The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary

Matti Lassas*
Rolf Nevanlinna Institute
University of Helsinki
Helsinki, P. O. Box 4
FIN-00014, Finland

Michael Taylor†
Mathematics Department
University of North Carolina
Chapel Hill NC 27599-3250
USA

Gunther Uhlmann‡
Department of Mathematics
University of Washington
Seattle, WA 98195-4350
USA

Abstract. *We study the problem of determining a complete Riemannian manifold with boundary from the Cauchy data of harmonic functions. This problem arises in electrical impedance tomography, where one tries to find an unknown conductivity inside a given body from measurements done on the boundary of the body. Here, we show that one can reconstruct a complete, real-analytic, Riemannian manifold M with compact boundary from the set of Cauchy data, given on a non-empty open subset Γ of the boundary, of all harmonic functions with Dirichlet data supported in Γ, provided $\dim M \geq 3$. We note that for this result we need no assumption on the topology of the manifold other than connectedness, nor do we need a priori knowledge of all of ∂M.*

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1 Introduction

In this paper we study the inverse problem of determining a real-analytic Riemannian manifold (M, g) from the knowledge of Cauchy data of harmonic functions on $\Gamma \subset \partial M$. Here we assume that the manifold M is an n-dimensional, connected, complete, real-analytic Riemannian manifold with nonempty boundary ∂M. We assume $n \geq 3$. We make a mild assumption on the regularity of ∂M, namely that each point of ∂M be a regular boundary point for the Dirichlet problem, in the sense of Wiener. We assume ∂M has an open subset $\Gamma \subset \partial M$, and we assume more regularity on Γ. In fact we assume Γ is a real analytic piece of boundary, and that the metric tensor of M is real analytic up to Γ.

To formulate the problem precisely, let $f \in C(\partial M)$ with supp $f \subset \Gamma$. Let $u \in C(\overline{M}) \cap C^\infty(M)$ be the solution of Laplace-Beltrami equation

$$\Delta_g u = 0 \text{ on } M, \quad u|_{\partial M} = f. \tag{1.1}$$

When $\overline{M}$ is compact, this solution is unique. In the more general case considered here, we specify the solution as follows. Let $\overline{M}_j \subset\subset \overline{M}_{j+1}$ be an exhaustion of $\overline{M}$ by compact manifolds $\overline{M}_j$, with boundary $\partial M_j = \partial M \cup \Sigma_j$, and consider u_j, solving

$$\Delta_g u = 0 \text{ on } M_j, \quad u_j|_{\partial M} = f, \quad u_j|_{\Sigma_j} = 0. \tag{1.2}$$

Take $f \geq 0$. Then u_j is positive, monotonically increasing and bounded from above by sup f, so the sequence converges to a unique limit, which we denote $\mathrm{PI}\, f$. The limit is seen to be independent of the choice of exhaustion of $\overline{M}$, and it extends uniquely to a linear map $\mathrm{PI} : C(\partial M) \to C(\overline{M})$, which is the solution operator we take for (1.1).

Now we describe the inverse problem. We assume that we know the Cauchy data on Γ of all possible solutions of (1.1), for f supported on Γ. Equivalently, we assume known the Dirichlet-to-Neumann map

$$\Lambda_{g,\Gamma} : f \mapsto \partial_\nu \, \mathrm{PI}\, f|_\Gamma \tag{1.3}$$

where ∂_ν is the exterior normal derivative of u and $f \in C_0^\infty(\Gamma)$. In this paper we address the question: *Is it possible to determine (M, g) by knowing a non-empty open subset of the boundary $\Gamma \subset \partial M$ as a differentiable manifold and the boundary operator $\Lambda_{g,\Gamma}$?* We show that this is the case if (M, g) is real-analytic up to Γ in dimension $n \geq 3$. See Theorem 1.1 below for a precise statement.

This problem arises in Electrical Impedance Tomography (EIT). The question in EIT is whether one can determine the (anisotropic) electrical conductivity of a medium Ω in Euclidean space by making voltage and current measurements at the boundary of the medium. Calderón proposed this problem [C] motivated by geophysical prospection. The electrical conductivity in an open subset Ω of $\mathbb{R}^n$ is represented by a positive definite matrix $\gamma = (\gamma^{ij})$. The Dirichlet-to-Neumann map is the voltage to current map, which maps a voltage potential at the boundary of the medium to the induced current flux at the boundary of the medium. The reason why the Riemannian manifolds appear naturally in the study of EIT (see [LeU]) is that in dimension $n \geq 3$ the EIT problem is equivalent to the problem of determining a Riemannian metric g from Λ_g with

$$g_{ij} = (\det \gamma^{kl})^{1/(n-2)}(\gamma^{ij})^{-1}. \tag{1.4}$$

Let us denote the closure of Ω by $\overline{\Omega}$. Then, if $\psi : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism which is the identity at the boundary, $\Lambda_{\psi^* g} = \Lambda_g$. Thus, the electrical boundary measurements for the conductivities corresponding to metrics g and $\psi^* g$ are identical. This means that one cannot reconstruct uniquely the conductivity from boundary measurements. The natural conjecture is that this diffeomorphism is the only obstruction to unique identifiability of the Riemannian metric (see Conjecture A in [LeU]).

Because of these observations, the EIT-problem can be studied in two parts. The first one is to determine the abstract manifold structure which corresponds to the conductivity. The second is to choose an appropriate embedding of the abstract manifold structure to the Euclidean domain Ω. We note that the second step is not unique, but in many cases one may have complementary information which can be used to make the choice of the embedding unique.

For "isotropic" metrics on $\mathbb{R}^n$ (i.e., $g_{ij} = \alpha(x)\delta_{ij}$ with δ_{ij} the Kronecker delta and α a positive function) the conjecture in dimension $n \geq 3$, is that the metric can be identified uniquely from the Dirichlet-to-Neumann map. This was proved for smooth isotropic metrics g_{ij} in $\Omega \subset \mathbb{R}^n$, $n \geq 3$ in [SU] and for $C^{3/2}$ isotropic metrics in [PPU].

In the anisotropic case in dimension $n \geq 3$ Lee and Uhlmann proved in [LeU] that the conjecture is valid for simply connected real-analytic Riemannian manifolds with boundary that are in addition geodesically convex.

In the two-dimensional case, the EIT problem for domains in Euclidean space was solved by Adrian Nachman in [N] for isotropic C^2-conductivities on $\Omega \subset \mathbb{R}^2$. This was extended to Lipschitz conductivities in [BU]. The problem for anisotropic conductivities in $\Omega \subset \mathbb{R}^2$ can be reduced to the case of isotropic ones by using an analog of isothermal coordinates as observed in [S].

In [LaU] the inverse problem is studied for a compact Riemannian manifold, assumed to be real-analytic in the case $n \geq 3$ and C^∞-smooth in the two dimensional case. There it is shown that the Dirichlet-to-Neumann map, measured only on an open subset of the boundary, determines the isometry class of the manifold uniquely in dimensions $n \geq 3$ and the conformal structure in dimension $n = 2$.

Here we extend the results of [LaU] in several respects. In particular we relax the hypothesis that $\overline{M}$ be compact to a completeness hypothesis, though we continue to assume ∂M is compact. Also we allow ∂M to be fairly rough away from Γ. We also have produced a different approach to the inverse problem, avoiding the use of sheaves. As in [LaU] the Green functions play a central role, but the role here lies in providing an infinite dimensional embedding of M that reveals its geometry. The following is our main result.

Theorem 1.1 *Let M_1 and M_2 be complete, connected, real-analytic Riemannian manifolds, with boundary. Assume the manifolds M_j have dimension $n \geq 3$. Assume the boundaries ∂M_j are compact and all boundary points are regular, in the sense of Wiener.*

Furthermore, assume that ∂M_1 and ∂M_2 contain a non-empty open set $\Gamma_1 = \Gamma_2 = \Gamma$, on which each boundary is real analytic, with the metric tensors analytic up to Γ_j.

Finally, assume the Dirichlet-to-Neumann maps Λ_{Γ,g_1} and Λ_{Γ,g_2} coincide.

Then M_1 and M_2 are isometric.

Strictly speaking in the statement above we mean by the set Γ the sets $\Gamma_1 \subset \partial M_1$ and $\Gamma_2 \subset \partial M_2$, which are identified by a diffeomorphism.

We outline the structure of the rest of this paper. In section 2 we extend the manifolds M_j to larger manifolds $\widetilde{M}_j$ and we show that the Green functions are determined on an open subset of $\widetilde{M}_j \setminus M_j$ if we know the Dirichlet-to-Neumann map on Γ. In section 3 we show that the Green functions provide embeddings of $\widetilde{M}_j$ into a Sobolev space. These embeddings will provide the appropriate isometry of the two manifolds if the Dirichlet-to-Neumann maps are the same on Γ. In section 4 we give some complementary results and examples, involving complete manifolds without boundary and noncompact, complete Riemann surfaces with boundary.

2 Construction of the metric on and near Γ

Near Γ we use boundary normal coordinates (s, h) where $s \in \Gamma$ is the point nearest to x and $h = \mathrm{dist}(x, s)$. Let $\xi = \xi(s)$ be local coordinates of Γ near a given boundary point $s_0 \in \Gamma$. Thus near s_0 we have in M coordinates $(\xi, h) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$. In these coordinates we represent the metric by the tensor $g_{ij}(\xi, h)$, $i, j = 1, \ldots, n$.

In [LeU] it is shown for a compact Riemannian manifold of the dimension $n \geq 3$ that $\Lambda_{g,\Gamma}$ determines all the normal derivatives $\partial_h^k g_{ij}(\xi, 0)$, $k \geq 0$ of the metric tensor at Γ. This result is based on the local fact that when $\Lambda_{g,\Gamma}$ is considered as a pseudodifferential operator, its symbol determines the derivatives of the metric in boundary normal coordinates. Because of the local nature of this construction, this result is valid also for non-compact manifolds.

Let $x_0 \in \Gamma$ and consider boundary normal coordinates with a coordinate function $\phi : V \to \{z \in \mathbb{R}^n : \ z_n \geq 0\}$, $\phi(x_0) = 0$ in a neighborhood $V \subset M$ of x_0. Then the metric tensor g_{ij} in these coordinates is a real-analytic function. Since the Taylor series of g_{ij} converge in some small ball $B(0, \rho)$, we can consider M as a subset of a larger real-analytic manifold $\widetilde{M}$ which has x_0 as an interior point. For instance the manifold $\widetilde{M}$ can be obtained by taking the disjoint union of M and the ball $B(0, \rho)$ with metric g_{ij}, and identifying the points in $\{z \in B(0, \rho) : \ z_n \geq 0\}$ with points of M. We set $\mathcal{O} = B(0, \rho) \subset \widetilde{M}$ and $U = \mathcal{O} \setminus \overline{M}$.

Since the partial derivatives of the metric g_{ij} in boundary normal coordinates near x_0 are determined, we have determined the metric g_{ij} uniquely in $\mathcal{O}$.

By [LT] there exists a minimal non-negative Green's function of $\widetilde{M}$, satisfying

$$\begin{aligned} \Delta_g G(\cdot\,, y) &= -\delta_y \text{ in } \widetilde{M}, \\ G(\cdot\,, y)|_{\partial \widetilde{M}} &= 0. \end{aligned} \tag{2.1}$$

By [LT], the minimal non-negative Green function can be obtained as the limit

$$G(\cdot\,, y) = \lim_{j \to \infty} G^j(\cdot\,, y) \tag{2.2}$$

where $G^j(x, y)$ are the Dirichlet Green functions of an exhaustion $\widetilde{M}_j$ of $\widetilde{M}$, such as described in the introduction. As shown in [LT], as long as

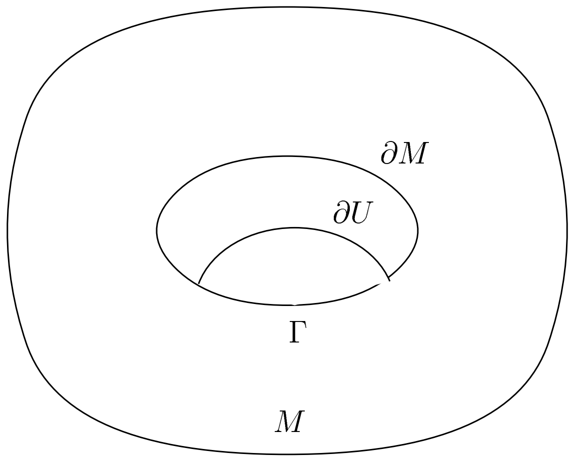

Figure 1: The complete manifold M, the set $\Gamma \subset \partial M$ and the extension of the manifold over Γ.

$\widetilde{M}$ is complete with nonempty compact boundary, the increasing sequence $G^j(\cdot, y)$ has a uniform upper bound on $\widetilde{M} \setminus \{y\}$. Given this, standard interior regularity estimates imply that

$$\lim_{j\to\infty} ||G(\cdot\,,\cdot\,) - G^j(\cdot\,,\cdot\,)||_{C^m(B\times B)} = 0 \tag{2.3}$$

for any compact $B \subset \widetilde{M}^{\rm int}$ and $m \geq 0$. It holds with $m = 0$ for any compact $B \subset \widetilde{M}$.

We recall some properties of the Green function that we will need later. It is known that the Green function $G(x, y)$ is a real-analytic function of x when $x \notin \{y\} \cup \partial\widetilde{M}$ (see, e.g., [H]). Moreover, when x is near to a given y it has the asymptotics (see [T])

$$G(x, y) \;=\; c_n d_{\widetilde{M}}(y, x)^{2-n} + \mathcal{O}(d_{\widetilde{M}}(y, x)^{3-n}) \tag{2.4}$$

where the constants $c_n \neq 0$ depend only on n and $d_{\widetilde{M}}$ is the distance in $\widetilde{M}$. When $n = 3$, the remainder might also include a log term.

Let us next consider two manifolds M_1 and M_2 for which we have identified $\Gamma_1 = \Gamma_2 = \Gamma$ and $\Lambda^1_{g_1,\Gamma} = \Lambda^1_{g_2,\Gamma}$. Using the previous construction of the set U and the metric tensor on U, which is the same for both manifolds, we can attach this set and the metric on it to both manifolds, i.e.,

$$\widetilde{M}_1 = M_1 \cup U,\;\; \widetilde{M}_2 = M_2 \cup U.$$

Now we consider the minimal non-negative Green functions of $\widetilde{M}_j$, satisfying

$$\begin{aligned} \Delta_j G_j(\cdot\,,y) &= -\delta_y \text{ in } \widetilde{M}_j, \\ G_j(\cdot\,,y)|_{\partial\widetilde{M}_j} &= 0, \end{aligned} \tag{2.5}$$

where Δ_j denotes the Laplace-Beltrami operator on $(\widetilde{M}_j, g_j)$.

Lemma 2.1 *The Green functions $G_j(x,y)$ satisfy*

$$G_1(x,y) = G_2(x,y), \quad (x,y) \in U \times U.$$

Proof. Pick $y \in U$, and define $V_0 \in C(\partial M_2)$ by

$$V_0(x) = G_1(x,y), \quad x \in \Gamma \cap \mathcal{O}; \quad V_0(x) = 0, \quad x \in \partial M_2 \setminus \mathcal{O}. \tag{2.6}$$

Now let V be the minimal non-negative solution on M_2 to

$$\Delta_2 V = 0 \text{ on } M_2, \quad V = V_0 \text{ on } \partial M_2.$$

The hypothesis that $\Lambda_{\Gamma,g_1} = \Lambda_{\Gamma,g_2}$ implies

$$\nabla_x V(x) = \nabla_x G_1(x,y), \quad x \in \Gamma,$$

and hence, by unique continuation, V continues analytically to $\tilde{V} \in C^\infty(\widetilde{M}_2 \setminus \{y\})$, with $\tilde{V}(x) = G_1(x,y)$ for $x \in U \setminus \{y\}$. This satisfies

$$\Delta_2 \tilde{V} = -\delta_y \text{ on } \widetilde{M}_2, \quad \tilde{V}|_{\partial\widetilde{M}_2} = 0.$$

We claim that $\tilde{V}(x) = G_2(x,y)$ for $x \in \widetilde{M}_2 \setminus \{y\}$. This is obvious in case $\widetilde{M}_2$ has compact closure. Under the more general hypothesis of completeness that we have made, we need to argue a little more.

What it clear from the analysis done so far is that $0 \le G_2(x,y) \le \tilde{V}(x)$. Restricting to $x \in \Gamma$, we have $0 \le G_2(x,y) \le G_1(x,y)$, for $x \in \Gamma$. Now we can switch the roles of $\widetilde{M}_1$ and $\widetilde{M}_2$, and deduce that

$$G_1(x,y) = G_2(x,y), \text{ for } y \in U, \ x \in \Gamma.$$

Hence, taking $W(x) = G_1(x,y) - G_2(x,y)$ for $x \in U$, we have

$$\Delta_2 W = 0 \text{ on } U, \quad W|_{\partial U} = 0,$$

so $W = 0$ on U, which proves Lemma 2.1. ∎

3 Embedding of the manifold into a Sobolev space

We will prove Theorem 1.1 via certain embeddings of $\widetilde{M}_j$ into a Sobolev space, defined by the Green functions G_j. We introduce the maps

$$\mathcal{G}_j : \widetilde{M}_j \longrightarrow H^s(U), \quad (\text{any } s < 2 - n/2), \tag{3.1}$$

defined by

$$\mathcal{G}_j(x)(y) = G_j(x, y), \quad x \in \widetilde{M}_j,\ y \in U. \tag{3.2}$$

Since $\delta_x \in H^{s-2}(U)$ depends continuously on x, we conclude that $\mathcal{G}_j(x) \in H^s(U)$ depends continuously on x. Similarly, we have for $s < 1 - n/2$ that the maps $\mathcal{G}_j, j = 1, 2$ are C^1. In the following we assume that $s < 1 - n/2$. Note that the derivative of $\mathcal{G}_j$

$$D\mathcal{G}_j(x) : T_x\widetilde{M}_j \longrightarrow H^s(U) \tag{3.3}$$

is given by

$$D\mathcal{G}_j(x)v = vG_j(x, \cdot) = v^k \frac{\partial}{\partial x^k} G_j(x, \cdot)|_x, \tag{3.4}$$

where $v = v^k(\partial/\partial x^k) \in T_x\widetilde{M}$.

Furthermore, since $G_j(x, y)$ are real-analytic functions of x in $\widetilde{M}_j \setminus \{y\}$, we see that the maps G_j are real analytic on M_j.

Lemma 3.1 *The map $D\mathcal{G}_j(x)$ is injective for each $x \in \widetilde{M}_j$.*

Proof. Suppose that the map $\mathcal{G}_j(x)$ annihilates a nonzero $v \in T_x\widetilde{M}_j$, then $v^k(\partial/\partial x^k)G_j(x, y) = 0$ for all $y \in U$. This implies that $v^k(\partial/\partial x^k)G_j(x, y) = 0$ for all $y \in \widetilde{M}_j \setminus \{x\}$ by real-analyticity of the Green functions. By considering paths $x(t)$ for which $x(0) = y$ and the functions $G_j(x(t), y)$ along them, we obtain a contradiction with the asymptotics (2.4). $\square$

We use this result to show:

Lemma 3.2 *The map $\mathcal{G}_j : \widetilde{M}_j \to H^s(U)$ is an embedding.*

Proof. It remains to show that $x_1 \neq x_2$ in $\widetilde{M}_j$ implies that $\mathcal{G}_j(x_1) \neq \mathcal{G}_j(x_2)$. Assume this is not the case, then

$$G_j(x_1, y) = G_j(x_2, y) \tag{3.5}$$

for all $y \in U$, hence, by analyticity, (3.5) holds for all $y \in \widetilde{M}_j \setminus \{x_1, x_2\}$. But $G_j(x_1, \cdot)$ is singular only at $y = x_1$ and $G_j(x_2, \cdot)$ is singular only at $y = x_2$. From this we conclude that $x_1 = x_2$. $\square$

Our next goal is to establish the following result, which will imply Theorem 1.1.

Theorem 3.3 *Assume that $\mathcal{G}_1$ and $\mathcal{G}_2$ coincide in the set U. Then the sets $\mathcal{G}_1(\widetilde{M}_1)$ and $\mathcal{G}_2(\widetilde{M}_2)$ are identical subsets of $H^s(U)$. Moreover, the map $\mathcal{G}_2 \circ \mathcal{G}_1^{-1} : \widetilde{M}_1 \to \widetilde{M}_2$ is an isometry.*

The proof of Theorem 3.3 occupies the remainder of this section.

First we show that $\mathcal{G}_1(\widetilde{M}_1) \subset \mathcal{G}_2(\widetilde{M}_2)$. Let

$$\begin{aligned} N(\varepsilon_0) &= \{x \in \widetilde{M}_1 : \ d_{\widetilde{M}_1}(x, \partial\widetilde{M}_1) \leq \varepsilon_0\}, \\ C(\varepsilon_0) &= \{x \in \widetilde{M}_1 : \ d_{\widetilde{M}_1}(x, \partial\widetilde{M}_1) > \varepsilon_0\} \end{aligned} \tag{3.6}$$

where $\varepsilon_0 > 0$ is small enough so that $C(\varepsilon_0)$ is connected. Let, $x_0 \in U \cap C(\varepsilon_0)$ and $B_1 \subset C(\varepsilon_0)$ be the largest connected open set containing x_0 such that $\mathcal{G}_1(x) \in \mathcal{G}_2(\widetilde{M}_2)$ for $x \in B_1$. Therefore, we can define the map

$$J = \mathcal{G}_2^{-1}\mathcal{G}_1 : B_1 \to \widetilde{M}_2.$$

Let $D_1 \subset B_1$ be the largest connected open set containing x_0 for which J is a local isometry, that is, $g_1 = J^* g_2$. Finally, let x_1 be the closest point of $\widetilde{M}_1 \setminus (N(\varepsilon_0) \cup D_1)$ to x_0. Clearly we have that $x_1 \in \partial D_1$.

Lemma 3.4 *Let $x_1 \in \partial D_1$ be the closest point in $\widetilde{M}_1 \setminus (N(\varepsilon_0) \cup D_1)$ to x_0. Then there exists $x_2 \in \widetilde{M}_2^{int}$ such that $\mathcal{G}_2(x_2) = \mathcal{G}_1(x_1)$. Moreover, there is a sequence $p_k \in D_1$ such that*

$$\lim_{k\to\infty} p_k = x_1, \quad \lim_{k\to\infty} J(p_k) = x_2. \tag{3.7}$$

Proof. We know that there exist $p_k \in D_1,\ q_k \in \widetilde{M}_2$ such that $p_k \to x_1$ and $\mathcal{G}_2(q_k) = \mathcal{G}_1(p_k)$. If some sequence $\{q_k\}$ has a limit point x_2 in $\widetilde{M}_2^{int}$, then we are done by the continuity of $\mathcal{G}_1$ and $\mathcal{G}_2$. If there are no limit points, then we have that for all sequences $\{q_k\}$ either

$$d_{\widetilde{M}_2}(q_k, x_0) \longrightarrow \infty, \tag{3.8}$$

or

$$q_k \longrightarrow q_0 \in \partial \widetilde{M}_2. \tag{3.9}$$

First we consider the case (3.8).

If $x_1 \notin \partial N(\varepsilon_0)$, let ξ be the direction of the shortest curve from x_1 to x_0 in $\widetilde{M}_1$. Since D_1 is open and connected and thus path-connected, it is easy to see that there is a path $\gamma([0,l])$ from x_1 to x_0, $\gamma'(0) = \xi$ such that $\gamma(]0,l]) \subset D_1$.

On the other hand, if $x_1 \in \partial N(\varepsilon_0)$, let $\eta \in T_{x_1}\widetilde{M}_1$ be the direction of the shortest curve from x_1 to x_0 in $\widetilde{M}_1$ and $\omega \in T_{x_1}\widetilde{M}_1$ be the direction of the shortest curve from x_1 to the boundary $\widetilde{M}_1$. Then the directions η and ω cannot be the same. Therefore there is $\xi \in T_{x_1}\widetilde{M}_1$ such that $\langle \xi, \omega \rangle < 0$ and $\langle \xi, \eta \rangle > 0$. Since x_1 is a nearest point of $\widetilde{M}_1 \setminus (N(\varepsilon_0) \cup D_1)$ to x_0, we see that again that there is a path $\gamma([0,l])$ from x_1 to x_0, $\gamma'(0) = \xi$ such that $\gamma(]0,l]) \subset D_1$.

Since the map J is isometry in D_1, the length of the paths γ and $J(\gamma)$ is the same. Thus, for the sequence $p_k = \gamma(l - 1/k) \subset D_1$ and $q_k = J(p_k)$ the distance $d_{\widetilde{M}_2}(q_k, x_0) \leq \text{length}(\gamma([0,l]))$ is uniformly bounded. Thus there is a sequence for which (3.8) is not valid. Let us consider this sequence. If the condition (3.9) is valid, $G_2(q_k, y) \to G_2(q_0, y) = 0$ in $H^s(U)$ (given q_0 a regular boundary point), which yields that $\mathcal{G}_2(q_k) \to 0$. This would give $\mathcal{G}_1(p_k) \to 0$, and hence

$$\mathcal{G}_1(x_1) = 0.$$

Therefore

$$G_1(x_1, y) = 0$$

for all $y \in U$, and by unique continuation $G_1(x_1, y) = 0$ for all $y \neq x_1$. Again, by using the asymptotics of the Green functions near $y = x_1$, we obtain a contradiction, which shows that the condition (3.9) is not valid.

Thus the limit point $x_2 \in \widetilde{M}_2^{int}$ exists and the limit (3.7) is valid. $\square$

By the previous considerations, the Green functions have the following property:

Lemma 3.5 *For each nonempty open set $\Omega \subset U$, the maps*

$$\mathcal{G}_j^\Omega : \widetilde{M}_j \longrightarrow H^s(\Omega) \quad (s < 1 - n/2), \tag{3.10}$$

given by composing (3.1)–(3.2) with the operation of restriction to Ω, are embedding. These maps are real-analytic on $\widetilde{M}_j \setminus \overline{\Omega}$.

Note that, given $x_j \in \widetilde{M}_j$, we have

$$\mathcal{G}_1(x_1) = \mathcal{G}_2(x_2) \Longleftrightarrow \mathcal{G}_1^\Omega(x_1) = \mathcal{G}_2^\Omega(x_2). \tag{3.11}$$

Now let us get back to $x_1 \in \partial D_1 \cap \widetilde{M}_1^{int}$ which is a closest point of $\widetilde{M}_1 \setminus (N(\varepsilon_0) \cup D_1)$ to x_0. We have $x_2 \in \widetilde{M}_2^{int}$ with $\mathcal{G}_2(x_2) = \mathcal{G}_1(x_1)$, hence

$$\mathcal{G}_2^\Omega(x_2) = \mathcal{G}_1^\Omega(x_1) = u \in H^s(\Omega). \tag{3.12}$$

Pick $\Omega \subset U$ disjoint from x_1 (in $\widetilde{M}_1$) and from x_2 (in $\widetilde{M}_2$). Then $\mathcal{G}_j^\Omega$ is an analytic embedding in a neighborhood of x_j. Let us consider the image of the map $\mathcal{G}_j^\Omega$, denoted by $\mathcal{R}(\mathcal{G}_j^\Omega)$, in $H^s(\Omega)$. We show next that their tangent spaces coincide at the point u, that is,

$$T_u\mathcal{R}(\mathcal{G}_1^\Omega) = T_u\mathcal{R}(\mathcal{G}_2^\Omega) \subset H^s(\Omega). \tag{3.13}$$

Indeed, let $v = (\mathcal{G}_1^\Omega)(q)$ where q is an interior point of D_1, and let $p = J(q) \in \widetilde{M}_2$ be the point for which $\mathcal{G}_2^\Omega(p) = v$. By assumption we know that $\mathcal{G}_1^\Omega$ and $\mathcal{G}_2^\Omega \circ J$ coincide in D_1, and hence their differentials coincide, too. By previous considerations, we know that there is a sequence $p_k \in D_1$ such that $p_k \to x_1$ and $q_k = J(p_k) \to x_2$. Denote $v_k = \mathcal{G}_1^\Omega(p_k)$. Then

$$T_{v_k}\mathcal{R}(\mathcal{G}_1^\Omega) = D\mathcal{G}_1^\Omega(T_{p_k}\widetilde{M}_1) = D\mathcal{G}_2^\Omega(T_{q_k}\widetilde{M}_2) = T_{v_k}\mathcal{R}(\mathcal{G}_2^\Omega).$$

Since the differential DG_1^Ω and $D\mathcal{G}_2^\Omega$ are continuous on $\widetilde{M}_1$ and $\widetilde{M}_2$, we conclude that

$$D\mathcal{G}_1^\Omega(T_{x_1}\widetilde{M}_1) = D\mathcal{G}_2^\Omega(T_{x_2}\widetilde{M}_2).$$

Hence there is a finite dimensional space

$$\mathcal{V} = T_u\mathcal{R}(\mathcal{G}_1^\Omega) = T_u\mathcal{R}(\mathcal{G}_2^\Omega) \subset H^s(\Omega). \tag{3.14}$$

Let $\mathcal{L}$ denote a linear subspace orthogonal to $\mathcal{V}$ in the inner product of $H^s(\Omega)$. Now, let

$$P : H^s(\Omega) \to \mathcal{V}$$

be the orthogonal projection to the space $\mathcal{V} \subset H^s(\Omega)$. Consider the map

$$P\mathcal{G}_j^\Omega : \widetilde{M}_j \to \mathcal{V}, \quad x \mapsto P(G_j(x,\cdot)).$$

The derivative of $P\mathcal{G}_j^\Omega$ at x_j is the map $P \circ (D\mathcal{G}_1^\Omega)$ which is surjective and thus it is invertible. Hence it follows from the implicit function theorem, that there is an open neighborhood $\mathfrak{A} \subset \mathcal{V}$ of Pu and a real-analytic map $H_j : \mathfrak{A} \to \widetilde{M}_j$ such that

$$P(G_j(H_j(v),\cdot)) = v.$$

Thus we can represent the graph of the function $\mathcal{G}_j^\Omega(\widetilde{M}_2)$ locally, near u, as graphs of real-analytic functions

$$\Phi_j : \mathfrak{A} \to \mathcal{L}, \quad \Phi_j(v) = G_j(H_j(v),\cdot). \tag{3.15}$$

The real-analytic maps Φ_j coincide in an open subset $P\mathcal{G}_1(D_1) \subset \mathfrak{A}$ and thus in the whole set $\mathfrak{A}$. This yields that $x_1 = H_1(Pu)$ is an interior point of the set B_1. Moreover, the maps $\mathcal{G}_1^\Omega$ and $\mathcal{G}_2^\Omega \circ J$ coincide near x_1. Now the map J has the representation $J = H_2 \circ H_1^{-1}$ and it is real analytic. We have shown that the Green functions $G_1(x,y)$ and $G_2(J(x),J(y))$ coincide when x is near x_1 and $y \in \Omega$. Since J is real-analytic in D_1 and near x_1 and also the Green functions are real-analytic, it follows that $G_1(x,y)$ and $G_2(J(x),J(y))$ coincide when x and y are near to x_1. By analyzing the behavior of the Green functions when x is near y, we can construct the metric tensor in local coordinates. Therefore the map J is an isometry near x_1 which is a contradiction with the assumption that x_1 is the boundary point of D_1. Since ε_0 in (3.6) can be chosen arbitrarily small, this proves Theorem 3.3 and hence Theorem 1.1.

4 Further results and examples

Here we provide some complements to the results treated in the previous sections. First we consider the case of complete, noncompact Riemannian manifolds without boundary. In such a case there might not exist a positive Green function, but nevertheless as shown in [LT] there is a symmetric Green function (possibly non-unique). The following result can be regarded as an extension of Theorem 3.3.

Theorem 4.1 *Assume M_1 and M_2 are complete, connected Riemannian manifolds without boundary, of dimension $n \geq 3$, with real analytic metric tensors, with Green functions G_1 and G_2. Assume there exists a nonempty open $U = U_1 = U_2$ such that $G_1 = G_2$ on $U \times U$. Then M_1 and M_2 are isometric.*

The proof of this result is basically a subset of the proof of Theorem 3.3. The only difference between the results is that here we assume the boundaries are empty, and we do not have the positivity of the Green functions to work with. However, in the proof of Theorem 3.3 the one place this positivity played a role was in an argument contradicting the possibility that $q_k \to q_0 \in \partial \widetilde{M}_2$. In the present case this phenomenon need not be dealt with, so we have a proof of Proposition 4.1.

Now we bring back the boundary but drop down to dimension 2. As recalled in the introduction, it was shown in [LaU] that if $\overline{M}$ is a compact, connected, 2-dimensional Riemannian manifold, with nonempty smooth boundary, and $\Gamma \subset \partial M$ a nonempty open set, then the Dirichlet-to-Neumann map on Γ determines the conformal class of M. Here we give examples of complete 2-dimensional manifolds $\overline{M}_j$ with boundary that are not conformally equivalent but that have equivalent Dirichlet-to-Neumann maps.

For the sake of simplicity, we start as follows. Let $\overline{M}$ be a compact 2-dimensional Riemannian manifold, with smooth boundary. In fact, we will even suppose the boundary ∂M is analytic and the metric analytic up to ∂M. Again for simplicity, let us take $\Gamma = \partial M$. We have a uniquely defined solution operator

$$\mathrm{PI} : C(\partial M) \to C(\overline{M})$$

to the Dirichlet problem $\Delta_g u = 0$ on M, $u|_{\partial M} = f$, where g denotes the given Riemannian metric tensor on M.

Now let $K \subset M$ be any compact subset of logarithmic capacity zero. On $\overline{M} \setminus K$ we can take a complete Riemannian metric tensor, conformally equivalent to g, say $h = \varphi g$, with positive $\varphi \in C^\infty(\overline{M} \setminus K)$. We can even arrange that φ be real analytic on $\overline{M} \setminus K$. To get this, let $\widetilde{M}$ be a neighborhood of $\overline{M}$. Then $\widetilde{M} \setminus K$ is conformally covered by the Poincaré disk. One can take $h_0 = \psi g$ to be the metric on $\widetilde{M} \setminus K$ induced by the Poincaré metric, then restricted to $\overline{M} \setminus K$. Furthermore, we can find positive β, real analytic on $\overline{M}$, equal to $1/\psi$ on ∂M, e.g., $\beta = \mathrm{PI}(1/\psi)$, and then taking $\varphi = \beta\psi$ gives a complete Riemannian metric tensor $h = \varphi g$ on $\overline{M} \setminus K$, real analytic, with $\varphi = 1$ on ∂M.

Given this metric $h = \varphi g$ on $\overline{M} \setminus K$, we have the solution operator to the Dirichlet problem

$$\mathrm{PI}_K : C(\partial M) \to C(\overline{M} \setminus K).$$

We claim that, as long as K has logarithmic capacity zero, for all $f \in C(\partial M)$,

$$\mathrm{PI}_K f = (\mathrm{PI}\, f)\big|_{M \setminus K}. \tag{4.1}$$

In fact, it is clear that $\mathrm{PI}_K\, f$ is bounded and continuous on $M \setminus K$. On the other hand it is classical that K is a removable set of singularities for bounded harmonic functions; cf. [Car]. This established (4.1). As long as $\varphi = 1$ on ∂M, this implies the Dirichlet-to-Neumann map for $\overline{M} \setminus K$, with metric $h = \varphi g$, coincides with that for $\overline{M}$.

References

[BU] Brown, R. and Uhlmann, G., Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions, *Comm. PDE* **22** (1997), 1009–1027.

[C] Calderón, A.P., On an inverse boundary value problem, *Seminar on Numerical Analysis and its applications to continuum physics, Soc. Brasileira de Matemática*, Río de Janeiro (1980), 65–73.

[Car] Carleson, L., *Selected problems on exceptional sets*, Van Nostrand, Princeton, N.J., 1967.

[G] Grigor'yan, A., Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bull. AMS* **36** (1999), 135-249.

[H] Hörmander, L., *The analysis of linear partial differential operators III,* Springer-Verlag, New York, 1985.

[I] Isakov, V., *Inverse problems for partial differential equations.* Springer-Verlag, New York, 1998.

[KV] Kohn, R. and Vogelius, M., Determining conductivity by boundary measurements II. Interior results, *Comm. Pure Appl. Math,* **38** (1985), 643–667.

[LaU] Lassas, M. and Uhlmann, G., On determining a Riemannian manifold from the Dirichlet-to-Neumann map, to appear Annales Scientifiques d l'Ecole Normale Superieure.

[LeU] Lee, J., Uhlmann, G., Determining anisotropic real-analytic conductivities by boundary measurements, *Comm. Pure. Appl. Math,* **42** (1989), 1097-1112.

[LT] Li, P., Tam, L.-F., Symmetric Green's functions on complete manifolds, *Amer. J. Math.* **109** (1987), 1129-1154.

[N] Nachman, A., A global uniqueness for a two-dimensional inverse problem, *Ann. Math* (2) **142** (1996), 71-96.

[PPU] Päivärinta, L., Panchencko, A., Uhlmann, G., Complex geometrical optics solutions for Lipschitz conductivities, preprint.

[P] Petersen, P., *Riemannian geometry,* Springer-Verlag, New York, 1998.

[S] Sylvester, J., An anisotropic inverse boundary value problem, *Comm. Pure. Appl. Math.*, **38** (1990), 201-232.

[SU] Sylvester, J. and Uhlmann, G., A uniqueness theorem for an inverse boundary problem, *Ann. Math.* (2)**125** (1987), 153-169.

[T] Taylor, M., *Partial differential equations, I–III*, Springer-Verlag, New York, 1996.

[U] Uhlmann, G., Developments in inverse problems since Calderón's foundational paper, *Harmonic analysis and partial differential equations*, University of Chicago Press (1999) 295–245. Essays in honor of Alberto P. Calderón, University of Chicago Press, edited by M. Christ, C. Kenig and C. Sadosky.