})_i, \tag{6}$$

and

$$W_{il} = \delta_{il}\frac{\partial^2}{\partial t^2} + A_{il} + \text{l.o.t.}, \quad A_{il} = -\frac{\partial}{\partial x_j}\frac{c_{ijkl}(x)}{\rho(x)}\frac{\partial}{\partial x_k}. \tag{7}$$

Here, $x \in \mathbb{R}^n$ and the subscripts $i, j, k, l \in \{1, \ldots, n\}$; ρ is the density of mass while c_{ijkl} denotes the stiffnesss tensor. The system of partial differential equations is assumed to be of principal type. It supports different wave types (modes), one 'compressional' and $n-1$ 'shear'. We label the modes by $M, N, \ldots$.

For waves in mode M, singularities are propagated along bicharacteristics, that are determined by Hamilton's equations generated by a Hamiltonian B_M,

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\lambda} &= \frac{\partial}{\partial \xi} B_M(x,\xi) \quad, \quad \frac{\mathrm{d}t}{\mathrm{d}\lambda} = 1, \\ \frac{\mathrm{d}\xi}{\mathrm{d}\lambda} &= -\frac{\partial}{\partial x} B_M(x,\xi)\ , \quad \frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = 0. \end{aligned} \tag{8}$$

The B_M follow from the diagonalization of the principal symbol matrix of A_{il}, as the square roots of its eigenvalues. Clearly, the solution may be parameterized by t. We denote the solution of (8) and initial values (x_0, ξ_0) at $t = 0$ by $(x_M(x_0, \xi_0, t), \xi_M(x_0, \xi_0, t))$.

In the contrast formulation the total value of the medium parameters ρ, c_{ijkl} is written as the sum of a smooth background constituent $\rho(x)$, $c_{ijkl}(x)$ and a singular perturbation $\delta\rho(x)$, $\delta c_{ijkl}(x)$, viz. $\rho + \delta\rho$, $c_{ijkl} + \delta c_{ijkl}$. This decomposition induces a perturbation of W_{il} (cf. (7)),

$$\delta W_{il} = \delta_{il}\frac{\delta\rho(x)}{\rho(x)}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x_j}\frac{\delta c_{ijkl}(x)}{\rho(x)}\frac{\partial}{\partial x_k}.$$

The scattered field, in the single scattering approximation, satisfies

$$W_{il}\delta u_l = -\delta W_{il} u_l.$$

Data are measurements of the scattered wave field δu, which we relate here to the Green's function perturbation: They are assumed to be representable by $\delta G_{MN}(\widehat{x}, \widetilde{x}, t)$ for $(\widehat{x}, \widetilde{x}, t)$ in some acquisition manifold, which contains the receiver and source points and time. Let $y \mapsto (\widehat{x}(y), \widetilde{x}(y), t(y))$ be a coordinate transformation, such that $y = (y', y'')$ and the acquisition manifold, Y say, is given by $y'' = 0$. We assume that the dimension of y'' is $2 + c$, where c is the codimension of the acquisition geometry. For example, for marine acquisition in seismic reflection data, $c = 1$, while also in global seismology-for many, but not all regions $c = 1$ – seismologists recognize this as lack of 'azimuthal' coverage. An example of $c = 2$ is provided by the common-'offset' acquisition geometry. In this framework, the data are modeled by

$$\left(\frac{\delta\rho(x)}{\rho(x)}, \frac{\delta c_{ijkl}(x)}{\rho(x)}\right) \mapsto \delta G_{MN}(\widehat{x}(y', 0), \widetilde{x}(y', 0), t(y', 0)). \tag{9}$$

We investigate the propagation of singularities by this mapping. Let $\tau = \mp B_M(x_0, \widehat{\xi}_0)$, and

$$\widehat{x} = x_M(x_0, \widehat{\xi}_0, \pm\widehat{t}), \quad \widetilde{x} = x_N(x_0, \widetilde{\xi}_0, \pm\widetilde{t}), \quad t = \widehat{t} + \widetilde{t},$$
$$\widehat{\xi} = \xi_M(x_0, \widehat{\xi}_0, \pm\widehat{t}), \quad \widetilde{\xi} = \xi_N(x_0, \widetilde{\xi}_0, \pm\widetilde{t}).$$

We then obtain $(y(x_0, \widehat{\xi}_0, \widetilde{\xi}_0, \widehat{t}, \widetilde{t}), \eta(x_0, \widehat{\xi}_0, \widetilde{\xi}_0, \widehat{t}, \widetilde{t}))$ by transforming $(\widehat{x}, \widetilde{x}, \widehat{t}+\widetilde{t}, \widehat{\xi}, \widetilde{\xi}, \tau)$ to (y, η) coordinates. We invoke the following assumptions that concern scattering over π and rays grazing the acquisition manifold,

Assumption 1. *There are no elements* $(y', 0, \eta', \eta'')$ *with* $(y', \eta') \in T^*Y\backslash 0$ *such that there is a direct bicharacteristic from* $(\widehat{x}(y', 0), \widehat{\xi}(y', 0, \eta', \eta''))$ *to* $(\widetilde{x}(y', 0), -\widetilde{\xi}(y', 0, \eta', \eta''))$ *with arrival time* $t(y', 0)$.

Assumption 2. *The matrix*

$$\frac{\partial y''}{\partial(x_0, \widehat{\xi}_0, \widetilde{\xi}_0, \widehat{t}, \widetilde{t})} \textit{ has maximal rank.} \tag{10}$$

The propagation of singularities by (9) is governed by the canonical relation

$$\begin{aligned}\Lambda_{MN} = \{(y'(x_0, \widehat{\xi}_0, \widetilde{\xi}_0, \widehat{t}, \widetilde{t}), \eta'(x_0, \widehat{\xi}_0, \widetilde{\xi}_0, \widehat{t}, \widetilde{t}); x_0, \widehat{\xi}_0 + \widetilde{\xi}_0) \,|\,, \\ B_M(x_0, \widehat{\xi}_0) = B_N(x_0, \widetilde{\xi}_0) = \mp\tau,\ y''(x_0, \widehat{\xi}_0, \widetilde{\xi}_0, \widehat{t}, \widetilde{t}) = 0\} \subset T^*Y\backslash 0 \times T^*X\backslash 0.\end{aligned} \tag{11}$$

The condition $y''(x_0, \widehat{\xi}_0, \widetilde{\xi}_0, \widehat{t}, \widetilde{t}) = 0$ determines the traveltimes $\widehat{t}$ for given $(x_0, \widehat{\xi}_0)$ and $\widetilde{t}$ for given $(x_0, \widetilde{\xi}_0)$. Following Maslov and Fedoriuk [21], we choose coordinates for Λ_{MN} of the form

$$(y'_I, x_0, \eta'_J), \tag{12}$$

where $I \cup J$ is a partition of $\{1, \ldots, 2n - 1 - c\}$, with associated generating function $S_{MN} = S_{MN}(y'_I, x_0, \eta'_J)$. The phase function in these coordinates becomes $\Phi_{MN} = \Phi_{MN}(y', x_0, \eta'_J)$.

Let $\tau = \frac{1}{2}(\widehat{\tau} + \widetilde{\tau})$ and $\bar{\tau} = \widehat{\tau} - \widetilde{\tau}$. The map

$$(x_0, \widehat{\xi}_0, \widetilde{\xi}_0, \widehat{t}, \widetilde{t}) \mapsto (x_0, y'_I, y'', \eta'_J, \bar{\tau}) \tag{13}$$

is bijective. Thus, for $y''=0$ and $\bar{\tau}=0$ we can express $(\widehat{\xi}_0, \widetilde{\xi}_0)$ as functions of (y'_I, x_0, η'_J). The amplitude associated with Λ_{MN}, to leading order, can be written in the form

$$|b_{MN}(y'_I, x_0, \eta'_J)| = (2\pi)^{-\frac{n+1+c}{4}} \left| \det \frac{\partial(\widehat{x}, \widetilde{x}, t)}{\partial(y', y'')} \right|^{-1/2} \left| \det \frac{\partial(x_0, \widehat{\xi}_0, \widetilde{\xi}_0, \widehat{t}, \widetilde{t})}{\partial(x_0, y'_I, y'', \eta'_J, \bar{\tau})} \right|^{1/2}_{y''=0, \bar{\tau}=0} \frac{1}{4\tau^2} \quad (14)$$

if

We assume that $\left(\frac{\delta\rho}{\rho}, \frac{\delta c_{ijkl}}{\rho}\right)$ are described by conormal distributions. We consider the case of a single interface, and a jump discontinuity in $(\delta\rho, \delta c_{ijkl})$ across this interface. Let $\kappa : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto z$ be a coordinate transformation such that the interface is given by $z_n = 0$. The corresponding cotangent vector is denoted by ζ, and transforms according to $\zeta_i(x, \xi) = ((\frac{\partial \kappa}{\partial x})^{-1})^t_{ij} \xi_j$, the z form coordinates on the manifold X, and we write $z = (z', z_n)$. We introduce the distributions $(\widetilde{\delta\rho}, \widetilde{\delta c_{ijkl}})$ by pull back with κ:

$$\widetilde{\delta\rho}(\kappa(x)) = \delta\rho(x), \quad \widetilde{\delta c_{ijkl}}(\kappa(x)) = \delta c_{ijkl}(x). \quad (15)$$

Then

$$\frac{\partial}{\partial x} \widetilde{\delta\rho} = \frac{\partial z_n}{\partial x} \rho' + \text{l.o.t.}, \quad \rho' = \frac{\partial}{\partial z_n} \widetilde{\delta\rho},$$

where ρ' contains a factor $\delta(z_n(x))$, and similarly for $\frac{\partial \widetilde{\delta c_{ijkl}}}{\partial x}$. Substituting (15) into the integral over X representing the high-frequency Born approximation for scattered waves, and integrating by parts, then yields an oscillatory integral representation in which

$$w_{MN;0}(y_I, x, \eta_J) \frac{\delta\rho(x)}{\rho(x)} + w_{MN;ijkl}(y_I, x, \eta_J) \frac{\delta c_{ijkl}(x)}{\rho(x)},$$

where w stands for the contrast-source radiation patterns derived from the pseudo-differential operators that diagonalize the elastodynamic system of equations, has been replaced by

$$2\mathrm{i}\tau R_{MN}(y_I, x, \eta_J) \left\| \frac{\partial z_n}{\partial x} \right\| \delta(z_n(x)).$$

Here we use that τ will be one of the components of η'_J. Also $\int (\cdots) \left\| \frac{\partial z_n}{\partial x} \right\| \delta(z_n(x))$ $\mathrm{d}x = \int_{z_n=0} (\cdots) \left| \det \frac{\partial x}{\partial z} \right| \left\| \frac{\partial z_n}{\partial x} \right\| \mathrm{d}z'$ becomes the Euclidean surface integral over the surface or manifold $z_n = 0$.

Theorem 1 [23]. *Suppose Assumptions 1, and 2 are satisfied microlocally for the relevant part of the data. Let $\Phi_{MN}(y, x, \eta_J)$ and $b_{MN}(y_I, x, \eta_J)$ be the phase function and amplitude introduced above. Then the mapping*

$$F : \left\| \frac{\partial z_n}{\partial x} \right\| \delta(z_n(x)) \mapsto G^{\text{refl}}_{MN}(y),$$

where

$$G^{\text{refl}}_{MN}(y) = (2\pi)^{-\frac{|J|}{2}-\frac{3n-1-c}{4}} \int\int_X (2\mathrm{i}\tau(\eta_J)\, b_{MN}(y_I, x, \eta_J) R_{MN}(y_I, x, \eta_J) + l.o.t.) \times \left\| \frac{\partial z_n}{\partial x} \right\| \delta(z_n(x)) \exp[\mathrm{i}\Phi_{MN}(y, x, \eta_J)]\, \mathrm{d}x\mathrm{d}\eta_J, \quad (16)$$

defines a Fourier integral operator with canonical relation Λ_{MN} *and of order* $\frac{n-1+c}{4} - 1$.

F models seismic reflection data. In the Kirchhoff approximation, one can identify the principal part of R_{MN} with the plane-wave reflection coefficient: Using (13) we find the $(x, \widehat{\xi_0}, \widetilde{\xi_0})$ associated with (y_I, x, η_J). A reflection from an incident N-mode with covector $\widetilde{\xi_0}$ into a scattered M-mode with covector $\widehat{\xi_0}$ takes place, at x, if the frequencies are equal and $\widehat{\xi_0} + \widetilde{\xi_0}$ is in the wavefront set of $\delta(z_n(x))$. Given $\widetilde{\xi_0}, \widehat{\xi_0}$ one can identify the down- and upgoing modes $\mu(M), \nu(N)$ relative to the interface, and define (at least to highest order) the reflection coefficient at x,

$$R_{MN} = R_{\mu(M),\nu(N)}(z'(x), \zeta'(x, \widetilde{\xi_0}), \tau) \quad \text{if } z_n(x) = 0, \quad (17)$$

see De Hoop and Bleistein [4] and Stolk and De Hoop [23]. The Kirchhoff approximation requires the following assumption

Assumption 3. *There are no rays tangent to the interface* $z_n = 0$*, i.e. elements in* Λ_{MN} *associated with* $(x(z', 0), \widetilde{\xi_0}(z', 0, \zeta', 0))$ *or with* $(x(z', 0), \widehat{\xi_0}(z', 0, \zeta', 0))$ *(cf. (11)).*

For a treatment of reflection and transmission of waves in the elastic case, using microlocal analysis, see Taylor [24]; for the acoustic case, see also Hansen [16]. Examples of conormal distributions, $\left\| \frac{\partial z_n}{\partial x} \right\| \delta(z_n(x))$, in the Earth sciences the reflections off which are observed, include the core-mantle boundary, thermal and chemical boundary layers in the deep mantle, fault zones, and geological interfaces in sedimentary basins.

3. Extension of the Scattering Operator

For simplicity of notation, from here on, we drop the subscripts MN and consider a single mode pair. In the single scattering approximation, subject to restriction to the acquisition manifold Y, the singular part of the medium parameters is a function of n variables, while the data are a function of $2n - 1 - c$ variables. Here, we discuss the extension of the scattering operator to act on distributions of $2n - 1(-c)$ variables, equal to the number of degrees of freedom in the acquisition.

3.1. The wavefront set of seismic data . The wavefront set of the modeled data is not arbitrary. This is a consequence of the fact that data consist of multiple experiments designed to provide a degree of redundancy, which we explain here.

Assumption 4 (Guillemin [13]). *The projection* π_Y *of* Λ *on* $T^*Y\backslash 0$ *is an embedding.*

This assumption is known as the Bolker condition. It admits the presence of caustics. Because Λ is a canonical relation that projects submersively on the subsurface variables (x, ξ) (using that the operator W_{il} is of principal type), the projection of (11) on $T^*Y\backslash 0$ is immersive [17, Lemma 25.3.6 and (25.3.4)]. In fact, only the injectivity part of the Bolker condition needs to be verified. The image $\mathcal{L}$ of π_Y is locally a coisotropic submanifold of $T^*Y\backslash 0$. Hence, for each $(y, \eta) \in \mathcal{L}$, $(T_{(y,\eta)}\mathcal{L})^\perp \subset T_{(y,\eta)}\mathcal{L}$. Setting $V_{(y,\eta)} = (T_{(y,\eta)}\mathcal{L})^\perp$, the vector bundle $V \to \mathcal{L}$ whose fiber at (y, η) is $(T_{(y,\eta)}\mathcal{L})^\perp$, is an integrable subbundle of $T\mathcal{L}$. Applying [14, Prop. 8.1], from the Bolker condition it follows that $\mathcal{L}$ satisfies their Axiom F: the foliation of $\mathcal{L}$ associated with V is fibrating, i.e. there exists a C^∞ Hausdorff manifold $\mathcal{X}$ and a smooth fiber map $\mathcal{L} \to \mathcal{X}$ whose fibers are the connected leaves of the foliation defined by V.

We choose coordinates revealing the mentioned fibration. Since the projection π_X of Λ on $T^*X\backslash 0$ is submersive, we can choose (x, ξ) as the first $2n$ local coordinates on Λ; the remaining $\dim\ Y - n = n - 1 - c$ coordinates are denoted by $e \in E$, E being a manifold itself. The sets $\mathcal{X} \ni (x, \xi) = \text{const.}$ are the isotropic fibers of the fibration of Hörmander [18], Theorem 21.2.6, see also Theorem 21.2.4. Duistermaat [8] calls them characteristic strips (see Theorem 3.6.2). Also, $\nu = \|\xi\|^{-1}\xi$ is then identified as the migration dip. The wavefront set of the data is contained in $\mathcal{L}$ and is a union of such fibers. The map $\pi_X \pi_Y^{-1}\colon \mathcal{L} \to \mathcal{X}$ is a canonical isotropic fibration, known to seismologists as map migration.

We consider again the canonical relation Λ and suppose that Assumption 4 is satisfied. We define Ω as the mapping $\pi_Y \pi_X^{-1}$,

$$\Omega\colon\ (x, \xi, e) \mapsto (y(x, \xi, e), \eta(x, \xi, e))\colon\ T^*X\backslash 0 \times E \to T^*Y\backslash 0,$$

which is known to seismologists as map demigration. This map conserves the symplectic form of $T^*X\backslash 0$. Indeed, let σ_Y denote the fundamental symplectic form on $T^*Y\backslash 0$. We consider the vector fields over an open subset of $\mathcal{L}$ with components $w_{x_i} = \frac{\partial(y,\eta)}{\partial x_i}$ and similarly for w_{ξ_i} and w_{e_i}. Then

$$\begin{aligned}
&\sigma_Y(w_{x_i}, w_{x_j}) = \sigma_Y(w_{\xi_i}, w_{\xi_j}) = 0,\\
&\sigma_Y(w_{\xi_i}, w_{x_j}) = \delta_{ij},\\
&\sigma_Y(w_{e_i}, w_{x_j}) = \sigma_Y(w_{e_i}, w_{\xi_j}) = \sigma_Y(w_{e_i}, w_{e_j}) = 0.
\end{aligned} \tag{18}$$

The (x, ξ, e) are 'symplectic coordinates' on the projection $\mathcal{L}$ of Λ on $T^*Y\backslash 0$. In the following lemma, we extend these coordinates to symplectic coordinates on an open neighborhood of $\mathcal{L}$, which is a manifestation of Darboux's theorem stating that T^*Y can be covered with symplectic local charts.

Lemma 1. *Let $\mathcal{L}$ be an embedded coisotropic submanifold of $T^*Y\backslash 0$, with coordinates (x, ξ, e) such that (18) holds. Denote $\mathcal{L} \ni (y, \eta) = \Omega(x, \xi, e)$. We can find a homogeneous canonical map G from an open part of $T^*(X \times E)\backslash 0$ to an open neighborhood of $\mathcal{L}$ in $T^*Y\backslash 0$, such that $G(x, e, \xi, \varepsilon = 0) = \Omega(x, \xi, e)$.*

3.2. An invertible Fourier integral operator. Let M be the canonical relation associated with the map G we constructed in Lemma 1, i.e.

$$M = \{(G(x, e, \xi, \varepsilon); x, e, \xi, \varepsilon)\} \subset T^*Y\backslash 0 \times T^*(X \times E)\backslash 0.$$

We now construct a Maslov-type phase function for M that is directly related to a phase function for Λ. Suppose (y_I, x, η_J) are suitable coordinates for Λ, at $\varepsilon = 0$. For ε small, the constant-ε subset of M allows the same set of coordinates, thus we can use coordinates $(y_I, \eta_J, x, \varepsilon)$ on M. Now there is (see Theorem 4.21 in Maslov and Fedoriuk [21]) a function $S(y_I, x, \eta_J, \varepsilon)$, called the generating function, such that M is given by

$$\begin{aligned} y_J &= -\frac{\partial S}{\partial \eta_J}\,, \quad & \eta_I &= \frac{\partial S}{\partial y_I}, \\ \xi &= -\frac{\partial S}{\partial x}\,, \quad & e &= \frac{\partial S}{\partial \varepsilon}. \end{aligned} \tag{19}$$

Thus a phase function for M is given by

$$\Psi(y, x, e, \eta_J, \varepsilon) = S(y_I, x, \eta_J, \varepsilon) + \langle \eta_J, y_J \rangle - \langle \varepsilon, e \rangle. \tag{20}$$

A phase function for Λ then follows as

$$\Phi(y, x, \eta_J) = \Psi(y, x, \tfrac{\partial S}{\partial \varepsilon}|_{\varepsilon=0}, \eta_J, 0) = S(y_I, x, \eta_J, 0) + \langle \eta_J, y_J \rangle.$$

We introduce the amplitude $b(y_I, x, \eta_J, \varepsilon)$ on M such that $b(y_I, x, \eta_J, \varepsilon = 0)$ coincides with the amplitude in Theorem 1. To leading order,

$$\frac{\partial}{\partial \varepsilon} b = 0$$

because the coordinates ε_i are in involution.

We construct a mapping from the reflectivity function to seismic data, extending the mapping from contrast to data. This is done by applying the results of Sect. 3.1 to the Kirchhoff modeling formula (16). We apply the change of coordinates on Λ from (y_I, x, η_J) to (x, ξ, e) to the symbol R_{MN} and write now $R_{MN} = R(x, \xi, e)$. To highest order, R does not depend on $\|\xi\|$ and is simply a function of (x, e).

Theorem 2 [23]. *Suppose microlocally that Assumptions 3 (no grazing rays at any interface), 1 (no scattering over π), 2 (transversality), and 4 (Bolker condition) are satisfied. Let H be the Fourier integral operator,*

$$H\colon \mathcal{E}'(X \times E) \to \mathcal{D}'(Y),$$

with canonical relation given by the extended map $G : (x, \xi, e, \varepsilon) \mapsto (y, \eta)$ constructed in Sect. 3.1, and with amplitude to highest order given by $(2\pi)^{n/2} 2i\tau(\eta_J) b(y_I, x, \eta_J, \varepsilon)$ expressible in terms of the coordinates (x, e, ξ, ε). Then the data, in both Born and Kirchhoff approximations, can be modeled by H acting on a distribution $r(x, e)$ of the form

$$r(x, e) = \mathsf{R}(x, D_x, e)\, \mathsf{c}(x), \tag{21}$$

where R stands for a smooth e-family of pseudodifferential operators and $\mathsf{c} \in \mathcal{E}'(X)$. For the Kirchhoff approximation the distribution c equals $\|\frac{\partial z_n}{\partial x}\| \delta(z_n(x))$, while the principal symbol of the pseudodifferential operator R equals $R(x, e)$, so to highest order

$$r(x, e) = R(x, e) \left\| \frac{\partial z_n}{\partial x} \right\| \delta(z_n(x)). \tag{22}$$

For the Born approximation the function $r(x, e)$ is given by a pseudodifferential operator R *with principal symbol* $(2i\tau(x, \xi, e))^{-1}(w_{MN;0}(x, \xi, e), w_{MN;ijkl}(x, \xi, e))$*, acting on a distribution* c *given by* $\left(\frac{\delta c_{ijkl}}{\rho}, \frac{\delta\rho}{\rho}\right)$*, so to highest order*

$$r(x, e) = (2i\tau(x, D_x, e))^{-1}\left[w_{MN;0}(x, D_x, e)\frac{\delta\rho(x)}{\rho(x)} + w_{MN;ijkl}(x, D_x, e)\frac{\delta c_{ijkl}(x)}{\rho(x)}\right].$$

The operator H is invertible.

Remark 1. Microlocally, we have obtained the following diagram (suggested by Symes, personal communication)

$$\begin{array}{rcl} & \mathcal{E}'(X \times E) \xrightarrow{H} \mathcal{D}'(Y) & \\ \mathsf{R}(x, D_x, e) & \uparrow \qquad\qquad \uparrow & \mathrm{Id} \\ & \mathcal{E}'(X) \xrightarrow{F} \mathcal{D}'(Y) & \end{array} \tag{23}$$

We note that $\mathsf{R}(x, D_x, e)$ is of order 0. H^{-1} maps data into what seismologists call common-image-point gathers (the integral over ε replaces the notion of beamforming; e plays the role of scattering angle and azimuth).

4. A Procedure for Data Continuation

4.1. The range of the scattering operator. If $n - 1 - c > 0$, there is a redundancy in the data parametrized by the variable e. The redundancy in the data manifests itself as a redundancy in images of the subsurface from these data. A smooth background is considered 'acceptable' if the data are contained in the range of F (or H). If a smooth background is acceptable, then applying the operator H^{-1} of Theorem 2 to the data results in a reflectivity distribution $r(x, e)$, the singular support (in x) of which does not depend on e.

One way to measure the agreement in singular supports between images of reflectivity $r(x, e)$ parametrized by e is by taking a derivative with respect to e. Taking (21) as the point of departure, we find that

$$\left(\mathsf{R}(x, D_x, e)\frac{\partial}{\partial e} - \frac{\partial \mathsf{R}}{\partial e}(x, D_x, e)\right) r(x, e) = \left[\mathsf{R}(x, D_x, e), \frac{\partial \mathsf{R}}{\partial e}(x, D_x, e)\right] \mathsf{c}(x). \tag{24}$$

Hence, microlocally where $\mathsf{R}(x, D_x, e)$ is elliptic,

$$\left(\mathsf{R}(x, D_x, e)\frac{\partial}{\partial e} - \frac{\partial \mathsf{R}}{\partial e}(x, D_x, e) - \left[\mathsf{R}(x, D_x, e), \frac{\partial \mathsf{R}}{\partial e}(x, D_x, e)\right] \mathsf{R}(x, D_x, e)^{-1}\right) r(x, e) = 0 \tag{25}$$

to all orders. We observe that the first operator acting on distributions in (x, e) in the sum is of order 1, the second operator is of order 0, while the third operator is of order -1. Falling back on (22) we exploit that, up to leading order, the operator R acts as a multiplication by $R(x, e)$. Clearly,

$$\left[R(x, e), \frac{\partial R}{\partial e}(x, e)\right] \left\| \frac{\partial z_n}{\partial x} \right\| \delta(z_n(x)) = 0,$$

cf. (24). Substituting (21) into (25) reveals that the operator in between parentheses on the left-hand side equals $(R(x,e)\frac{\partial}{\partial e} - \frac{\partial R}{\partial e}(x,e))$ up to the leading two orders. Hence,

$$\left(R(x,e)\frac{\partial}{\partial e} - \frac{\partial R}{\partial e}(x,e)\right) r(x,e) = 0 \tag{26}$$

up to the highest two orders.

Conjugating the operator in between parentheses in (26), or in (25), with the invertible Fourier integral operator H, we obtain a pseudodifferential operator on $\mathcal{D}'(Y)$ [23]

Lemma 2. *Let the pseudodifferential operators* $P_i(y, D_y) : \mathcal{D}'(Y) \to \mathcal{D}'(Y)$ *of order* 1 *be given by the composition*

$$P_i(y, D_y) = H\left(R(x,e)\frac{\partial}{\partial e_i} - \frac{\partial R}{\partial e_i}(x,e)\right)H^{-1}, \quad i = 1, \dots, \underbrace{n-1-c}_{r}.$$

Then for Kirchhoff data $d(y)$ *modeled by* F*, we have to the highest two orders,*

$$P_i(y, D_y)d(y) = 0, \quad i = 1, \dots, n-1-c. \tag{27}$$

Microlocally, for values of e where $R(x,e) \neq 0$*, the operator* $P_i(y', D_{y'})$ *can be modeled after* (25) *such that* (27) *is valid to all orders.*

The principal part of the symbol of P_i is denoted by p_i, while the next order term in the symbol's polyhomogeneous expansion is denoted by $p_{i;0}$. The subprincipal symbols (which show up naturally in the Weyl calculus of symbols), c_i, of the annihilators are then given by $c_i := p_{i;0} + \frac{\mathrm{i}}{2}\sum_j \frac{\partial^2 p_i}{\partial y_j \partial \eta_j}$.

Remark 2. The wavefront set of the data is contained in $\mathcal{L} = \pi_Y(\Lambda)$, which, in analogy with the eikonal equation, is also the submanifold of $T^*Y\backslash 0$ defined by

$$p_i(y, \eta) = 0, \quad i = 1, \dots, n-1-c, \tag{28}$$

where p_i is the principal symbol of P_i as before, and is of codimension $2[2(n-1) - c + 1] - (3n - 1 - c) = n - 1 - c$, which is also the dimension of the covector ε.

The operator F in Corollary 1 is continuous $H^{\frac{n-1}{2}}(X) \to L^2(Y)$. We now define the operator projection, $\pi : L^2(Y) \to L^2(Y)$, onto the range of the scattering operator F. Microlocally, $\pi^2 = \pi$. Since Assumption 4 is satisfied, using [14, Prop. 8.3], π is an elliptic minimal projector. By [14, Theorem 6.6], the kernel of

$$P = P_1^2 + \cdots + P_{n-1-c}^2 \tag{29}$$

is identical with the range of π. More precisely, let Q denote the global parametrix of P, then, by [14, Theorem 6.7],

$$\pi = I - QP + \text{smoothing operator}. \tag{30}$$

4.2. A global parametrix. The construction of a global parametrix, Q, for an operator of the type P is given by Guillemin and Uhlmann [15]. A natural parametrix for P would have as principal symbol $\dfrac{1}{p_1^2 + \cdots + p_{n-1-c}^2}$. However, this expression becomes singular at the set $\{p_1 = \cdots = p_{n-1-c} = 0\}$. A class of operators, containing pseudodifferential operators with singular symbols, was introduced by Guillemin and Uhlmann [15]. The wavefront set of the kernels of these operators consist of two Langrangian manifolds, Λ_0 and Λ_1 say, intersecting cleanly in a submanifold of given codimension. In our case, Λ_0 is the diagonal $\operatorname{diag}(T^*Y\backslash 0)$ in $T^*Y\backslash 0 \times T^*Y\backslash 0$, while Λ_1 is the fiber product $\mathcal{L} \overset{\mathcal{X}}{\times} \mathcal{L}$. The Lagrangian submanifold $\Lambda_1 \subset T^*Y\backslash 0 \times T^*Y\backslash 0$ precisely consists of points on the joint flowout from $\operatorname{diag}(T^*Y\backslash 0) \cap \{p_1 = \cdots = p_{n-1-c} = 0\}$ by the Hamiltonian flows of the $H_{p_1}, \ldots, H_{p_{n-1-c}}$, where H_{p_i} denotes the Hamiltonian field associated with the function p_i. The flowout is described by the solution to the Hamilton systems with parameters e_i,

$$\frac{\partial y_j}{\partial e_i} = \frac{\partial p_i}{\partial \eta_j}(y, \eta), \quad \frac{\partial \eta_j}{\partial e_i} = -\frac{\partial p_i}{\partial y_j}(y, \eta), \qquad 1 \le i,\ j \le n - 1 - c. \tag{31}$$

The Lagrangian submanifolds Λ_0 and Λ_1 intersect cleanly in a submanifold of codimension $n - 1 - c$, see Remark 2.

Guillemin and Uhlmann's construction relies on the introduction of the space of distributional half densities, $I^{p,l}(Y \times Y; \Lambda_0, \Lambda_1)$, defining a class of Fourier integral operators with singular symbols, with the properties $\cap_l I^{p,l}(Y \times Y; \Lambda_0, \Lambda_1) = I^p(Y \times Y, \Lambda_1)$ (defining standard Fourier integral operators with canonical relation Λ_1) for p fixed, and $\cap_p I^{p,l}(Y \times Y; \Lambda_0, \Lambda_1) = C_0^\infty(Y \times Y)$ for l fixed. Viewing the Schwartz kernel of the identity (I) as an element of $I^{0,0}(Y \times Y; \Lambda_0, \Lambda_1)$, Guillemin and Uhlmann's recursive construction results in

$$QP = I - \pi + R,$$

where the kernel of π belongs to $\cap_l I^{p,l}(Y \times Y; \Lambda_0, \Lambda_1)$, and R is a smoothing operator with kernel in $\cap_p I^{p,l}(Y \times Y; \Lambda_0, \Lambda_1)$. Here, we discuss the properties of Q.

We observe that $\Lambda_1 \circ \Lambda_1 = \Lambda_1$. The elliptic minimal projector π, introduced in the previous subsection, is a Fourier integral operator with canonical relation Λ_1,

$$\begin{array}{ccc} \mathcal{L} & \overset{(31)}{\longrightarrow} & \mathcal{L} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \end{array} \tag{32}$$

The wavefront set of Q is contained in $\Lambda_0 \cup \Lambda_1$. Q is a pseudodifferential operator on $\Lambda_0\backslash(\Lambda_0 \cap \Lambda_1)$ and its principal symbol there is given by $\sigma_0 = \dfrac{1}{p_1^2 + \cdots + p_{n-1-c}^2}$ up to Maslov factors and half densities. Q is a Fourier integral operator on $\Lambda_1\backslash(\Lambda_0 \cap \Lambda_1)$. Its principal symbol, σ_1, solves the transport equation

$$\sum_{i=1}^{n-1-c} \left(\mathrm{i} H_{p_i} - c_i\right)^2 \sigma_1 = \sigma_\pi$$

on $\Lambda_1\backslash(\Lambda_0 \cap \Lambda_1)$, where σ_π denotes the symbol of π. (We return to the evaluation of σ_π in Sect. 6.) The expression between parentheses is an elliptic differential operator of order 2 on each fiber of $\mathcal{L}$. The equation is Laplace's equation in every leaf of the foliation generated by the commuting vector fields H_{p_i}, $i = 1, \ldots, n-1-c$. The principal symbol σ_1 has a conormal singularity at $\Lambda_0 \cap \Lambda_1$, expressible by an appropriate Fourier transform of the singularity of σ_0, see [15, (5.14)].

4.3. Data continuation. We apply the results of the previous subsections to the problem of source-receiver continuation of seismic data: Seismic data are commonly measured on an open subset of the manifold of all possible observations. Continuation of these data from the open subset to the full acquisition manifold is desired for various data processing procedures, including imaging – in the seismic literature this continuation is referred to as the 'forward extrapolation' step within data regularization [9], or 'data healing'.

Theorem 3. *Suppose u is a distribution belonging to the range of the scattering operator F. Let $\chi = \chi(y, D_y)$ be a pseudodifferential operator of order 0 that acts as a cutoff in phase space $T^*Y\backslash 0$. Assume that we observe $u_0 = \chi u$ in accordance with the constraints of the acquisition geometry. Suppose χ is elliptic on a leaf of the foliation of $\mathcal{L}$, then* $\mathrm{WF}(\pi u_0)$ *intersected with this leaf is equal to* $\mathrm{WF}(u)$ *intersected with the same leaf. In this case, π heals the data on this leaf.*

Proof. We observe that $u = \pi v$ for some v. Then

$$u_0 = (\chi\pi)\, v. \tag{33}$$

Because $\pi^2 u = u$, it is natural to investigate πu_0, i.e.

$$\pi u_0 = (\pi\chi\pi)\, v. \tag{34}$$

If χ is elliptic on a leaf of the foliation of $\mathcal{L}$, then $(\pi - \pi\chi\pi)\, v = 0$, or $u - \pi u_0 = 0$, microlocally on this leaf. This implies the statement in the theorem. □

We implement π by making use of the following observation. In view of the Bolker condition, Assumption 4, the composition F^*F is an elliptic pseudodifferential operator of order $n-1$. Let Ξ denote the parametrix for F^*F. The operator $F\Xi F^*$ belongs to Guillemin and Sternberg's algebra $\mathcal{R}_{\mathcal{L}}$ [14] of Fourier integral operators with canonical relation Λ_1 [6]. Clearly, $(F\Xi F^*)^2 = F\Xi F^*$ microlocally, while $I - \Xi F^*F = I - F^*F\Xi$ is the orthogonal projection onto the kernel of F^*F. Indeed, $F\Xi F^*$ is precisely an elliptic minimal projector [14, Proof of Thm. 8.3] of the type introduced in Sect. 4. The symbol of this operator follows by the standard composition calculus.

Following the composition of F^* with F in $F\Xi F^*$, we represent the canonical relation Λ_1 as the composition of canonical relations Λ^* with Λ.

Remark 3. The transformation to zero offset (TZO) of seismic data, which is derived from Dip MoveOut, can be expressed in the form $R_0\pi$, where R_0 is the restriction of distributions on Y to an acquisition manifold with coinciding sources and receivers: In this case, $y = (y', y'')$ with $y' = (\frac{1}{2}(s + r), t)$ and $y'' = \frac{1}{2}(r - s)$ if (s, r, t) are the original local coordinates on Y. Assumption 2, subject to this substitution, guarantees that the composition, $R_0\pi$, is again a Fourier integral operator.

5. Goldin's Equation Revisited

In context of the simplest seismic scattering theory, in a background that essentially is constant, the following simplications are made. To begin with, the source, s, and receiver, r, points in Y are assumed to be contained in a flat surface, $\mathbb{R}^{n-1}$, while e is initially replaced by half source-receiver offset $h = \frac{1}{2}(r - s) \in \mathbb{R}^{n-1}$. Thus y is replaced by (s, r, t); we write $\eta = (\sigma, \rho, \tau)$. Essentially, we assume that the rays between reflector and acquisition surface are straight, see Fig. 1. We repeat the NMO correction,

$$\kappa : (s, r, t) \mapsto (z, t_n, h),\ z = \tfrac{1}{2}(r + s),\ h = \tfrac{1}{2}(r - s),\ t_n = \sqrt{t^2 - \frac{4h^2}{v^2}},$$

of the introduction. (The subscript $_n$ refers in this section to normal moveout.) Here, v could be thought of as the so-called NMO velocity, which can be introduced for 'pure mode' scattering (i.e. $M = N$), even in the anisotropic media under consideration here [12]; Goldin, however, restricts his analysis to an isotropic medium and compressional waves. We will reproduce Goldin's result here in the context of our analysis subject to the substitutions $n = 2$ and $c = 0$ (and $m = 1$).

NMO correction applied to the data yields $(\kappa^{-1})^* d$. Including a so-called geometrical spreading correction, a multiplication by time t, then leads to the map

$$d(s, r, t) \mapsto ((\kappa^{-1})^*(t\, d))(z, t_n, h) \tag{35}$$

that replaces $(H^{-1}d)(x, e)$; the point x has attained coordinates (z, t_n). The outcome is of the form

$$r(z, t_n, h) = R_n(z, h)\, \mathsf{c}_n(t_n - T_n(z, h)). \tag{36}$$

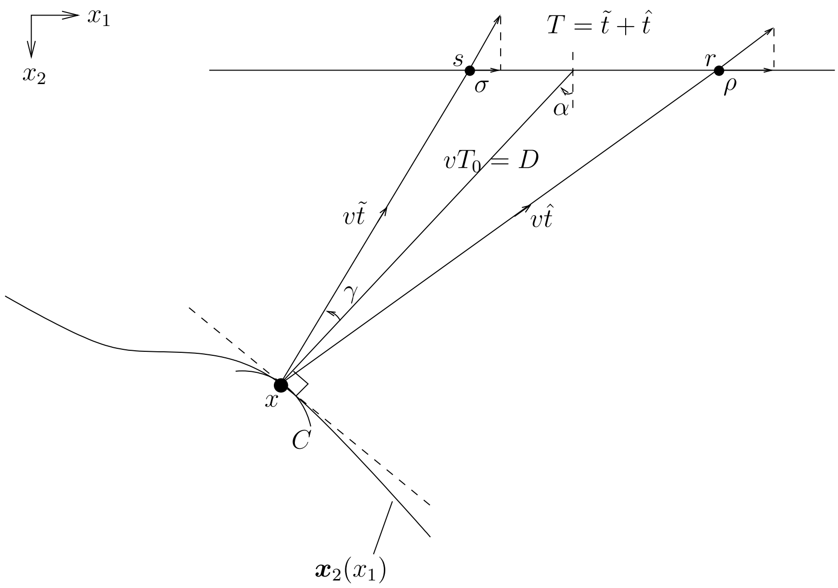

Fig. 1. Geometry underlying the annihilator symbol for constant coefficients

Equation (36) replaces (22). The reflection time, $T(s, r, x)$ maps under κ according to

$$T(s, r, x) \mapsto T_n(z, h) = \sqrt{(T(z-h, z+h, x))^2 - \frac{4h^2}{v^2}}.$$

We observe that, in the simplication considered, R_n is independent of t_n, while c_n is not only a function of (z, t_n) but also of h. Hence, a simple derivative of r with respect to h, motivated by (26), would not yield a vanishing outcome up to leading order. Instead, it is possible to construct a candidate operator P_1, acting on the data, directly. In Fig. 1 we introduce angles α and γ; in fact, (x, α, γ) can be identified with (x, ν, e). Using simple trigonometric identities (including the law of sines) and the geometry in Fig. 1 (observing that the total length of the reflected ray is $vt = (r-s)\,\frac{\cos\alpha}{\sin\gamma}$ with 2γ denoting the scattering angle and α denoting the incidence angle of the zero-offset ray at the surface), it follows that

$$p_1(s, r, t, \sigma, \rho, \tau) = \left(t^2 + \frac{(r-s)^2}{v^2}\right)(\sigma - \rho) - 2\,(r-s)\,t\left(\frac{\tau}{v^2} - \tau^{-1}\sigma\rho\right) = 0 \quad (37)$$

defines the points in $\mathcal{L}$ (Remark 2) in the simplification under consideration; p_1 is v dependent. Applying the coordinate transformation implied by κ to this symbol, and multiplying the result by frequency τ, yields

$$p'(z, t_n, h, \zeta, \tau_n, \varepsilon) = (\tau p_1)(\kappa(s, r, t), ((\kappa')^{-1})^t(\sigma, \rho, \tau)) \quad (38)$$

or

$$p'(z, t_n, h, \zeta, \tau_n, \varepsilon) = -t_n \tau_n\, \varepsilon + h\,(\zeta^2 - \varepsilon^2), \quad (39)$$

which defines the principal symbol of an operator P'; we observe that p' is v *independent*. We recover Goldin's equation,

$$\begin{aligned} &P'(z, t_n, h, D_z, D_{t_n}, D_h) r = 0, \\ &P'(z, t_n, h, D_z, D_{t_n}, D_h) = t_n \frac{\partial^2}{\partial t_n \partial h} - h\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial h^2}\right), \end{aligned} \quad (40)$$

which is valid up to highest order. It would be valid up to the next order if we had not applied the geometrical spreading correction in (35). Accounting for this correction leads to a subprincipal symbol contribution:

$$P'(z, t_n, h, D_z, D_{t_n}, D_h) := t_n \frac{\partial^2}{\partial t_n \partial h} - h\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial h^2}\right) - \frac{\partial}{\partial h}.$$

Through the coordinate transformation implied by κ, we obtain a subprincipal symbol contribution to the operator with principal symbol p_1 (cf. (37)). Note that the operator P' is of second order unlike the operator annihilating r in (26) which is of first order. However, in (38) we introduced a multiplication by τ, raising the order by one. A first-order operator derived from P' in (40) follows to be

$$P''(z, t_n, h, D_z, D_{t_n}, D_h) = \left(\frac{\partial}{\partial h}\right)^{-1} \frac{h}{t_n}\left(\frac{\partial^2}{\partial h^2} - \frac{\partial^2}{\partial z^2}\right) + \frac{\partial}{\partial t_n}.$$

We write down the Hamiltonian system describing the flowout as in (31); there is only one such system since $n - c - 1 = 1$. We use the first-order symbol, $p''(z, t_n, h, \zeta, \tau_n, \varepsilon) = \tau_n - \frac{h}{\varepsilon t_n}(\zeta^2 - \varepsilon^2)$ (we omitted a factor i), so that

$$\begin{aligned}
\frac{\partial z}{\partial e} &= -\frac{2h\zeta}{\varepsilon t_n} \quad , & \frac{\partial \zeta}{\partial e} &= 0, \\
\frac{\partial t_n}{\partial e} &= 1 \quad , & \frac{\partial \tau_n}{\partial e} &= -\frac{h}{\varepsilon t_n^2}(\zeta^2 - \varepsilon^2), \\
\frac{\partial h}{\partial e} &= -\frac{\zeta^2}{\varepsilon^2}\frac{h}{t_n} + \frac{2h}{t_n} \, , & \frac{\partial \varepsilon}{\partial e} &= \frac{1}{\varepsilon t_n}(\zeta^2 - \varepsilon^2).
\end{aligned} \tag{41}$$

We set $e = t_n$, and eliminate τ_n. To this end, we introduce the slowness vectors $\widehat{\zeta}$ and $\widehat{\varepsilon}$ according to $\zeta = \tau_n \widehat{\zeta}$ and $\varepsilon = \tau_n \widehat{\varepsilon}$. Substituting $\tau_n = \frac{h}{\varepsilon t_n}(\zeta^2 - \varepsilon^2)$ (using that $p'' = 0$), and the equation for $\frac{\partial \tau_n}{\partial e}$, then yields the system

$$\begin{aligned}
\frac{\partial z}{\partial t_n} &= -\frac{2h\widehat{\zeta}}{\widehat{\varepsilon}\, t_n} \quad , & \frac{\partial \widehat{\zeta}}{\partial t_n} &= \frac{\widehat{\zeta}}{t_n}, \\
\frac{\partial h}{\partial t_n} &= -\frac{1}{\widehat{\varepsilon}} + \frac{h}{t_n} \, , & \frac{\partial \widehat{\varepsilon}}{\partial t_n} &= \frac{\widehat{\zeta}^2}{\widehat{\varepsilon}\, t_n}.
\end{aligned} \tag{42}$$

We note that t_n and h are directly related to one another. Indeed, let the zero-offset reflecton time be given by $T_0 = T(z_0, z_0, x) = T_n(z_0, 0)$. Then, for given z,

$$\frac{T_n}{T_0} = \frac{h}{(h^2 - (z - z_0)^2)^{1/2}}.$$

Along bicharacteristics,

$$\frac{h}{\widehat{\varepsilon}\, t_n} = -\frac{v^2}{4 \sin^2 \alpha} \tag{43}$$

is invariant (i.e. its derivative with respect to e is zero). We can convert t_n to half scattering angle γ – keeping (x, α) fixed – according to the relation

$$\frac{T_n}{T_0} = \frac{\cos\alpha \, \cos\gamma}{(\cos^2 \alpha - \sin^2 \gamma)^{1/2}}, \qquad T_0 = \frac{2V}{v}. \tag{44}$$

We discuss in as much the kernel of P' determines the range of F under the simplication ('straight rays') under consideration. The range is described by wavefields of the form

$$\bar{d}(s, r, t) = \left(vt \, \frac{\sqrt{\cos^2 \gamma + VC}}{\cos \gamma} \right)^{-1} r(x, \gamma)\, \delta(t - T), \quad T = T(s, r, x), \tag{45}$$

obtained after preprocessing d for time signature (a 2.5D correction) and source or receiver radiation characteristics, applying an appropriate pseudodifferential operator. In (45), $V = \frac{1}{2} v T(z, z, x)$ denotes the length of the zero-offset ray, and $C = \boldsymbol{x}_2''(x_1) \cos^3 \alpha$ denotes the curvature of the reflector, see Fig. 1. We have assumed that the reflecting interface can be described by the graph $(x_1, \boldsymbol{x}_2(x_1))$. (If the interface is the zero level set of $z_2 = z_2(x_1, x_2)$ then we assume that $\frac{\partial z_2}{\partial x_2} \neq 0$.) Equation (45) is the outcome of a stationary phase calculation of the scattered field in the Kirchhoff approximation. We set $r(x, \gamma) \equiv 1$ and apply $(\kappa^{-1})^*$ to $\bar{d}(s, r, t)$ and obtain

$$\begin{aligned} \bar{r}(z, t_n, h) &= A_n(z, h)\, \delta(t_n - T_n), \\ T_n &= T_n(z, h), \\ A_n &= \left(vt\, \frac{\sqrt{\cos^2\gamma + VC}}{\cos\gamma} \right)^{-1} \frac{t}{t_n}, \end{aligned} \tag{46}$$

because $|\frac{\mathrm{d}t_n}{\mathrm{d}t}| = \frac{t}{t_n}$. Expression (46) is indeed of the form (36).

To verify whether the wavefields in (45), via (46), coincide with functions in the kernel of P', up to leading order, we first notice that by derivation $p'(z, T_n, h, -\mathrm{i}\tau_n \frac{\partial T_n}{\partial z}, \tau_n, -\mathrm{i}\tau_n \frac{\partial T_n}{\partial h}) = 0$. Secondly, we consider the transport equation derived from (40), which is given by

$$\left(T_n - 2h \frac{\partial T_n}{\partial h} \right) \frac{\partial A_n}{\partial h} + 2h \frac{\partial T_n}{\partial z} \frac{\partial A_n}{\partial z} + h A_n \left(\frac{\partial^2 T_n}{\partial z^2} - \frac{\partial^2 T_n}{\partial h^2} \right) = 0.$$

The velocity vector associated with a ray or characteristic is given by $(\frac{\partial T_n}{\partial z}, \frac{\partial T_n}{\partial h})$. Thus, along a characteristic, the transport equation becomes

$$-\frac{1}{A_n} \frac{\mathrm{d}A_n}{\mathrm{d}T_n} + h \left(T_n \frac{\partial T_n}{\partial h} \right)^{-1} \left(\frac{\partial^2 T_n}{\partial z^2} - \frac{\partial^2 T_n}{\partial h^2} \right) = 0, \tag{47}$$

where we made use of (42). We change variables according to (44), with

$$\frac{1}{T_n} \frac{\mathrm{d}T_n}{\mathrm{d}\gamma} = -\frac{\sin^2\alpha \sin\gamma}{(\cos^2\alpha - \sin^2\gamma)\, \cos\gamma}.$$

Furthermore, using the ray geometry, we find the identity

$$T_n \left(\frac{\partial^2 T_n}{\partial z^2} - \frac{\partial^2 T_n}{\partial h^2} \right) = 4 \left(T \frac{\partial^2 T}{\partial s \partial r} + \frac{\cos^2\gamma}{v^2} \right) = 4\, \frac{\cos^2\gamma}{v^2} \frac{\sin^2\alpha + VC}{\cos^2\gamma + VC}. \tag{48}$$

Substituting identity (48) and invariant (43) into (47), applying the change of variables (44), leads to the equation,

$$-\frac{1}{A_n} \frac{\mathrm{d}A_n}{\mathrm{d}\gamma} + \left(\frac{1}{(\cos^2\alpha - \sin^2\gamma)} - \frac{1}{\cos^2\gamma + VC} \right) \sin\gamma \cos\gamma = 0. \tag{49}$$

This equation can be directly integrated to yield solutions for A_n of the type (46). We conclude that the kernel of P' generates wavefields of the type (45) which comprise the range of the scattering operator subject to processing for time signature and setting $r(x, \gamma) \equiv 1$.

6. Numerical Example

The minimal elliptic projector π is a Fourier integral operator and is directly implementable and applicable to data. This is the subject of this section. Indeed, given a smooth background model, we can construct a minimal elliptic projector for data continuation by operator composition, $F \Xi F^*$. On the other hand, however, in Sect. 4 we showed that, given the annihilators of the data (in practice, just their principal parts), we can construct the global parametrix Q, from which the elliptic minimal projector follows. This procedure is related to what Guillemin and Sternberg call relative geometrical quantization.

We include an example to confirm the computability of our result. The algorithm used is designed and explained in [20]. In our example, $n = 2$ and χ is replaced by a smooth cutoff ψ_Y; the cutoff restricts the data to the set $\{(s, r, t) \mid s, r \in \mathbb{R}^{n-1},\ \|r-s\| > h_0,\ t \in (0, T)\}$. The goal is to continue the data to an acquisition manifold with the constraint $\|r - s\| > h_0$ removed. Elastic-wave data were simulated over a model illustrated in Fig. 2 (top). The P-wave velocities are shown in grey scale; a low-velocity Gaussian lens was inserted (white-to-grey). The continuation is illustrated for the P-wave constituents even though the simulated data contained S waves as well. By selecting the vertical

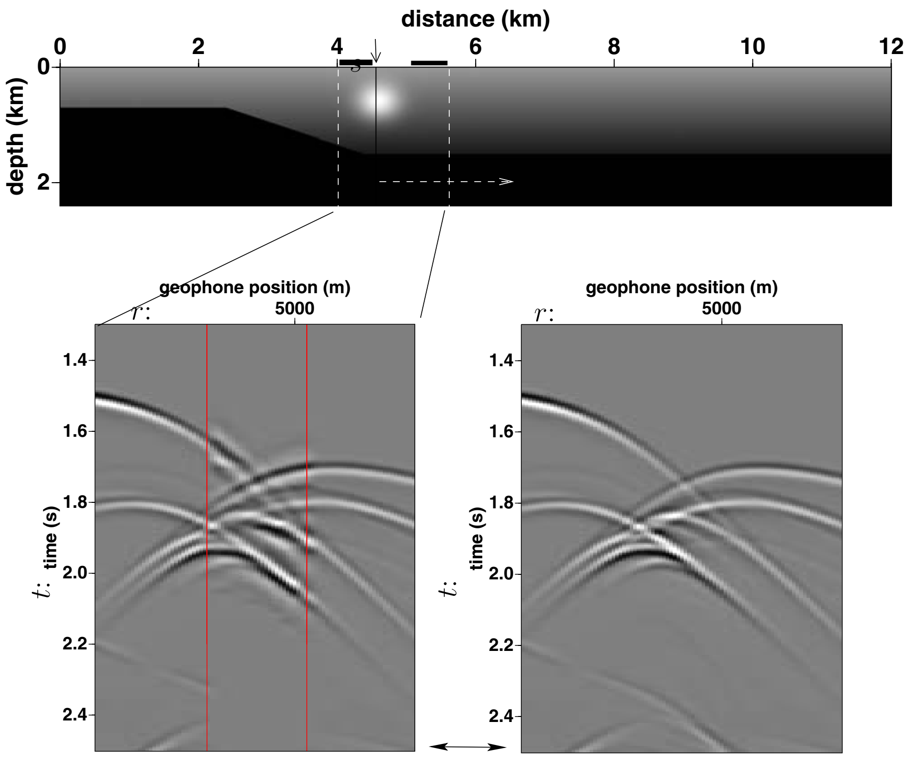

Fig. 2. A numerical example of data continuation: The top figure is the isotropic P-wave velocity model used in the reflection data simulation, containing a Gaussian (low velocity) lens (in white). The bottom two figures both show a shot record (receiver location versus time) with source location at the black vertical line in top figure; the left record shows the outcome of continuation (the data in between the two vertical lines was missing) while the right record shows the original simulated data

displacement component, most energy in the wavefield can be attributed to P waves. For one value of s the synthetic data as a function of r and t are shown in Fig. 2 (bottom, right). We set $T = 3$s and $h_0 = 500$m. The input to the continuation (u_0) were the data with r values in between the black vertical lines removed Fig. 2 (bottom, left). The result of the continuation is plotted in between the black vertical lines of the same figure and should be compared with Fig. 2 (bottom, right).

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