Electrical impedance tomography and Calderón's problem

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Abstract. We survey mathematical developments in the inverse method of Electrical Impedance Tomography which consists in determining the electrical properties of a medium by making voltage and current measurements at the boundary of the medium. In the mathematical literature this is also known as Calderón's problem from Calderón's pioneer contribution [23]. We concentrate this article around the topic of complex geometrical optics solutions that have led to many advances in the field. In the last section we review some counterexamples to Calderón's problems that have attracted a lot of interest because of connections with cloaking and invisibility.

AMS classification scheme numbers: Primary 35R30, 78A46 ; Secondary 58J05, 78A10

Submitted to: *Inverse Problems*

1. Introduction

In 1980 A. P. Calderón published a short paper entitled "On an inverse boundary value problem" [23]. This pioneer contribution motivated many developments in inverse problems, in particular in the construction of "complex geometrical optics" solutions of partial differential equations to solve several inverse problems. We survey some these developments in this paper. The problem that Calderón considered was whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is known as *Electrical Impedance Tomography* (EIT). Calderón was motivated by oil prospection. In the 40's he worked as an engineer for Yacimientos Petrolíferos Fiscales (YPF), the state oil company of Argentina, and he thought about this problem then although he did not publish his results until many years later. For use of electrical methods in geophysical prospection see [147]. Parenthetically Calderón said in his speech accepting the "Doctor Honoris Causa" of the Universidad Autónoma de Madrid that his work at YPF had been very interesting but he was not well treated there; he would have stayed at YPF otherwise [24]. It goes without saying that the bad treatment of Calderón by YPF was very fortunate for Mathematics!

EIT also arises in medical imaging given that human organs and tissues have quite different conductivities [71]. One exciting potential application is the early diagnosis of breast cancer [149]. The conductivity of a malignant breast tumor is typically 0.2 mho which is significantly higher than normal tissue which has been typically measured at 0.03 mho. Another application is to monitor pulmonary functions [62]. See the book [52] and the issue of Physiological Measurement [53] for other medical imaging applications of EIT. This inverse method has also been used to detect leaks from buried pipes [70].

We now describe more precisely the mathematical problem.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary (many of the results we will describe are valid for domains with Lipschitz boundaries). The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the equation for the potential is given by

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega \tag{1}$$

since, by Ohm's law, $\gamma \nabla u$ represents the current flux.

Given a potential $f \in H^{\frac{1}{2}}(\partial\Omega)$ on the boundary the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= 0 \text{ in } \Omega, \\ u\big|_{\partial\Omega} &= f. \end{aligned} \tag{2}$$

The Dirichlet to Neumann map, or voltage to current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega} \tag{3}$$

where ν denotes the unit outer normal to $\partial\Omega$.

The inverse problem is to determine γ knowing Λ_γ. It is difficult to find a systematic way of prescribing voltage measurements at the boundary to be able to find the conductivity. Calderón took instead a different route.

Using the divergence theorem we have

$$Q_\gamma(f) := \int_\Omega \gamma |\nabla u|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) f \, dS \tag{4}$$

where dS denotes surface measure and u is the solution of (2). In other words $Q_\gamma(f)$ is the quadratic form associated to the linear map $\Lambda_\gamma(f)$, and to know $\Lambda_\gamma(f)$ or $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ is equivalent. $Q_\gamma(f)$ measures the energy needed to maintain the potential f at the boundary. Calderón's point of view is that if one looks at $Q_\gamma(f)$ the problem is changed to finding enough solutions $u \in H^1(\Omega)$ of the equation (1) in order to find γ in the interior. We will explain this approach further in the next section where we study the linearization of the map

$$\gamma \xrightarrow{Q} Q_\gamma. \tag{5}$$

Here we consider Q_γ as the bilinear form associated to the quadratic form (4).

In section 2 we describe Calderón's paper and how he used complex exponentials to prove that the linearization of (5) is injective at constant conductivities. He also gave an approximation formula to reconstruct a conductivity which is, a priori, close to a constant conductivity. In section 3 we give an application of Calderón's method to determine cavities and inclusions. In section 4 we describe results about uniqueness, stability and reconstruction, for the boundary values of a conductivity and its normal derivative.

In section 5 we describe the construction by Sylvester and Uhlmann [132], [131] of complex geometrical optics solutions for the Schrödinger equation associated to a bounded potential. These solutions behave like Calderón's complex exponential solutions for large complex frequencies. In section 6 we use these solutions to prove, in dimension $n \geq 3$, a global identifiability result, stability estimates and a reconstruction method for the inverse problem. We also describe an extension of the identifiability result to non-linear conductivities [124] and give other applications of complex geometrical optics solutions.

In section 7 we consider the partial data problem, that is the case when the DN map is measured on a part of the boundary. We describe the results of [75] for the non-linear problem in dimension three or larger. This uses a larger class of CGO solutions, having a non-linear phase function that are constructed using Carleman estimates. We also review the article [29] for the linearized problem with partial data.

In section 8 we consider the two dimensional case. In particular we describe briefly the recent work of Astala and Päivärinta proving uniqueness for bounded measurable coefficients, and the work of Bukhgeim proving uniqueness for a potential from Cauchy data associated to the Schrödinger equation. Finally we describe the work of Imanuvilov, Uhlmann and Yamamoto on the partial data problem [59].

Sections 2-8 deal with the case of *isotropic* conductivities. In section 9 we consider the case of *anisotropic* conductivities, i.e. the conductivity depends also on direction. In two dimensions that there has been substantial progress in the understanding of anisotropic problems since one can usually reduce the problem to the isotropic case by using isothermal coordinates. In dimension three the problem as pointed out in [85] is of geometric nature. We review the results of [83], [31].

In section 10 we describe a connection between Calderón' problems and the problem of recovering the sound speed of a medium from the travel time between boundary points. Finally in section 11 we discuss the ideas of *transformation optics* and invisibility applied to the case of electrostatics.

We end the introduction by mentioning that we don't discuss several other important developments in EIT including the work of Borcea et al on discrete resistors networks and the approximation to the continuous case [17] and also the case when the boundary is unknown [78]. We concentrate mainly in the applications involving complex geometrical optics solutions. For other reviews of EIT see [16], [48], [27] and [140].

2. Calderón's paper

Calderón proved in [23] that the map Q is analytic. The Fréchet derivative of Q at $\gamma = \gamma_0$ in the direction h is given by

$$dQ|_{\gamma=\gamma_0}(h)(f,g) = \int_\Omega h\nabla u \cdot \nabla v \, dx \tag{6}$$

where $u, v \in H^1(\Omega)$ solve

$$\begin{cases} \nabla \cdot (\gamma_0 \nabla u) = \nabla \cdot (\gamma_0 \nabla v) = 0 \text{ in } \Omega \\ u\Big|_{\partial\Omega} = f \in H^{\frac{1}{2}}(\partial\Omega), \quad v\Big|_{\partial\Omega} = g \in H^{\frac{1}{2}}(\partial\Omega). \end{cases} \tag{7}$$

So the linearized map is injective if the products of $H^1(\Omega)$ solutions of $\nabla \cdot (\gamma_0 \nabla u) = 0$ is dense in, say, $L^2(\Omega)$.

Calderón proved injectivity of the linearized map in the case $\gamma_0 =$ constant, which we assume for simplicity to be the constant function 1. The question is reduced to whether the product of gradients of harmonic functions is dense in, say, $L^2(\Omega)$.

Calderón took the following harmonic functions

$$u = e^{x\cdot\rho}, \quad v = e^{-x\cdot\overline{\rho}} \tag{8}$$

where $\rho \in \mathbb{C}^n$ with

$$\rho \cdot \rho = 0. \tag{9}$$

We remark that the condition (9) is equivalent to the following

$$\begin{aligned} &\rho = \frac{\eta + ik}{2}, \eta, k \in \mathbb{R}^n, \\ &|\eta| = |k|, \eta \cdot k = 0. \end{aligned} \tag{10}$$

Then plugging the solutions (8) into (6) we obtain if $dQ|_{\gamma_0=1}(h) = 0$

$$|k|^2 (\chi_\Omega h)^\wedge(k) = 0 \qquad \forall\, k \in \mathbb{R}^n$$

where χ_Ω denotes the characteristic function of Ω and $^\wedge$ denotes Fourier transform. Then we conclude by the Fourier inversion formula that $h = 0$ on Ω. However, one cannot apply the implicit function theorem to conclude that γ is invertible near a constant since conditions on the range of Q that would allow use of the implicit function theorem are either false or not known.

Calderón also observed that using the solutions (8) one can find an approximation for the conductivity γ if

$$\gamma = 1 + h \tag{11}$$

and h small enough in the L^∞ norm.

We are given

$$G_\gamma = Q_\gamma \left(e^{x\cdot\rho}\Big|_{\partial\Omega}, e^{-x\cdot\overline{\rho}}\Big|_{\partial\Omega} \right)$$

with $\rho \in \mathbb{C}^n$ as in (2.4). Now

$$G_\gamma = \int_\Omega (1+h)\nabla u \cdot \nabla v \, dx + \int_\Omega h(\nabla \delta u \cdot \nabla v + \nabla u \cdot \nabla \delta v)\, dx + \int_\Omega (1+h)\nabla \delta u \cdot \nabla \delta v \, dx \tag{12}$$

with u, v as in (8) and

$$\begin{array}{rcl} \nabla \cdot (\gamma \nabla (u + \delta u)) & = & \nabla \cdot (\gamma \nabla (v + \delta v)) = 0 \text{ in } \Omega \\ \delta u\Big|_{\partial\Omega} & = & \delta v\Big|_{\partial\Omega} = 0. \end{array} \tag{13}$$

Now standard elliptic estimates applied to (13) show that

$$\|\nabla \delta u\|_{L^2(\Omega)}, \quad \|\nabla \delta v\|_{L^2(\Omega)} \leq C\|h\|_{L^\infty(\Omega)}|k|e^{\frac{1}{2}r|k|} \tag{14}$$

for some $C > 0$ where r denotes the radius of the smallest ball containing Ω.

Plugging u, v into (2.7) we obtain

$$\widehat{\chi_\Omega \gamma}(k) = -2\frac{G_\gamma}{|k|^2} + R(k) = \widehat{F}(k) + R(k) \tag{15}$$

where F is determined by G_γ and therefore known. Using (14), we can show that $R(k)$ satisfies the estimate

$$|R(k)| \leq C\|h\|^2_{L^\infty(\Omega)} e^{r|k|}. \tag{16}$$

In other words we know $\widehat{\chi_\Omega \gamma}(k)$ up to a term that is small for k small enough. More precisely, let $1 < \alpha < 2$. Then for

$$|k| \leq \frac{2-\alpha}{r} \log \frac{1}{\|h\|_{L^\infty}} =: \sigma \tag{17}$$

we have

$$|R(k)| \leq C\|h\|^\alpha_{L^\infty(\Omega)} \tag{18}$$

for some $C > 0$.

We take $\widehat{\eta}$ a C^∞ cut-off so that $\widehat{\eta}(0) = 1$, $\operatorname{supp} \widehat{\eta}(k) \subset \{k \in \mathbb{R}^n, |k| \leq 1\}$ and $\eta_\sigma(x) = \sigma^n \eta(\sigma x)$. Then we obtain

$$\widehat{\chi_\Omega \gamma}(k)\widehat{\eta}\left(\frac{k}{\sigma}\right) = \frac{-2G_\gamma \gamma}{|k|^2}\widehat{\eta}\left(\frac{k}{\sigma}\right) + R(k)\widehat{\eta}\left(\frac{k}{\sigma}\right).$$

Using this we get the following estimate

$$|l(x)| \leq C\|h\|^\alpha_{L^\infty(\Omega)} \left[\log \frac{1}{\|h\|_{L^\infty(\Omega)}}\right]^n \tag{19}$$

where $l(x) = (\chi_\Omega \gamma * \eta_\sigma)(x) - (F * \eta_\sigma)(x)$. Formula (19) gives then an approximation to the smoothed out conductivity, $\chi_\Omega \gamma * \eta_\sigma$, for h sufficiently small.

This approximation estimate of Calderón and modifications of it have been tried out numerically [60] .

Another quite different inverse problems where these exponential solutions have been used is to the inverse transport problem with angularly averaged measurements [10].

This estimate uses the harmonic exponentials for low frequencies. In section 4 we consider high (complex) frequency solutions of the conductivity equation

$$L_\gamma = \nabla \cdot (\gamma \nabla u) = 0.$$

3. Determination of Cavities and Inclusions

Calderón's exponential harmonic functions have the property that they grow exponentially in a direction where the inner product of the real part of the complex phase with the direction is strictly positive, they are exponentially decaying if this inner product is negative and oscillatory if the inner product is zero. This was exploited by Ikehata in [56], [57] to give a reconstruction procedure from the DN map of a cavity D with strongly convex C^2 boundary ∂D inside a conductive medium Ω with conductivity 1 such that $\Omega \setminus \overline{D}$ is connected. We sketch some of the details here. We define the DN map Λ_D by

$$\Lambda_D(f) := \frac{\partial u(f)}{\partial \nu}|_{\partial\Omega}, \tag{20}$$

where $u(f) \in H^2(\Omega)$ is the solution to

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \frac{\partial u}{\partial \nu}|_{\partial D} = 0, \\ u|_{\partial\Omega} = f \in H^{3/2}(\partial\Omega) \end{cases} \tag{21}$$

and ν is the unit normal of ∂D. If $D = \emptyset$, we denote Λ_D by Λ_0. Let $\omega, \omega^\perp$ be unit real vectors perpendicular to each other. For $\tau > 0$, consider the Calderón harmonic functions

$$v(x, \tau, \omega, \omega^\perp) = e^{-t\tau} e^{\tau x \cdot (\omega + i\omega^\perp)}. \tag{22}$$

Note that this function grows exponentially in the half space $x \cdot \omega > t$ and decays exponentially in the half space $x \cdot \omega < t$. For $t \in \mathbb{R}$, define the indicator function by

$$I_{\omega,\omega^\perp}(\tau, t) := \int_{\partial\Omega} ((\Lambda_D - \Lambda_0) v|_{\partial\Omega}) \overline{v|_{\partial\Omega}} \, dS. \tag{23}$$

We also define the support function $h_D(\omega)$ of D by

$$h_D(\omega) := \sup_{x \in D} x \cdot \omega. \tag{24}$$

Ikehata characterizes the support function in terms of the indicator function. More precisely we have

$$h_D(\omega) - t = \lim_{\tau \to \infty} \frac{I_{\omega,\omega^\perp}(\tau, t)}{2\tau}. \tag{25}$$

Hence, by taking many ω's, we can recover the shape of D. See [56], [58] for more details and references, including numerical implementation of this method.

4. Boundary Determination

Kohn and Vogelius proved the following identifiability result at the boundary [81].

Theorem 4.1 *Let $\gamma_i \in C^\infty(\overline{\Omega})$ be strictly positive. Assume $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then*

$$\partial^\alpha \gamma_1 \Big|_{\partial\Omega} = \partial^\alpha \gamma_2 \Big|_{\partial\Omega}, \qquad \forall |\alpha|.$$

This settled the identifiability question for the non-linear problem in the real-analytic category. They extended the identifiability result to piecewise real-analytic conductivities in [82].

Sketch of proof of Theorem 4.1. We outline an alternative proof to the one given by Kohn and Vogelius of 4.1. In the case $\gamma \in C^\infty(\overline{\Omega})$ we know, by another result of Calderón [25], that Λ_γ is a classical pseudodifferential operator of order 1. Let (x', x^n) be coordinates near a point $x_0 \in \partial\Omega$ so that the boundary is given by $x^n = 0$. The function $\lambda_\gamma(x', \xi')$ denotes the full symbol of Λ_γ in these coordinates. It was proved in [133] that

$$\lambda_\gamma(x', \xi') = \gamma(x', 0)|\xi'| + a_0(x', \xi') + r(x', \xi') \tag{26}$$

where $a_0(x', \xi')$ is homogeneous of degree 0 in ξ' and is determined by the normal derivative of γ at the boundary and tangential derivatives of γ at the boundary. The term $r(x', \xi')$ is a classical symbol of order -1. Then $\gamma\big|_{\partial\Omega}$ is determined by the principal symbol of Λ_γ and $\frac{\partial\gamma}{\partial x^n}\big|_{\partial\Omega}$ by the principal symbol and the term homogeneous of degree 0 in the expansion of the full symbol of Λ_γ. More generally the higher order normal derivatives of the conductivity at the boundary can be determined recursively. In [85] one can find a more general approach to the calculation of the full symbol of the Dirichlet to Neumann map that applies to more general situations.

We note that this gives also a reconstruction procedure. We first can reconstruct γ at the boundary since $\gamma\big|_{\partial\Omega}|\xi'|$ is the principal symbol of Λ_γ (see (26)). In other words in coordinates (x', x^n) so that $\partial\Omega$ is locally given by $x^n = 0$ we have

$$\gamma(x', 0)a(x') = \lim_{s\to\infty} e^{-is<x',\omega'>}\frac{1}{s}\Lambda_\gamma(e^{is<x',\omega'>}a(x'))$$

with $\omega' \in \mathbb{R}^{n-1}$ and $|\omega'| = 1$ and a a smooth and compactly supported function.

In a similar fashion, using (26), one can find $\frac{\partial\gamma}{\partial\nu}\big|_{\partial\Omega}$ by computing the principal symbol of $(\Lambda_\gamma - \gamma\big|_{\partial\Omega}\Lambda_1)$ where Λ_1 denotes the Dirichlet to Neumann map associated to the conductivity 1. The other terms can be reconstructed recursively in a similar fashion.

We also observe, by taking an appropriate cut-off function a above, that this procedure is *local*, that is we only need to know the DN map in an open set of the boundary to determine the Taylor series of the conductivity in that open set.

This method also leads to stability estimates at the boundary [133].

Theorem 4.2 *Suppose that γ_1 and γ_2 are C^∞ functions on $\overline{\Omega} \subseteq \mathbb{R}^n$ satisfying*

i) $0 < \frac{1}{E} \leq \gamma_i \leq E$

ii) $\|\gamma_i\|_{C^2(\overline{\Omega})} \leq E$

Given any $0 < \sigma < \frac{1}{n+1}$, there exists $C = C(\Omega, E, n, \sigma)$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},\frac{-1}{2}} \tag{27}$$

and

$$\left\|\frac{\partial\gamma_1}{\partial\nu} - \frac{\partial\gamma_2}{\partial\nu}\right\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{\sigma}_{\frac{1}{2},\frac{-1}{2}}. \tag{28}$$

This result implies that we don't need the conductivity to be smooth to determine the conductivity and its normal derivative at the boundary. In the case γ is continuous on $\overline{\Omega}$ we can determine γ at the boundary by using the stability estimate i) above and an approximation argument. In the case that $\gamma \in C^1(\overline{\Omega})$ we can determine, knowing the DN map, γ and its normal derivative at the boundary using the estimate ii) above and an approximation argument. For other results and approaches to boundary determination of the conductivity see [4], [18], [92], [97]. In one way or another the boundary determination involves testing the DN map against highly oscillatory functions at the boundary.

5. Complex geometrical optics solutions with a linear phase

Motivated by Calderón exponential solutions described in section 2, Sylvester and Uhlmann [131, 132] constructed in dimension $n \geq 2$ complex geometrical optics (CGO) solutions of the conductivity equation for C^2 conductivities that behave like Calderón exponential solutions for large frequencies. This can be reduced to constructing solutions in the whole space (by extending $\gamma = 1$ outside a large ball containing Ω) for the Schrödinger equation with potential. We describe this more precisely below.

Let $\gamma \in C^2(\mathbb{R}^n)$, γ strictly positive in $\mathbb{R}^n$ and $\gamma = 1$ for $|x| \geq R$, $R > 0$. Let $L_\gamma u = \nabla \cdot \gamma \nabla u$. Then we have

$$\gamma^{-\frac{1}{2}} L_\gamma (\gamma^{-\frac{1}{2}}) = \Delta - q \tag{29}$$

where

$$q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{30}$$

Therefore, to construct solutions of $L_\gamma u = 0$ in $\mathbb{R}^n$ it is enough to construct solutions of the Schrödinger equation $(\Delta - q)u = 0$ with q of the form (30). The next result proven in [131, 132] states the existence of complex geometrical optics solutions for the Schrödinger equation associated to any bounded and compactly supported potential.

Theorem 5.1 *Let $q \in L^\infty(\mathbb{R}^n)$, $n \geq 2$, with $q(x) = 0$ for $|x| \geq R > 0$. Let $-1 < \delta < 0$. There exists $\epsilon(\delta)$ and such that for every $\rho \in \mathbb{C}^n$ satisfying*

$$\rho \cdot \rho = 0$$

and

$$\frac{\|(1+|x|^2)^{1/2} q\|_{L^\infty(\mathbb{R}^n)} + 1}{|\rho|} \leq \epsilon$$

there exists a unique solution to

$$(\Delta - q)u = 0$$

of the form

$$u = e^{x\cdot\rho}(1 + \psi_q(x,\rho)) \tag{31}$$

with $\psi_q(\cdot,\rho) \in L^2_\delta(\mathbb{R}^n)$. Moreover $\psi_q(\cdot,\rho) \in H^2_\delta(\mathbb{R}^n)$ and for $0 \leq s \leq 2$ there exists $C = C(n,s,\delta) > 0$ such that

$$\|\psi_q(\cdot,\rho)\|_{H^s_\delta} \leq \frac{C}{|\rho|^{1-s}}. \tag{32}$$

Here

$$L^2_\delta(\mathbb{R}^n) = \{f; \int (1+|x|^2)^\delta |f(x)|^2 dx < \infty\}$$

with the norm given by $\|f\|^2_{L^2_\delta} = \int (1+|x|^2)^\delta |f(x)|^2 dx$ and $H^m_\delta(\mathbb{R}^n)$ denotes the corresponding Sobolev space. Note that for large $|\rho|$ these solutions behave like Calderón's exponential solutions $e^{x\cdot\rho}$. The equation for ψ_q is given by

$$(\Delta + 2\rho\cdot\nabla)\psi_q = q(1+\psi_q). \tag{33}$$

The equation (33) is solved by constructing an inverse for $(\Delta + 2\rho\cdot\nabla)$ and solving the integral equation

$$\psi_q = (\Delta + 2\rho\cdot\nabla)^{-1}(q(1+\psi_q)). \tag{34}$$

Lemma 5.2 *Let $-1 < \delta < 0$, $0 \le s \le 1$. Let $\rho \in \mathbb{C}^n - 0$, $\rho\cdot\rho = 0$. Let $f \in L^2_{\delta+1}(\mathbb{R}^n)$. Then there exists a unique solution $u_\rho \in L^2_\delta(\mathbb{R}^n)$ of the equation*

$$\Delta_\rho u_\rho := (\Delta + 2\rho\cdot\nabla)u_\rho = f. \tag{35}$$

Moreover $u_\rho \in H^2_\delta(\mathbb{R}^n)$ and

$$\|u_\rho\|_{H^s_\delta(\mathbb{R}^n)} \le \frac{C_{s,\delta}\|f\|_{L^2_{\delta+1}}}{|\rho|^{s-1}}$$

for $0 \le s \le 1$ and for some constant $C_{s,\delta} > 0$.

The integral equation (33) can then be solved in $L^2_\delta(\mathbb{R}^n)$ for large $|\rho|$ since

$$(I - (\Delta + 2\rho\cdot\nabla)^{-1}q)\psi_q = (\Delta + 2\rho\cdot\nabla)^{-1}q$$

and $\|(\Delta + 2\rho\cdot\nabla)^{-1}q\|_{L^2_\delta\to L^2_\delta} \le \frac{C}{|\rho|}$ for some $C > 0$ where $\|\cdot\|_{L^2_\delta\to L^2_\delta}$ denotes the operator norm between $L^2_\delta(\mathbb{R}^n)$ and $L^2_\delta(\mathbb{R}^n)$. We will not give details of the proof of Lemma 5.2 here. We refer to the papers [131, 132] .

We note that there has been other approaches to construct CGO solutions for the Schrödinger equation [65], [47]. These constructions don't give uniqueness of the CGO solutions that are used in the reconstruction method of the conductivity from the DN map (see section 6).

If 0 is not a Dirichlet eigenvalue for the Schrödinger equation we can also define the DN map

$$\Lambda_q(f) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$$

where u solves

$$(\Delta - q)u = 0; \quad u|_{\partial\Omega} = f.$$

More generally we can define the set of Cauchy data for the Schrödinger equation. Let $q \in L^\infty(\Omega)$. We define the Cauchy data as the set

$$\mathcal{C}_q = \left\{ \left(u\Big|_{\partial\Omega}, \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega} \right) \right\}, \tag{36}$$

where $u \in H^1(\Omega)$ is a solution of

$$(\Delta - q)u = 0 \text{ in } \Omega. \tag{37}$$

We have $\mathcal{C}_q \subseteq H^{\frac12}(\partial\Omega) \times H^{-\frac12}(\partial\Omega)$. If 0 is not a Dirichlet eigenvalue of $\Delta - q$, then in fact $\mathcal{C}_q$ is a graph, namely

$$\mathcal{C}_q = \{(f, \Lambda_q(f)) \in H^{\frac12}(\partial\Omega) \times H^{-\frac12}(\partial\Omega)\}.$$

Complex geometrical optics for first order equations and systems under different regularity assumptions of the coefficients have been constructed in [98], [101], [137], [72], [113], [112] We refer to the article [112] for the more up to date developments on this topic and the references given there.

6. The Calderón problem in dimension $n \geq 3$

In this section we summarize some of the basic theoretical results for Calderón's problem in dimension three or higher.

6.1. Uniqueness

The identifiability question was resolved in [131] for smooth enough conductivities. The result is

Theorem 6.1 *Let $\gamma_i \in C^2(\overline{\Omega})$, γ_i strictly positive, $i = 1, 2$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.*

In dimension $n \geq 3$ this result is a consequence of a more general result. Let $q \in L^\infty(\Omega)$.

Theorem 6.2 *Let $q_i \in L^\infty(\Omega)$, $i = 1, 2$. Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, then $q_1 = q_2$.*

We now show that Theorem 6.2 implies Theorem 6.1.

Using (29) we have

$$\mathcal{C}_{q_i} = \left\{ \left(f, \left(\frac{1}{2} \gamma_i^{-\frac{1}{2}} \Big|_{\partial\Omega} \frac{\partial \gamma_i}{\partial \nu} \Big|_{\partial\Omega} \right) f + \gamma_i^{-\frac{1}{2}} \Big|_{\partial\Omega} \Lambda_{\gamma_i} \left(\gamma^{-\frac{1}{2}} \Big|_{\partial\Omega} f \right) \right), \quad f \in H^{\frac{1}{2}}(\partial\Omega) \right\}.$$

Then we conclude $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ using the the boundary identifiability result of Kohn and Vogelius [81] and its extension [133].

Proof of Theorem 6.2. Let $u_i \in H^1(\Omega)$ be a solution of

$$(\Delta - q_i) u_i = 0 \text{ in } \Omega, \qquad i = 1, 2.$$

Then using the divergence theorem we have

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = \int_{\partial\Omega} \left(\frac{\partial u_1}{\partial \nu} u_2 - u_1 \frac{\partial u_2}{\partial \nu} \right) dS. \tag{38}$$

Now it is easy to prove that if $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ then the LHS of (38) is zero

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0. \tag{39}$$

Now we extend $q_i = 0$ in Ω^c. We take solutions of $(\Delta - q_i) u_i = 0$ in $\mathbb{R}^n$ of the form

$$u_i = e^{x \cdot \rho_i} (1 + \psi_{q_i}(x, \rho_i)), \qquad i = 1, 2 \tag{40}$$

with $|\rho_i|$ large, $i = 1, 2$, with

$$\rho_1 = \frac{\eta}{2} + i \left(\frac{k + l}{2} \right) \tag{41}$$

$$\rho_2 = -\frac{\eta}{2} + i \left(\frac{k - l}{2} \right)$$

and $\eta, k, l \in \mathbb{R}^n$ such that

$$\eta \cdot k = k \cdot l = \eta \cdot l = 0 \tag{42}$$

$$|\eta|^2 = |k|^2 + |l|^2.$$

Condition (6.5) guarantees that $\rho_i \cdot \rho_i = 0$, $i = 2$. Substituting (6.3) into (6.2) we conclude

$$\widehat{(q_1 - q_2)}(-k) = -\int_\Omega e^{ix\cdot k}(q_1 - q_2)(\psi_{q_1} + \psi_{q_2} + \psi_{q_1}\psi_{q_2})dx. \tag{43}$$

Now $\|\psi_{q_i}\|_{L^2(\Omega)} \leq \frac{C}{|\rho_i|}$. Therefore by taking $|l| \to \infty$ we obtain

$$\chi_\Omega \widehat{(q_1 - q_2)}(k) = 0 \qquad \forall\, k \in \mathbb{R}^n$$

concluding the proof.

Theorem 6.1 has been extended to conductivities having 3/2 derivatives in some sense in [106], [19]. Uniqueness for conormal conductivies in $C^{1+\epsilon}$ was shown in [41]. It is an open problem whether uniqueness holds in dimension $n \geq 3$ for Lipschitz or less regular conductivities.

Theorem 6.2 was extended to potentials in $L^{n/2}$ and small potentials in the Fefferman-Phong class in [26]. For conormal potentials with singularities not in $L^{n/2}$, for instance almost a delta function of an hypersurface, uniqueness was shown in [41].

6.2. Non-linear conductivities

We now give an extension of this result to conductivities that depend of the voltage.

Let $\gamma(x,t)$ be a function with domain $\overline{\Omega} \times \mathbb{R}$. Let α be such that $0 < \alpha < 1$. We assume

$$\gamma \in C^{1,\alpha}(\overline{\Omega} \times [-T,T]), \quad \forall\, T, \tag{44}$$

$$\gamma(x,t) > 0,\ \forall\, (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{45}$$

Given $f \in C^{2,\alpha}(\partial\Omega)$, there exists a unique solution of the Dirichlet problem (see [38])

$$\nabla \cdot (\gamma(x,u)\nabla u) = 0 \text{ in } \Omega, \tag{46}$$

$$u\Big|_{\partial\Omega} = f.$$

Then the Dirichlet to Neumann map is defined by

$$\Lambda_\gamma(f) = \gamma(x,f)\Big|_{\partial\Omega} \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \tag{47}$$

where u is a solution to (6.9). Sun [124] proved the following result.

Theorem 6.3 *Let $n \geq 3$. Assume $\gamma_i, \in C^{1,1}(\overline{\Omega} \times [-T,T])\ \forall\, T > 0$, $i = 1,2$, and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then $\gamma_1(x,t) = \gamma_2(x,t)$ on $\overline{\Omega} \times \mathbb{R}$.*

The main idea is to linearize the Dirichlet to Neumann map at constant boundary data equal to the parameter t (then the solution of (6.9) is equal to t). Isakov [64] was the first to use a linearization technique to study an inverse parabolic problems associated to non-linear equations. The case of the Dirichlet to Neumann map associated to the Schrödinger equation with a non-linear potential was considered in [58] under some assumptions on the potential. We note that, in contrast to the linear case, one cannot reduce the study of the inverse problem of the conductivity equation (46) to the Schrödinger equation with a non-linear potential since a reduction from the conductivity equation to the Schrödinger equation is not possible in this case. The main technical lemma in the proof of Theorem 6.3 is

Lemma 6.4 *Let $\gamma(x,t)$ be as in (44) and (45). Let $1 < p < \infty$, $0 < \alpha < 1$. Let us define*

$$\gamma^t(x) = \gamma(x,t). \tag{48}$$

Then for any $f \in C^{2,\alpha}(\partial\Omega)$, $t \in \mathbb{R}$

$$\lim_{s\to 0} \|\frac{1}{s}\Lambda_\gamma(t+sf) - \Lambda_{\gamma^t}(f)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = 0. \tag{49}$$

The proof of Theorem 6.3 follows immediately from the lemma. Namely (6.12) and the hypotheses $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \Lambda_{\gamma_1^t} = \Lambda_{\gamma_2^t}$ for all $t \in \mathbb{R}$. Then using the linear result, Theorem 6.1, we conclude that $\gamma_1^t = \gamma_2^t$ proving the result.

We remark that the reduction from the non-linear problem to the linear is also valid in the two dimensional case [66]. Using the result of Astala and Päivärinta [8], which is reviewed in section 8, one can extend Theorem 6.3 to $L^\infty(\Omega)$ conductivities in the two dimensional case.

There are several open questions when the conductivity also depends on ∇u, see [126] for a survey of results and open problems in this direction.

6.3. Other applications

We give a short list of other applications to inverse problems using the CGO solutions described above for the Schrödinger equation.

- Quantum Scattering. In dimension $n \geq 3$ and in the case of a compactly supported electric potential, uniqueness for the fixed energy scattering problem was proven in [92], [102], [110]. In the earlier paper [103] this was done for small potentials. For compactly supported potentials, knowledge of the scattering amplitude at fixed energy is equivalent to knowing the Dirichlet-to-Neumann map for the Schrödinger equation measured on the boundary of a large ball containing the support of the potential (see [139], [141] for an account). Then Theorem 6.2 implies the result. Melrose [88] suggested a related proof that uses the density of products of scattering solutions. Applications of CGO solutions to the 3-body problem were given in [142].
- Optics. The DN map associated to the Helmholtz equation $-\Delta + k^2 n(x)$ with an isotropic index of refraction n determines uniquely a bounded index of refraction in dimension $n \geq 3$.
- Optical tomography in the diffusion approximation. In this case we have $\nabla \cdot \mathcal{D}(x)\nabla u - \sigma_a(x)u - i\omega u = 0$ in Ω where u represents the density of photons, $\mathcal{D}$ the diffusion coefficient, and σ_a the optical absorption. Using the result of [131] one can show in dimension three or higher that if $\omega \neq 0$ one can recover both $\mathcal{D}$ and σ_a from the corresponding DN map. If $\omega = 0$ then one can recover one of the two parameters.
- Electromagnetics. For Maxwell's equations the analog of the DN map is the admittance map that maps the tangential component of the electric field to the tangential component of the magnetic field [122]. The admittance map for isotropic Maxwell's equations determines uniquely the isotropic electric permittivity, magnetic permeability and conductivity [104]. This system can in fact be reduced to the Schrödinger equation $\Delta - Q$ with Q an 8×8 system and Δ the Laplacian times the identity matrix [105].

- Determination of Inclusions and Obstacles. The CGO solutions constructed in Theorem 5.1 have been applied to determine inclusions for Helmholtz equations in [57] and Maxwell's equations in [148] using the enclosure method [57].

6.4. Stability

The arguments used in the proofs of Theorems 6.1, 6.2, 4.1 can be pushed further to prove the following stability estimates proven in [3].

Theorem 6.5 *Let $n \geq 3$. Suppose that $s > \frac{n}{2}$ and that γ_1 and γ_2 are C^∞ conductivities on $\overline{\Omega} \subseteq \mathbb{R}^n$ satisfying*

i) $0 < \frac{1}{E} \leq \gamma_j \leq E, j = 1, 2.$

ii) $\|\gamma_j\|_{H^{s+2}(\Omega)} \leq E, j = 1, 2.$

Then there exists $C = C(\Omega, E, n, s)$ and $0 < \sigma < 1$ ($\sigma = \sigma(n, s)$) such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\big(|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},\frac{-1}{2}}|^{-\sigma} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},\frac{-1}{2}}\big) \tag{50}$$

where $\| \quad \|_{\frac{1}{2},\frac{-1}{2}}$ denotes the operator norm as operators from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$.

Notice that this is logarithmic type stability estimates indicates that the problem is severely ill-posed. Mandache [87] has shown that this estimate is optimal up to to the value of the exponent. There is the question of whether under some additional a-priori condition one can improve this logarithmic type stability estimate. Alessandrini and Vessella [6] have shown that this is indeed the case and one has a Lipschitz type stability estimate if the conductivity is piecewise constant with jumps on a finite number of domains. Rondi [111] has subsequently shown that the constant in the estimate grows exponentially with the number of domains.

It is conjectured, and this is supported by numerical experiments, that the stability estimate should be "better" near the boundary and gets increasingly worse as one penetrated deeper into the domain (Theorem 4.1 shows that at the boundary we have Lipschitz type stability estimate.) This type of depth dependence stability estimate has been proved in [95] for the case of some electrical inclusions.

For a recent review of stability issues in EIT see [5].

Theorem 6.5 is a consequence of Theorem 4.2 and the following result.

Theorem 6.6 *Assume 0 is not a Dirichlet eigenvalue of $\Delta - q_i$, $i = 1, 2$. Let $s > \frac{n}{2}$, $n \geq 3$ and*

$$\|q_j\|_{H^s(\Omega)} \leq M.$$

Then there exists $C = C(\Omega, M, n, s)$ and $0 < \sigma < 1$ ($\sigma = \sigma(n, s)$) such that

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C\big(|\log \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\frac{1}{2},\frac{-1}{2}}|^{-\sigma} + \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\frac{1}{2},\frac{-1}{2}}\big). \tag{51}$$

6.5. Reconstruction

The complex geometrical optics solution of Theorems 6.1 and 6.2 were also used by A. Nachman [92] and R. Novikov [102] to give a reconstruction procedure of the conductivity from Λ_γ.

As we have already noticed in section 4 we can reconstruct the conductivity and its normal derivative from the DN map. Therefore if we know Λ_γ we can determine

Λ_q. We will then show how to reconstruct q from Λ_q. Once this is done, to find $\sqrt{\gamma}$, we solve the problem

$$\begin{aligned} \Delta u - qu &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= \sqrt{\gamma}\Big|_{\partial\Omega}. \end{aligned} \tag{52}$$

Let $q_1 = q$, $q_2 = 0$ in formula (38). Then we have

$$\int_\Omega quvdx = \int_{\partial\Omega} (\Lambda_q - \Lambda_0)\left(v\big|_{\partial\Omega}\right) u\big|_{\partial\Omega} dS \tag{53}$$

where $u, v \in H^1(\Omega)$ solve$\Delta u - qu = 0$, $\Delta v = 0$ in Ω. Here Λ_0 denotes the Dirichlet to Neumann map associated to the potential $q = 0$. We choose $\rho_i, i = 1, 2$ as in (42) and (6.5).

Take $v = e^{x\cdot\rho_1}$, $u := u_\rho = e^{x\cdot\rho_2}(1 + \psi_q(x, \rho_2))$ as in Theorem 5.1. By taking $|l| \to \infty$ in (53) we conclude

$$\widehat{q}(-k) = \lim_{|l|\to\infty} \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(e^{x\cdot\rho_1}\Big|_{\partial\Omega}) u_\rho\Big|_{\partial\Omega} dS.$$

So the problem is then to recover the boundary values of the solutions u_ρ from Λ_q.

The idea is to find $u_\rho\Big|_{\partial\Omega}$ by looking at the exterior problem. Namely by extending $q = 0$ outside Ω, u_ρ solves

$$\begin{aligned} \Delta u_\rho &= 0 \text{ in } \mathbb{R}^n - \Omega \\ \frac{\partial u_\rho}{\partial \nu}\Big|_{\partial\Omega} &= \Lambda_q(u_\rho\Big|_{\partial\Omega}). \end{aligned} \tag{54}$$

Also note that

$$e^{-x\cdot\rho_2} u_\rho - 1 \in L^2_\delta(\mathbb{R}^n). \tag{55}$$

Let $\rho \in \mathbb{C}^n - 0$ with $\rho \cdot \rho = 0$. Let $G_\rho(x, y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ denote the Schwartz kernel of the operator Δ_ρ^{-1}. Then we have that

$$g_\rho(x) = e^{x\cdot\rho} G_\rho(x) \tag{56}$$

is a Green's kernel for Δ, namely

$$\Delta g_\rho = \delta_0. \tag{57}$$

We write the solution of (6.19) and (55) in terms of single and double layer potentials using this Green's kernel. This is also called Faddeev Green's kernel [35] who considered it in the context of scattering theory.

We define the single and double layer potentials

$$S_\rho f(x) = \int_{\partial\Omega} g_\rho(x - y) f(y) dS_y, \qquad x \in \mathbb{R}^n - \Omega, \tag{58}$$

$$D_\rho f(x) = \int_{\partial\Omega} \frac{\partial g_\rho}{\partial \nu}(x - y) f(y) dS_y, \qquad x \in \mathbb{R}^n - \Omega \tag{59}$$

$$B_\rho f(x) = p.v. \int_{\partial\Omega} \frac{\partial g_\rho}{\partial \nu}(x - y) f(y) dS_y, \qquad x \in \partial\Omega. \tag{60}$$

Nachman showed that $f_\rho = u_\rho\Big|_{\partial\Omega}$ is a solution of the integral equation

$$f_\rho = e^{x\cdot\rho} - (S_\rho \Lambda_q - B_\rho - \frac{1}{2} I) f_\rho. \tag{61}$$

Moreover (61) is an inhomogeneous integral equation of Fredholm type for f_ρ and it has a unique solution in $H^{\frac{3}{2}}(\partial\Omega)$. The uniqueness of the homogeneous equation follows from the uniqueness of the CGO solutions in Theorem 6.2.

7. The Partial Data Problem

In several applications in EIT one can only measure currents and voltages on part of the boundary. Substantial progress has been made recently on the problem of whether one can determine the conductivity in the interior by measuring the DN map on part of the boundary. We review here the articles [75] and [29].

The paper [22] used the method of Carleman estimates with a linear weight to prove that, roughly speaking, knowledge of the DN map in "half" of the boundary is enough to determine uniquely a C^2 conductivity. The regularity assumption on the conductivity was relaxed to $C^{1+\epsilon}, \epsilon > 0$ in [76]. Stability estimates for the uniqueness result of [22] were given in [49]. Stability estimates for the magnetic Schrödinger operator with partial data in the setting of [22] can be found in [138].

The result [22] was substantially improved in [75]. The latter paper contains a global identifiability result where it is assumed that the DN map is measured on any open subset of the boundary of a strictly convex domain for all functions supported, roughly, on the complement. We state the theorem more precisely below. The key new ingredient is the construction of a larger class of CGO solutions than the ones considered in section 5.

Let $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$, where $\mathrm{ch}\,(\Omega)$ denotes the convex hull of Ω. Define the front and the back faces of $\partial\Omega$ by

$$F(x_0) = \{x \in \partial\Omega;\, (x - x_0) \cdot \nu \leq 0\},\ B(x_0) = \{x \in \partial\Omega;\, (x - x_0) \cdot \nu > 0\}.$$

The main result of [75] is the following:

Theorem 7.1 *Let $n > 2$. With Ω, x_0, $F(x_0)$, $B(x_0)$ defined as above, let $q_1, q_2 \in L^\infty(\Omega)$ be two potentials and assume that there exist open neighborhoods $\widetilde{F}, \widetilde{B} \subset \partial\Omega$ of $F(x_0)$ and $B(x_0) \cup \{x \in \partial\Omega; (x - x_0) \cdot \nu = 0\}$ respectively, such that*

$$\Lambda_{q_1} u = \Lambda_{q_2} u \text{ in } \widetilde{F},\ \text{for all } u \in H^{\frac{1}{2}}(\partial\Omega) \cap \mathscr{E}'(\widetilde{B}). \tag{62}$$

Then $q_1 = q_2$.

Here $\mathscr{E}'(\widetilde{B})$ denotes the space of compactly supported distributions in $\widetilde{B}$.

The proof of this result uses Carleman estimates for the Laplacian with limiting Carleman weights (LCW). The Carleman estimates allow one to construct, for large τ, a larger class of CGO solutions for the Schrödinger equation than previously used. These have the form

$$u = e^{\tau(\phi + i\psi)}(a + r), \tag{63}$$

where $\nabla\phi \cdot \nabla\psi = 0, |\nabla\phi|^2 = |\nabla\psi|^2$ and ϕ is the LCW. Moreover a is smooth and non-vanishing and $\|r\|_{L^2(\Omega)} = O(\frac{1}{\tau})$, $\|r\|_{H^1(\Omega)} = O(1)$. Examples of LCW are the linear phase $\phi(x) = x \cdot \omega, \omega \in S^{n-1}$, used previously, and the non-linear phase $\phi(x) = \ln|x - x_0|$, where $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$ which was used in [75]. Any conformal transformation of these would also be a LCW. Below we give a characterization of all the LCW in $\mathbb{R}^n$, $n > 2$, see [31]. In two dimensions any harmonic function is a LCW [144].

The CGO solutions used in [75] are of the form

$$u(x, \tau) = e^{\log|x - x_0| + id(\frac{x - x_0}{|x - x_0|}, \omega)}(a + r) \tag{64}$$

where x_0 is a point outside the convex hull of Ω, ω is a unit vector and $d(\frac{x - x_0}{|x - x_0|}, \omega)$ denote distance. We take directions ω so that the distance function is smooth for $x \in \overline{\Omega}$.

7.1. Limiting Carleman weights

We only recall here the main ideas in the construction of the CGO solutions. We will denote $\tau = \frac{1}{h}$ in order to use the standard semiclassical notation. Let $P_0 = -h^2\Delta$, where $h > 0$ is a small semi-classical parameter. The weighted L^2-estimate

$$\|e^{\phi/h}u\| \leq C\|e^{\phi/h}P_0u\|$$

is of course equivalent to the unweighted estimate for a conjugated operator:

$$\|v\| \leq C\|e^{\phi/h}P_0e^{-\phi/h}v\|.$$

The semi-classical principal symbol of P_0 is $p(x,\xi) = \xi^2$, and that of the conjugated operator $e^{\phi/h}P_0e^{-\phi/h}$ is

$$p(x,\xi + i\phi'(x)) = a(x,\xi) + ib(x,\xi),$$

where

$$a(x,\xi) = \xi^2 - \phi'(x)^2, \ b(x,\xi) = 2\xi\cdot\phi'(x).$$

Here we denote by ϕ' the gradient of ϕ.

Write the conjugated operator as $A + iB$, with A and B formally selfadjoint and with a and b as their associated principal symbols. Then

$$\|(A + iB)u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u|u).$$

The principal symbol of $i[A,B]$ is $h\{a,b\}$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket. In order to get enough negativity to satisfy Hörmander's solvability condition we require that

$$a(x,\xi) = b(x,\xi) = 0 \Rightarrow \{a,b\} \leq 0.$$

It is then indeed possible to get an a-priori estimate for the conjugated operator. We are led to the limiting case since we need to have CGO solutions for both ϕ and $-\phi$.

Definition 7.2 *ϕ is a limiting Carleman weight (LCW) on some open set Ω if $\nabla\phi(x)$ is non-vanishing there and we have*

$$a(x,\xi) = b(x,\xi) = 0 \Rightarrow \{a,b\}(x) = 0, x \in \Omega.$$

We remark that if ϕ is a LCW so is $-\phi$.

In [31] we have classified locally all the LCW in Euclidean space.

Theorem 7.3 *Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 3$. The limiting Carleman weights in Ω are locally of the form*

$$\phi(x) = a\phi_0(x - x_0) + b$$

where $a \in \mathbb{R}\setminus\{0\}$ and ϕ_0 is one of the following functions:

$$\langle x,\xi\rangle, \qquad \arg\langle x,\omega_1 + i\omega_2\rangle,$$

$$\log|x|, \quad \frac{\langle x,\xi\rangle}{|x|^2}, \quad \arg\left(e^{i\theta}(x + i\xi)^2\right), \quad \log\frac{|x+\xi|^2}{|x-\xi|^2}$$

with ω_1,ω_2 orthogonal unit vectors, $\theta \in [0,2\pi)$ and $\xi \in \mathbb{R}^n\setminus\{0\}$.

As noted earlier, in two dimensions, any harmonic function with a non-vanishing gradient is a limiting Carleman weight.

7.2. Construction of CGO Solutions

A key ingredient in the construction of a richer family of CGO solutions is the following Carleman estimate.

Proposition 7.4 *Let $\phi \in C^\infty(\mathrm{neigh}\,(\overline{\Omega}))$ be an LCW, $P = -h^2\Delta + h^2 q$, $q \in L^\infty(\Omega)$. Then, for $u \in C^\infty(\overline{\Omega})$, with $u_{|\partial\Omega} = 0$, we have*

$$\begin{aligned} -\frac{h^3}{C}((\phi'_x \cdot \nu)e^{\phi/h}\partial_\nu u|e^{\phi/h}\partial_\nu u)_{\partial\Omega_-} + \frac{h^2}{C}(\|e^{\phi/h}u\|^2 + \|e^{\phi/h}h\nabla u\|^2) \\ \leq Ch^3((\phi'_x \cdot \nu)e^{\phi/h}\partial_\nu u|e^{\phi/h}\partial_\nu u)_{\partial\Omega_+} + \|e^{\phi/h}Pu\|^2, \end{aligned} \tag{65}$$

where norms and scalar products are in $L^2(\Omega)$ unless a subscript A (like for instance $A = \partial\Omega_-$) indicates that they should be taken in $L^2(A)$. Here

$$\partial\Omega_\pm = \{x \in \partial\Omega;\ \pm\nu(x)\cdot\phi'(x) \geq 0\}.$$

The proof of existence of solutions of the form (63) follows by using the Hahn-Banach theorem for the adjoint equation $e^{-\phi/h}Pe^{\phi/h}u = v$.

Let ϕ be a LCW and write $p(x, \xi + \phi'(x)) = a(x,\xi) + ib(x,\xi)$. Then we know that a and b are in involution on their common zero set, and in this case it is well-known and exploited in [32] that we can find plenty of local solutions to the Hamilton-Jacobi system

$$a(x, \psi'(x)) = 0, \qquad b(x, \psi'(x)) = 0 \Leftrightarrow \psi'^2 = \phi'^2, \qquad \psi'\cdot\phi' = 0 \tag{66}$$

We need the following more global statement:

Proposition 7.5 *Let $\phi \in C^\infty(\mathrm{neigh}\,(\overline{\Omega}))$ be a LCW, where Ω is a domain in $\mathbb{R}^n$ and define the hypersurface $G = p^{-1}(C_0)$ for some fixed value of C_0. Assume that each integral curve of $\phi'\cdot\nabla_x$ through a point in Ω also intersects G and that the corresponding projection map $\Omega \to G$ is proper. Then we get a solution of (66) in $C^\infty(\Omega)$ by solving first $g'(x)^2 = \phi'(x)^2$ on G and then defining ψ by $\psi_{|G} = g$, $\phi'(x)\cdot\partial_x\psi = 0$. The vector fields $\phi'\cdot\partial_x$ and $\psi'\cdot\partial_x$ commute.*

This result will be applied with a new domain Ω that contains the original one.

Next consider the WKB-problem

$$P_0(e^{\frac{1}{h}(-\phi+i\psi)}a(x)) = e^{\frac{1}{h}(-\phi+i\psi)}\mathcal{O}(h^2). \tag{67}$$

The transport equation for a is of Cauchy-Riemann type along the two-dimensional integral leaves of $\{\phi'\cdot\partial_x, \psi'\cdot\partial_x\}$. We have solutions that are smooth and everywhere $\neq 0$. (See [32].)

The existence result for $e^{\phi/h}Pe^{-\phi/h}$ mentioned in one of the remarks after Proposition 7.4 permits us to to replace the right hand side of (67) by zero; more precisely, we can find $r = \mathcal{O}(h)$ in the semi-classical Sobolev space H^1 equipped with the norm $\|r\| = \|\langle hD\rangle r\|$, such that

$$P(e^{\frac{1}{h}(-\phi+i\psi)}(a + r)) = 0. \tag{68}$$

7.3. The uniqueness proof

We sketch the proof for the case that $\widetilde{B} = \partial\Omega$. All the arguments in this section are in dimension $n > 2$. The arguments are similar to those of [22] using the new CGO solutions. Let ϕ be an LCW for which the constructions of section 7.2 are available. Let $q_1, q_2 \in L^\infty(\Omega)$ be as in Theorem 7.1 with

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f) \text{ in } \partial\Omega_{-,\epsilon_0},\ \forall f \in H^{\frac{1}{2}}(\partial\Omega), \tag{69}$$

where

$$\partial\Omega_{-,\epsilon_0} = \{x \in \partial\Omega;\, \nu(x)\cdot\phi'(x) < \epsilon_0\}$$
$$\partial\Omega_{+,\epsilon_0} = \{x \in \partial\Omega;\, \nu(x)\cdot\phi'(x) \geq \epsilon_0\}.$$

Let

$$u_2 = e^{\frac{1}{h}(\phi+i\psi_2)}(a_2 + r_2)$$

solve

$$(\Delta - q_2)u_2 = 0 \text{ in } \Omega,$$

with $\|r_2\|_{H^1} = \mathcal{O}(h)$. Let u_1 solve

$$(\Delta - q_1)u_1 = 0 \text{ in } \Omega,\ u_{1|_{\partial\Omega}} = u_{2|_{\partial\Omega}}.$$

Then according to the assumptions in the theorem, we have $\partial_\nu u_1 = \partial_\nu u_2$ in $\partial\Omega_{-,\epsilon_0}$ if $\epsilon_0 > 0$ has been fixed sufficiently small and we choose $\phi(x) = \ln|x - x_0|$.

Put $u = u_1 - u_2$, $q = q_2 - q_1$, so that

$$(\Delta - q_1)u = qu_2,\ u_{|_{\partial\Omega}} = 0,\ \operatorname{supp}(\partial_\nu u_{|_{\partial\Omega}}) \subset \partial\Omega_{+,\epsilon_0}. \tag{70}$$

For $v \in H^1(\Omega)$ with $\Delta v \in L^2$, we get from Green's formula

$$\begin{aligned}\int_\Omega q u_2 \overline{v} dx &= \int_\Omega (\Delta - q_1)u\overline{v}dx \\ &= \int_\Omega u\,\overline{(\Delta - \overline{q}_1)v}dx + \int_{\partial\Omega_{+,\epsilon_0}} \partial_\nu u\overline{v}dS.\end{aligned} \tag{71}$$

Similarly, we choose

$$v = e^{-\frac{1}{h}(\phi+i\psi_1)}(a_1 + r_1),$$

with

$$(\Delta - \overline{q}_1)v = 0.$$

Then

$$\int_\Omega q e^{\frac{i}{h}(\psi_1+\psi_2)}(a_2 + r_2)(\overline{a_1 + r_1})dx = \int_{\partial\Omega_{+,\epsilon_0}} \partial_\nu u\, e^{-\frac{1}{h}(\phi - i\psi_1)}(\overline{a_1 + r_1})dS. \tag{72}$$

Assume that ψ_1, ψ_2 are slightly h-dependent with

$$\frac{1}{h}(\psi_1 + \psi_2) \to f,\ h \to 0.$$

The left hand side of (72) tends to

$$\int_\Omega q e^{if} a_2 \overline{a}_1 dx,$$

when $h \to 0$. The modulus of the right hand side is

$$\leq \|a_1 + r_1\|_{\partial\Omega_{+,\epsilon_0}} \big(\int_{\partial\Omega_{+,\epsilon_0}} e^{-2\phi/h} |\partial_\nu u|^2 dS \big)^{\frac{1}{2}}.$$

Here the first factor is bounded when $h \to 0$. In the Carleman estimate (65) we can replace ϕ by $-\phi$ and make the corresponding permutation of $\partial\Omega_-$ and $\partial\Omega_+$. Applying this variant to the equation (70), we see that the second factor tends to 0, when $h \to 0$. Thus,

$$\int_\Omega e^{if(x)} a_2(x) \overline{a_1(x)} q(x) dx = 0.$$

Here we can arrange it so that f, a_2, a_1 are real-analytic and so that a_1, a_2 are non-vanishing. Moreover if f can be attained as a limit of $(\psi_1 + \psi_2)/h$ when $h \to 0$, so can λf for any $\lambda > 0$. Thus we get the conclusion

$$\int_\Omega e^{i\lambda f(x)} a_2(x) \overline{a_1(x)} q(x) dx = 0. \tag{73}$$

To show that $q = 0$ one uses arguments of analytic microlocal analysis [75].

In [7] it was shown that if the potential is known in a neighborhood of the boundary and the DN map is measured on any open subset with Dirichlet data supported in the same set, the potential can be reconstructed from this data. It is an open problem whether this is valid in the general case. Isakov [63] proved a uniqueness result in dimension three or higher when the DN map is given on an arbitrary part of the boundary assuming that the remaining part is an open subset of a plane or a sphere and the DN map is measured on the plane or sphere.

The DN map with partial data for the magnetic Schrödinger operator was studied in [30], [77], [138]. We also mention that in [44] (resp. [69]) CGO approximate solutions are concentrated near planes (resp. spheres) and provided some local results related to the local DN map. It would be very interesting to extend the partial data result to systems. The only result available at the present time is the very interesting one of [114] for Dirac systems.

Using methods of hyperbolic geometry similar to [69] it is shown in [55] that one can reconstruct inclusions from the *local* DN map using CGO solutions that decay exponentially inside a ball and grow exponentially outside. These are called *complex spherical waves*. A numerical implementation of this method has been done in [55]. The construction of complex spherical waves can also be done using the CGO solutions constructed in [75]. This was done in [143] in order to detect elastic inclusions, in [144] to detect inclusions in the two dimensional case for a large class of systems with inhomogeneous background, and in [115] for the case of inclusions contained in a slab. We mention that methods of hyperbolic geometry have been also studied earlier in the works [11], [37], [116].

7.4. The Linearized Calderón Partial Data Problem

It is an open problem in dimension $n \geq 3$ that if the Dirichlet to Neumann map for the conductivity or potential is measured on an open non-empty subset of the boundary for Dirichlet data supported in that set we can determine uniquely the potential.

In this section we consider the linearized version of this problem, generalizing Calderón's approach. We add the constraint that the restriction of the harmonic functions to the boundary vanishes on any fixed closed proper subset of the boundary. We show that the product of these harmonic functions is dense. More precisely

Theorem 7.6 *Let Ω be a connected bounded open set in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary. The set of products of harmonic functions on Ω which vanish on a closed proper subset $\Gamma \subsetneq \partial\Omega$ of the boundary is dense in $L^1(\Omega)$.*

Sketch of the Proof. We take $f \in L^1(\Omega)$. Assume

$$\int_\Omega fuvdx = 0, \tag{74}$$

for all harmonic functions u, v with $u|_\Gamma = v|_\Gamma = 0$.

First one proves a local result. Fix a point x_0 on the boundary. It is shown that if that $f = 0$ in a neighborhood of x_0 then $f = 0$ in the whole domain. See [29] for the proof.

We now extend $f = 0$ outside Ω. We reduce the problem to the case where the point x_0 has a hyperplane tangent to the boundary at the point x_0. We use Calderón's exponential solutions for all the possible complex frequencies ρ such that $\rho \cdot \rho = 0$ (previously we used in sections 2 and 5 the cancellation of the real parts when taking the products). Using these solutions and the identity (6.2) one obtains a decay of the Bargmann-Segal transform of f (see [120])

$$Tf(z) = \int_{\mathbb{R}^n} e^{-\frac{1}{2h}}(z-y)^2 f(y)dy \tag{75}$$

for certain complex directions. Using the watermelon approach [73], [121], one then shows that there is an exponential decay of this transform for other directions implying that the point (x_0, ν), where ν is the normal to the the point x_0, is not in the analytic wave front set of f in contradiction to the microlocal version of Holmgren's uniqueness theorem [54], [120]. We explain below some of the details of the proof below.

One can assume that $\Omega \setminus \{x_0\}$ is on one side of the tangent hyperplane $T_{x_0}(\Omega)$ at x_0 by making a conformal transformation. Pick $a \in \mathbb{R}^n \setminus \overline{\Omega}$ which sits on the line segment in the direction of the outward normal to $\partial\Omega$ at x_0; there is a ball $B(a,r)$ such that $\partial B(a,r) \cap \overline{\Omega} = \{x_0\}$, and there is a conformal transformation

$$\begin{aligned} \psi : \mathbb{R}^n \setminus B(a,r) &\to \overline{B(a,r)} \\ x &\mapsto \frac{x-a}{|x-a|^2} r^2 + a \end{aligned}$$

which fixes x_0 and exchanges the interior and the exterior of the ball $B(a,r)$. The hyperplane $H : (x - x_0) \cdot (a - x_0) = 0$ is tangent to $\psi(\Omega)$, and the image $\psi(\Omega) \setminus \{x_0\}$ by the conformal transformation lies inside the ball $B(a,r)$, therefore on one side of H. The fact that functions are supported on the boundary close to x_0 is left unchanged. Since a function is harmonic on Ω if and only if its Kelvin transform

$$u^* = r^{n-2}|x-a|^{-n+2} u \circ \psi$$

is harmonic on $\psi(\Omega)$, (7.12) becomes

$$0 = \int_\Omega fuv\,dx = \int_{\psi(\Omega)} r^4 |x-a|^{-4} f \circ \psi\, u^* v^*\, dx$$

for all harmonic functions u^*, v^* on $\psi(\Omega)$. If one can prove that if $|x-a|^{-4} f \circ \psi$ vanishes close to x_0 then so does f. Moreover, by scaling one can assume that Ω is contained in a ball of radius 1.

Our setting will therefore be as follows: $x_0 = 0$, the tangent hyperplane at x_0 is given by $x_1 = 0$ and

$$\Omega \subset \big\{x \in \mathbb{R}^n : |x + e_1| < 1\big\}, \quad \Gamma = \big\{x \in \partial\Omega : x_1 \geq -2c\big\}. \tag{76}$$

The prime will be used to denote the last $n-1$ variables so that $x = (x_1, x')$ for instance. The Laplacian on $\mathbb{R}^n$ has $p(\xi) = \xi^2$ as a principal symbol, if we still denote by $p(\zeta) = \zeta^2$ the continuation of this principal symbol on $\mathbb{C}^n$, we consider

$$p^{-1}(0) = \big\{\zeta \in \mathbb{C}^n : \zeta^2 = 0\big\}.$$

In dimension $n = 2$, this set is the union of two complex lines

$$p^{-1}(0) = \mathbb{C}\gamma \cup \mathbb{C}\overline{\gamma}$$

where $\gamma = ie_1 + e_2 = (i, 1) \in \mathbb{C}^2$. Note that $(\gamma, \overline{\gamma})$ is a basis of $\mathbb{C}^2$: the decomposition of a complex vector in this basis reads

$$\zeta = \zeta_1 e_1 + \zeta_2 e_2 = \frac{\zeta_2 - i\zeta_1}{2}\,\gamma + \frac{\zeta_2 + i\zeta_1}{2}\,\overline{\gamma}. \tag{77}$$

Similarly for $n \geq 2$, the differential of the map

$$\begin{aligned} s : p^{-1}(0) \times p^{-1}(0) &\to \mathbb{C}^n \\ (\zeta, \eta) \quad &\mapsto \zeta + \eta \end{aligned}$$

at (ζ_0, η_0) is surjective

$$\begin{aligned} Ds(\zeta_0, \eta_0) : T_{\zeta_0}p^{-1}(0) \times T_{\eta_0}p^{-1}(0) \quad &\to \mathbb{C}^n \\ (\zeta, \eta) &\mapsto \zeta + \eta \end{aligned}$$

provided $\mathbb{C}^n = T_{\zeta_0}p^{-1}(0) + T_{\eta_0}p^{-1}(0)$, i.e. provided ζ_0 and η_0 are linearly independent. In particular, this is the case if $\zeta_0 = \gamma$ and $\eta_0 = -\overline{\gamma}$; as a consequence all $z \in \mathbb{C}^n$, $|z - 2ie_1| < 2\varepsilon$ may be decomposed as a sum of the form

$$z = \zeta + \eta, \qquad \text{with } \zeta, \eta \in p^{-1}(0),\ |\zeta - \gamma| < C\varepsilon,\ |\eta + \overline{\gamma}| < C\varepsilon \tag{78}$$

provided $\varepsilon > 0$ is small enough.

The exponentials with linear weights

$$e^{-\frac{i}{h}x\cdot\zeta}, \quad \zeta \in p^{-1}(0)$$

are harmonic functions. We need to add a correction term in order to obtain harmonic functions u satisfying the boundary requirement $u|_\Gamma = 0$. Let $\chi \in \mathcal{D}(\mathbb{R}^n)$ be a cutoff function which equals 1 on Γ, and consider the solution w to the Dirichlet problem

$$\Delta w = 0 \quad \text{in } \Omega, \qquad\qquad w|_{\partial\Omega} = -(e^{-\frac{i}{h}x\cdot\zeta}\chi)|_{\partial\Omega}. \tag{79}$$

The function

$$u(x, \zeta) = e^{-\frac{i}{h}x\cdot\zeta} + w(x, \zeta)$$

is harmonic and satisfies $u|_\Gamma = 0$. We have the following bound on w:

$$\begin{aligned} \|w\|_{H^1(\Omega)} &\leq C_1 \|e^{-\frac{i}{h}x\cdot\zeta}\chi\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq C_2(1 + h^{-1}|\zeta|)^{\frac{1}{2}}\, e^{\frac{1}{h}H_K(\mathrm{Im}\zeta)} \end{aligned} \tag{80}$$

where H_K is the supporting function of the compact subset $K = \operatorname{supp}\chi \cap \partial\Omega$ of the boundary

$$H_K(\xi) = \sup_{x\in K} x\cdot\xi, \quad \xi \in \mathbb{R}^n.$$

In particular, if we take χ to be supported in $x_1 \le -c$ and equal to 1 on $x_1 \le -2c$ then the bound (80) becomes

$$\|w\|_{H^1(\Omega)} \le C_2(1+h^{-1}|\zeta|)^{\frac{1}{2}}\, e^{-\frac{c}{h}\mathrm{Im}\zeta_1}\, e^{\frac{1}{h}|\mathrm{Im}\zeta'|} \quad \text{when } \mathrm{Im}\zeta_1 \ge 0. \tag{81}$$

The starting point is the cancellation of the integral

$$\int_\Omega f(x)u(x,\zeta)u(x,\eta)\,dx = 0, \quad \zeta,\eta \in p^{-1}(0) \tag{82}$$

which may be rewritten in the form

$$\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx = -\int_\Omega f(x)e^{-\frac{i}{h}x\cdot\zeta}w(x,\eta)\,dx - \int_\Omega f(x)e^{-\frac{i}{h}x\cdot\eta}w(x,\zeta)\,dx - \int_\Omega f(x)w(x,\zeta)w(x,\eta)\,dx.$$

This allows to give a bound on the left-hand side

$$\left|\int_\Omega f(x) \quad e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\right| \le \|f\|_{L^\infty(\Omega)}\big(\|e^{-\frac{i}{h}x\cdot\zeta}\|_{L^2(\Omega)}\|w(x,\eta)\|_{L^2(\Omega)} + \|e^{-\frac{i}{h}x\cdot\eta}\|_{L^2(\Omega)}\|w(x,\zeta)\|_{L^2(\Omega)} + \|w(x,\eta)\|_{L^2(\Omega)}\|w(x,\zeta)\|_{L^2(\Omega)}\big).$$

Thus using (81)

$$\left|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\right| \le C_3\|f\|_{L^\infty(\Omega)}(1+h^{-1}|\eta|)^{\frac{1}{2}}(1+h^{-1}|\zeta|)^{\frac{1}{2}} \times e^{-\frac{c}{h}\min(\mathrm{Im}\zeta_1,\mathrm{Im}\eta_1)}\, e^{\frac{1}{h}(|\mathrm{Im}\zeta'|+|\mathrm{Im}\eta'|)}$$

when $\mathrm{Im}\zeta_1 \ge 0, \mathrm{Im}\eta_1 \ge 0$ and $\zeta,\eta \in p^{-1}(0)$. In particular, if $|\zeta - a\gamma| < C\varepsilon a$ and $|\eta + a\overline{\gamma}| < C\varepsilon a$ with $\varepsilon \le 1/2C$ then

$$\left|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\right| \le C_4 h^{-1}\|f\|_{L^\infty(\Omega)}e^{-\frac{ca}{2h}}\, e^{\frac{2C\varepsilon a}{h}}.$$

Take $z \in \mathbb{C}^n$ with $|z - 2ae_1| < 2\varepsilon a$ and with ε small enough. Once rescaled the decomposition (78) gives

$$z = \zeta + \eta, \quad \zeta,\eta \in p^{-1}(0),\ |\zeta - a\gamma| < C\varepsilon a,\ |\eta + a\overline{\gamma}| < C\varepsilon a,$$

and we therefore get the estimate

$$\left|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot z}\,dx\right| \le C_4 h^{-1}\|f\|_{L^\infty(\Omega)}e^{-\frac{ca}{2h}}\, e^{\frac{2C\varepsilon a}{h}} \tag{83}$$

for all $z \in \mathbb{C}^n$ such that $|z - 2aie_1| < 2\varepsilon a$.

This implies that the Bargmann-Segal transform of f satisfies

$$|Tf(z)| \le C|f\|_{L^\infty(\Omega)}e^{\frac{1}{2h}(|Imz|^2 - |Rez|^2 - \frac{ca}{2})} \tag{84}$$

for some $\epsilon, a, c > 0$ and for all $z \in \mathbb{C}^n$ such that $|z - 2aie_1| < 2\varepsilon a$.

By the definition of the analytic wave front set, the last estimate says that the point $(0, 2ae_1)$ is not in the analytic wave front set of f. By Kashiwara's watermelon theorem [121], [73], since f is supported in the half space $x_1 \leq 0$, if 0 is in the support of f then $(0, \nu)$ with ν the unit normal to the boundary is also in the analytic wave front set but this is a contradiction since $2ae_1$ is also normal to $x_1 = 0$. Therefore 0 is not in the support of f and f vanishes in a neighborhood of 0.

8. The Calderón Problem in Two Dimensions

Astala and Päivärinta [8], in a seminal contribution, have recently extended significantly the uniqueness result of [91] for conductivities having two derivatives in an appropriate sense and the result of [20] for conductivities having one derivative in appropriate sense, by proving that any L^∞ conductivity in two dimensions can be determined uniquely from the DN map. We remark that the method of [91] and [20] uses, besides CGO solutions, the $\overline{\partial}$ method introduced in one dimension by Beals and Coifman [13] and generalized to several dimensions in [1], [93], [14], [137]. The $\overline{\partial}$ method has been used in numerical reconstruction procedures in two dimensions in [61], [119] among others.

The proof of [8] relies also on construction of CGO solutions for the conductivity equation with L^∞ coefficients and the $\overline{\partial}$ method. This is done by transforming the conductivity equation to a quasi-regular map. Let $\mathcal{D}$ be the unit disk in the plane. Then we have

Lemma 8.1 *Assume $u \in H^1(\mathcal{D})$ is real valued and satisfies the conductivity equation on $\mathcal{D}$. Then there exists a function $v \in H^1(\mathcal{D})$, unique up to a constant, such that $f = u + iv$ satisfies the Beltrami equation*

$$\overline{\partial} f = \mu \overline{\partial f}, \tag{85}$$

where $\mu = (1 - \gamma)/(1 + \gamma)$.

Conversely, if $f \in H^1(\mathcal{D})$ satisfies (85) with a real-valued μ, then $u = Ref$ and $v = Imf$ satisfy

$$\nabla \cdot \gamma \nabla u = 0 \quad \textit{and} \quad \nabla \cdot \frac{1}{\gamma} \nabla v = 0, \tag{86}$$

respectively, where $\gamma = (1 - \mu)/(1 + \mu)$.

Let us denote $\kappa = ||\mu||_{L^\infty} < 1$. Then (85) means that f is a quasi-regular map. The function v is called the γ-harmonic conjugate of u and it is unique up to a constant.

Astala and Päivärinta consider the μ-Hilbert transform $\mathcal{H}_\mu : H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ that is defined by

$$\mathcal{H}_\mu : u\big|_{\partial\Omega} \mapsto v\big|_{\partial\Omega}$$

and show that the DN map Λ_γ determines $\mathcal{H}_\mu$ and vice versa.

Below we use the complex notation $z = x_1 + ix_2$. Moreover, for the equation (85), it is shown that for every $k \in \mathbb{C}$ there are complex geometrical optics solutions of the Beltrami equation that have the form

$$f_\mu(z, k) = e^{ikz} M_\mu(z, k), \tag{87}$$

where

$$M_\mu(z, k) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \rightarrow \infty. \tag{88}$$

More precisely, they prove that

Theorem 8.2 *For each $k \in \mathbb{C}$ and for each $2 < p < 1 + 1/\kappa$ the equation (85) admits a unique solution $f \in W^{1,p}_{loc}(\mathbb{C})$ of the form (87) such that the asymptotic formula (88) holds true.*

In the case of non-smooth coefficients the function M_μ grows sub-exponentially in k. Astala and Päivärinta introduce the "transport matrix" to deal with this problem. Using a result of Bers connecting pseudoanalytic functions with quasi-regular maps they show that this matrix is determined by the Hilbert transform H_μ and therefore the DN map. Then they use the transport matrix to show that Λ_γ determines uniquely γ. See [8] for more details. Logarithmic type stability estimates for Hölder conductivities of positive exponent have been given in [12].

8.1. Bukhgeim's Result

In a recent breakthrough, Bukhgeim [21] proved that a potential in $W^{2,p}(\Omega), p > 2$ can be uniquely determined from the set of Cauchy data as defined in (36). An earlier result [128] gave this for a generic class of potentials. As before, if two potentials q_1, q_2 have the same set of Cauchy data, we have

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0 \tag{89}$$

where $u_i, i = 1, 2$, are solutions of the Schrödinger equation.

Assume now that $0 \in \Omega$. Bukhgeim takes CGO solutions of the form

$$u_1(z,k) = e^{z^2 k}(1 + \psi_1(z,k)), \quad u_2(z,k) = e^{-\overline{z}^2 k}(1 + \psi_2(z,k)) \tag{90}$$

where $z, k \in \mathbb{C}$ and we have used the complex notation $z = x_1 + ix_2$. Moreover ψ_1 and ψ_2 decay uniformly in Ω, in an appropriate sense, for $|k|$ large.

Note that the weight $z^2 k$ in the exponential is a limiting Carleman weight since it is a harmonic function but it is singular at 0 since its gradient vanishes there.

Substituting (90) into 89 we obtain

$$\int_\Omega e^{2i\tau x_1 x_2}(q_1 - q_2)(1 + \psi_1 + \psi_2 + \psi_1\psi_2) dx = 0.$$

Now using the decay of ψ_i in τ, $i = 1, 2$, and applying stationary phase (the phase function $x_1 x_2$ that has a non-degenerate critical point at 0) we obtain $q_1(0) = q_2(0) = 0$ in Ω. Of course we can do this at any point of Ω proving the result.

This result also shows that complex conductivities can be determined uniquely from the DN map. Francini has shown in [36] that this was the case for conductivities with small imaginary part. It also implies unique determination of a potential from the fixed energy scattering amplitude in two dimensions.

8.2. Partial Data Problem in 2D

It is shown in [59] that for a two dimensional bounded domain the Cauchy data for the Schrödinger equation measured on an arbitrary open subset of the boundary determines uniquely the potential. This implies, for the conductivity equation, that if one measures the current fluxes at the boundary on an arbitrary open subset of the boundary produced by voltage potentials supported in the same subset, one can determine uniquely the conductivity. The paper [59] uses Carleman estimates with weights which are harmonic functions with non-degenerate critical points to construct appropriate complex geometrical optics solutions to prove the result. We describe this more precisely below.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain which consists of N smooth closed curves γ_j, $\partial\Omega = \cup_{j=}^{N} \gamma_j$.

As before we define the set of Cauchy data for a bounded potential q by:

$$\widehat{C_q} = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \mid (\Delta - q)u = 0 \text{ on } \Omega, \quad u \in H^1(\Omega) \right\}. \tag{91}$$

Let $\Gamma \subset \partial\Omega$ be a non-empty open subset of the boundary. Denote $\Gamma_0 = \partial\Omega \setminus \overline{\Gamma}$.

The main result of [59] gives global uniqueness by measuring the Cauchy data on Γ. Let $q_j \in C^{2+\alpha}(\overline{\Omega})$, $j = 1,2$ for some $\alpha > 0$ and let q_j be complex-valued. Consider the following sets of Cauchy data on Γ:

$$\mathcal{C}_{q_j} = \left\{ \left(u|_{\Gamma}, \frac{\partial u}{\partial \nu}\Big|_{\Gamma} \right) \mid (\Delta - q_j)u = 0 \, \text{in} \, \Omega, \; u|_{\Gamma_0} = 0, \; u \in H^1(\Omega) \right\}, \quad j = 1,2. \tag{92}$$

Theorem 8.3 *Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. Then $q_1 \equiv q_2$.*

Using Theorem 8.3 one concludes immediately, as a corollary, the following global identifiability result for the conductivity equation (2). This result uses that knowledge of the Dirichlet-to-Neumann map on an open subset of the boundary determines γ and its first derivatives on Γ (see [80], [133]).

Corollary 8.4 *With some $\alpha > 0$, let $\gamma_j \in C^{4+\alpha}(\overline{\Omega})$, $j = 1,2$, be non-vanishing functions. Assume that*

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \; on \, \Gamma \; for \; all \; f \in H^{\frac{1}{2}}(\Gamma), \; supp \, f \subset \Gamma.$$

Then $\gamma_1 = \gamma_2$.

It is easy to see that Theorem 8.3 implies the analogous result to [75] in the two dimensional case.

Notice that Theorem 8.3 does not assume that Ω is simply connected. An interesting inverse problem is whether one can determine the potential or conductivity in a region of the plane with holes by measuring the Cauchy data only on the accessible boundary. This is also called the obstacle problem.

Let Ω, D be domains in $\mathbb{R}^2$ with smooth boundaries such that $\overline{D} \subset \Omega$. Let $V \subset \partial\Omega$ be an open set. Let $q_j \in C^{2+\alpha}(\overline{\Omega \setminus D})$, for some $\alpha > 0$ and $j = 1,2$. Let us consider the following set of partial Cauchy data

$$\widetilde{C}_{q_j} = \{(u|_V, \frac{\partial u}{\partial \nu}|_V) | (\Delta - q_j)u = 0 \text{ in } \Omega \setminus \overline{D}, \, u|_{\partial D \cup \partial\Omega \setminus V} = 0, u \in H^1(\Omega \setminus \overline{D})\}.$$

Corollary 8.5 *Assume $\widetilde{C}_{q_1} = \widetilde{C}_{q_2}$. Then $q_1 = q_2$.*

A similar result holds for the conductivity equation.

Corollary 8.6 *Let $\gamma_j \in C^{4+\alpha}(\overline{\Omega \setminus D})$ $j = 1,2$ be non vanishing functions. Assume*

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \; on \; V \quad \forall f \in H^{\frac{1}{2}}(\partial(\overline{\Omega} \setminus D)), \; supp \, f \subset V$$

Then $\gamma_1 = \gamma_2$.

The two dimensional case has special features since one can construct a much larger set of complex geometrical optics solutions than in higher dimensions. On the other hand, the problem is formally determined in two dimensions and therefore more difficult. The proof of Theorem 8.3 is based on the construction of appropriate complex geometrical optics solutions by Carleman estimates with degenerate weight functions.

Sketch of the Proof. For the partial data problem we need a more general glass of CGO solutions than the ones constructed by Bukhgeim, since we would like to have the imaginary part of the phase vanish on Γ. So we consider more general holomorphic functions with non-degenerate critical points as phases.

Let the function $\Phi(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2) \in C^2(\overline{\Omega})$ be holomorphic in Ω and $\mathrm{Im}\ \Phi|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$. Notice that this implies $\nabla\varphi \cdot \nu = 0$ on $\partial\Omega \setminus \widetilde{\Gamma}$.

We denote the set of critical points of Φ by

$$\mathcal{H} = \{z \in \overline{\Omega} | \partial_z \Phi(z) = 0\}.$$

We assume that Φ has a finite number of non-degenerate critical points in $\overline{\Omega}$, that is $\partial_z^2 \Phi(z) \neq 0$, $z \in \mathcal{H}$. We denote the critical points by $\mathcal{H} = \{\widetilde{x}_1, ..., \widetilde{x}_\ell\}$

As in the partial data problem considered in section 7 we construct appropriate CGO solutions by proving a Carleman estimate.

Carleman estimate

$u \in H_0^1(\Omega)$, real valued. Then for all large $\tau > 0$:

$$\tau \|ue^{\tau\varphi}\|^2_{L^2(\Omega)} + \|ue^{\tau\varphi}\|^2_{H^1(\Omega)} + \left\|\frac{\partial u}{\partial \nu} e^{\tau\varphi}\right\|^2_{L^2(\partial\Omega\setminus\widetilde{\Gamma})} + \tau^2 \left\|\left|\frac{\partial \Phi}{\partial z}\right| ue^{\tau\varphi}\right\|^2_{L^2(\Omega)}$$
$$\leq C\left(\|(\Delta u)e^{\tau\varphi}\|^2_{L^2(\Omega)} + \tau \int_{\widetilde{\Gamma}} \left|\frac{\partial u}{\partial \nu}\right|^2 e^{2\tau\varphi} d\sigma\right)$$

with σ the standard measure on $\partial\Omega$.

The Carleman estimate implies the existence of a solution to the boundary value problem for the Schrödinger equation

$$(\Delta - q)u = f \text{ in } \Omega; \quad u|_{\partial\Omega\setminus\widetilde{\Gamma}} = g \tag{93}$$

and that it satisfies an estimate. More precisely we have

Proposition 8.7 *Let $q \in L^\infty(\Omega)$. There exists $\tau_0 > 0$ such that for all $|\tau| > \tau_0$ there exists a solution of (93) such that*

$$\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \leq C\big(\|fe^{-\tau\varphi}\|_{L^2(\Omega)}/\tau + \|ge^{-\tau\varphi}\|_{L^2(\partial\Omega\setminus\widetilde{\Gamma})}\big).$$

We next find CGO solutions of

$$(\Delta - q)u = 0 \quad \text{in } \Omega; \quad u|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0 \tag{94}$$

of the form

$$u(x) = e^{\tau\Phi(z)}(a(z)+a_0(z)/\tau)+e^{\tau\overline{\Phi(z)}}\overline{(a(z) + a_1(z)/\tau)}+e^{\tau\varphi}u_1+e^{\tau\varphi}u_2. \tag{95}$$

The functions $a, a_0, a_1 \in C^2(\overline{\Omega})$ are holomorphic in Ω and $\mathrm{Re}\, a|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$.

Moreover

$$\|u_j\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right), \quad \tau \to \infty, j = 1, 2. \tag{96}$$

Now we take two potentials q_1 and q_2 satisfying the hypothesis of Theorem 8.3. We take for the potential q_1 a solution u of the corresponding Schrödinger equation of the form (95) and for the Schrödinger equation associated to q_2 a solution v of the form

$$v(x) = e^{-\tau\Phi(z)}(a(z) + b_0(z)/\tau) + e^{-\tau\overline{\Phi(z)}}\overline{(a(z) + b_1(z)/\tau)} + e^{-\tau\varphi}v_1 + e^{-\tau\varphi}v_2 \quad (97)$$

with v_1, v_2 satisfying the same decay for large τ as u_1, u_2. Using arguments similar to those of section 7 we get

$$\int_\Omega (q_1 - q_2)uvdx = 0. \quad (98)$$

Substituting (94) and (97) into this identity and applying stationary phase we conclude

Proposition 8.8 *Let $\{\widetilde{x}_1, \ldots, \widetilde{x}_\ell\}$ be the set of critical points of the function $Im\,\Phi$. Then for any potentials q_1, q_2 satisfying the hypotheses of Theorem 8.3 and for any holomorphic function a, we have*

$$2\sum_{k=1}^{\ell} \frac{\pi((q_1 - q_2)|a|^2)(\widetilde{x}_k) Re\, e^{2i\tau Im\,\Phi(\widetilde{x_k})}}{|(det\, Im\,\Phi'')(\widetilde{x_k})|^{\frac{1}{2}}} = 0, \quad \tau > 0.$$

We can choose Φ such that

$$\mathrm{Im}\,\Phi(\widetilde{x_k}) \neq \mathrm{Im}\,\Phi(\widetilde{x_j}),\ j \neq k.$$

Let $a(\widetilde{x_k}) \neq 0$. Then Proposition 8.8 implies

$$q_1(\widetilde{x_k}) = q_2(\widetilde{x_k}).$$

We then show that the non-degenerate critical points of Φ can be chosen to be a dense set concluding the sketch of the proof of the theorem.

9. Anisotropic Conductivities

Anisotropic conductivities depend on direction. Muscle tissue in the human body is an important example of an anisotropic conductor. For instance cardiac muscle has a conductivity of 2.3 mho in the transverse direction and 6.3 in the longitudinal direction. The conductivity in this case is represented by a positive definite, smooth, symmetric matrix $\gamma = (\gamma^{ij}(x))$ on Ω.

Under the assumption of no sources or sinks of current in Ω, the potential u in Ω, given a voltage potential f on $\partial\Omega$, solves the Dirichlet problem

$$\begin{cases} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) &= 0 \text{ on } \Omega \\ u|_{\partial\Omega} &= f. \end{cases} \quad (99)$$

The DN map is defined by

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i\gamma^{ij}\frac{\partial u}{\partial x_j}\Big|_{\partial\Omega} \quad (100)$$

where $\nu = (\nu^1, \ldots, \nu^n)$ denotes the unit outer normal to $\partial\Omega$ and u is the solution of (99). The inverse problem is whether one can determine γ by knowing Λ_γ.

Unfortunately, Λ_γ doesn't determine γ uniquely. This observation is due to L. Tartar (see [80] for an account).

Let $\psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism with $\psi|_{\partial\Omega} = Id$ where Id denotes the identity map. We have

$$\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma \tag{101}$$

where

$$\widetilde{\gamma} = \left(\frac{(D\psi)^T \circ \gamma \circ (D\psi)}{|\mathrm{det} D\psi|} \right) \circ \psi^{-1}. \tag{102}$$

Here $D\psi$ denotes the (matrix) differential of ψ, $(D\psi)^T$ its transpose and the composition in (102) is to be interpreted as multiplication of matrices.

We have then a large number of conductivities with the same DN map: any change of variables of Ω that leaves the boundary fixed gives rise to a new conductivity with the same electrostatic boundary measurements. The question is then whether this is the only obstruction to unique identifiability of the conductivity.

In two dimensions this has been shown for $L^\infty(\Omega)$ conductivities in [9]. This is done by reducing the anisotropic problem to the isotropic one by using isothermal coordinates [2], [130] and using Astala and Päivärinta's result in the isotropic case [8]. Earlier results were for C^3 conductivities using the result of Nachman [91] and for Lipschitz conductivities in [127] using the techniques of [20]. An extension of some of these results to quasilinear anisotropic conductivities can be found in [129].

In three dimensions, as was pointed out in [85], this is a problem of geometrical nature and makes sense for general compact Riemannian manifolds with boundary.

Let (M, g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric g is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det g} g^{ij} \frac{\partial u}{\partial x_j} \right) \tag{103}$$

where (g^{ij}) is the matrix inverse of the matrix (g_{ij}). Let us consider the Dirichlet problem associated to (103)

$$\Delta_g u = 0 \text{ on } \Omega, \quad u|_{\partial\Omega} = f. \tag{104}$$

We define the DN map in this case by

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^i g^{ij} \frac{\partial u}{\partial x_j} \sqrt{\det g}|_{\partial\Omega} \tag{105}$$

The inverse problem is to recover g from Λ_g.

We have

$$\Lambda_{\psi^* g} = \Lambda_g \tag{106}$$

where ψ is any C^∞ diffeomorphism of $\overline{M}$ which is the identity on the boundary. As usual $\psi^* g$ denotes the pull back of the metric g by the diffeomorphism ψ.

In the case that M is an open, bounded subset of $\mathbb{R}^n$ with smooth boundary, it is easy to see ([85]) that for $n \geq 3$

$$\Lambda_g = \Lambda_\gamma \tag{107}$$

where

$$(g_{ij}) = (\mathrm{det}\gamma^{kl})^{\frac{1}{n-2}} (\gamma^{ij})^{-1}, \quad (\gamma^{ij}) = (\det g_{kl})^{\frac{1}{2}} (g_{ij})^{-1}. \tag{108}$$

In the two dimensional case there is an additional obstruction since the Laplace-Beltrami operator is conformally invariant. More precisely we have

$$\Delta_{\alpha g} = \frac{1}{\alpha}\Delta_g$$

for any function α, $\alpha \neq 0$. Therefore we have, for $n = 2$,

$$\Lambda_{\alpha(\psi^* g)} = \Lambda_g \tag{109}$$

for any smooth function $\alpha \neq 0$ so that $\alpha|_{\partial M} = 1$.

Lassas and Uhlmann ([83]) proved that (106) is the only obstruction to unique identifiability of the conductivity for real-analytic manifolds in dimension $n \geq 3$.

In the two dimensional case they showed that (109) is the only obstruction to unique identifiability for smooth Riemannian surfaces. Moreover these results assume that the DN map is measured only on an open subset of the boundary. We state the two basic results.

Let Γ be an open subset ∂M. We define for f, $\operatorname{supp} f \subseteq \Gamma$

$$\Lambda_{g,\Gamma}(f) = \Lambda_g(f)|_\Gamma.$$

Theorem 9.1 ($n = 2$) *Let (M, g) be a compact Riemannian surface with boundary. Let $\Gamma \subseteq \partial M$ be an open subset. Then $\Lambda_{g,\Gamma}$ determines uniquely the conformal class of (M, g).*

Theorem 9.2 ($n \geq 3$) *Let (M, g) be a real-analytic compact, connected Riemannian manifold with boundary. Let $\Gamma \subseteq \partial M$ be real-analytic and assume that g is real-analytic up to Γ. Then $\Lambda_{g,\Gamma}$ determines uniquely (M, g) up to an isometry.*

Einstein manifolds are real-analytic in the interior and it was conjectured by Lassas and Uhlmann that they were uniquely determined up to isometry by the DN map. This was proven in [45].

Notice that these results don't assume any condition on the topology of the manifold except for connectedness. An earlier result of [85] assumed that (M, g) was strongly convex and simply connected and $\Gamma = \partial M$ in both results. Theorem 9.2 was extended in [84] to non-compact, connected real-analytic manifolds with boundary.

9.1. The Calderón Problem on Manifolds

The invariant form on a Riemannian manifold with boundary (M, g) for an isotropic conductivity β is given by

$$\operatorname{div}_g(\beta\nabla_g)u = 0 \tag{110}$$

where div_g (resp. ∇_g) denotes divergence (resp. gradient) with respect to the Riemannian metric g. This includes the case considered by Calderón with g the Euclidean metric, the anisotropic case by taking $g^{ij} = \gamma^{ij}\beta$ and $\beta = \sqrt{\det g}$.

It was shown in [127] for bounded domains of Euclidean space in two that the isometric class of (β, g) is determined uniquely by the DN map associated to 110. In two dimensions, when the metric g is known, it is proven in [50] that one can uniquely determine the conductivity β. Guillarmou and Tzou [46] have shown that a potential is uniquely determined for the Schrödinger equation $\Delta_g - q$, with Δ_g the Laplace-Beltrami operator associated to the metric g, generalizing the result of [50].

In dimension $n \geq 3$ it is an open problem whether one can determine the isotropic conductivity β from the corresponding DN map associated to (110). As before one can consider the more general problem of recovering the potential q from the DN map associated to $\Delta_g - q$. We review below the progress that has been made on this problem based on [31].

9.2. Complex geometrical optics on manifolds

We review the recent construction of complex geometrical optics construction for a class of Riemannian manifolds based on [30].

In this paper those Riemannian manifolds which admit limiting Carleman weights, were characterized. All such weights in Euclidean space were listed in section 7.

Theorem 9.3 *If (M, g) is an open manifold having a limiting Carleman weight, then some conformal multiple of the metric g admits a parallel unit vector field. For simply connected manifolds, the converse is also true.*

Locally, a manifold admits a parallel unit vector field if and only if it is isometric to the product of an Euclidean interval and another Riemannian manifold. This is an instance of the de Rham decomposition [109]. Thus, if (M, g) has an LCW φ, one can choose local coordinates in such a way that $\phi(x) = x_1$ and

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$

where c is a positive conformal factor. Conversely, any metric of this form admits $\varphi(x) = x_1$ as a limiting weight.

In the case $n = 2$, limiting Carleman weights in (M, g) are exactly the harmonic functions with non-vanishing differential.

Let us now introduce the class of manifolds which admit limiting Carleman weights and for which one can prove uniqueness results. For this we need the notion of simple manifolds [117].

Definition 9.4 *A manifold (M, g) with boundary is* simple *if ∂M is strictly convex, and for any point $x \in M$ the exponential map $\exp_x$ is a diffeomorphism from some closed neighborhood of 0 in T_xM onto M.*

Definition 9.5 *A compact manifold with boundary (M, g), of dimension $n \geq 3$, is* admissible *if it is conformal to a submanifold with boundary of $\mathbb{R} \times (M_0, g_0)$ where (M_0, g_0) is a compact simple $(n-1)$-dimensional manifold.*

Examples of admissible manifolds include the following:

1. Bounded domains in Euclidean space, in the sphere minus a point, or in hyperbolic space. In the last two cases, the manifold is conformal to a domain in Euclidean space via stereographic projection.
2. More generally, any domain in a locally conformally flat manifold is admissible, provided that the domain is appropriately small. Such manifolds include locally symmetric 3-dimensional spaces, which have parallel curvature tensor so their Cotton tensor vanishes [33].
3. Any bounded domain M in $\mathbb{R}^n$, endowed with a metric which in some coordinates has the form
$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$
with $c > 0$ and g_0 simple, is admissible.
4. The class of admissible metrics is stable under C^2-small perturbations of g_0.

The first inverse problem involves the Schrödinger operator

$$\mathcal{L}_{g,q} = \Delta_g - q,$$

where q is a smooth complex valued function on (M, g). We make the standing assumption that 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g,q}$ in M. Then the Dirichlet problem

$$\left\{ \; L_{g,q} u = 0 \text{ in } M, \quad u = f \text{ on } \partial M \right.$$

has a unique solution for any $f \in H^{1/2}(\partial M)$, and we may define the DN map

$$\Lambda_{g,q} : f \mapsto \partial_\nu u|_{\partial M}.$$

Given a fixed admissible metric, one can determine the potential q from boundary measurements.

Theorem 9.6 *Let (M, g) be admissible, and let q_1 and q_2 be two smooth functions on M. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then $q_1 = q_2$.*

This result was known previously in dimensions $n \geq 3$ for the Euclidean metric [131] and for the hyperbolic metric [68]. It has been generalized to Maxwell's equations in [74].

10. The Boundary Rigidity Problem and the DN map

We give here a surprising connection between the DN map and the boundary rigidity problem in two dimensions, two seemingly quite different inverse problems.

The boundary rigidity problem is that of determining the Riemannian metric of a compact Riemannian manifold with boundary (M, g) by measuring the lengths of geodesics joining points on the boundary. The information is encoded in the distance function d_g between boundary points. This problem also arose in geophysics in determining the substructure of the Earth by measuring the travel times of seismic waves. The Riemannian metric in the isotropic case is given by

$$ds^2 = \frac{1}{c^2(x)} dx^2 \tag{111}$$

where $c(x)$ denotes the wave speed.

Herglotz [51] considered the case where M is spherically symmetric and the sound speed is smooth and depends only on the radius. Under the condition that $\frac{d}{dr}(\frac{r}{c(r)}) > 0$, i.e. there are no regions of low velocities, they gave a formula to find $c(r)$ from the lengths of geodesics. The anisotropic case has also been of interest since it has been shown that the inner core of the Earth exhibits anisotropic behavior [28].

We have, similar to the invariance discussed in the section 9, $d_{\psi^* g} = d_g$ for any diffeomorphism $\psi : M \to M$ that leaves the boundary pointwise fixed, i.e., $\psi|_{\partial M} = Id$. If this is the only obstruction the manifold is said to be boundary rigid.

It is easy to see that not all Riemannian manifolds are boundary rigid since the boundary distance function only takes into account the minimizing geodesics (first arrival time of waves) and not all geodesics. Some a-priori restriction is needed on the manifold. The most usual restriction assumed is simplicity of the manifold as in section 9.

Michel conjectured that all simple manifolds are boundary rigid [89]. For a review of recent results on this problem see [123]. Pestov and Uhlmann [108] have proven this conjecture in the two dimensional case by making a surprising connection with the DN map. The result is:

Theorem 10.1 *Two dimensional compact, simple Riemannian manifolds with boundary are boundary rigid.*

The main lemma in the proof states that, under the assumptions of the theorem, the boundary distance function determines the DN map associated to the Laplace-Beltrami operator. In other words $d_{g_1} = d_{g_2}$ implies that $\Lambda_{g_1} = \Lambda_{g_2}$. By Theorem 5.2, there exists a diffeomorphism $\psi : M \longrightarrow M$, $\psi|_{\partial M} = Id$ and a function $\beta \neq 0, \beta|_{\partial M} = 1$ such that $g_1 = \beta\psi^* g_2$. Mukhometov's theorem [90] implies that $\beta = 1$ showing that $g_1 = \psi^* g_2$ proving Theorem 7.1.

11. Invisibility for Electrostatics

We discuss here only invisibility results for electrostatics. For similar results for electromagnetic waves, acoustic waves, quantum waves, etc., see the review papers [39], [40] and the references given there.

The fact that the boundary measurements do not change, when a conductivity is pushed forward by a smooth diffeomorphism leaving the boundary fixed (see section 9), can already be considered as a weak form of invisibility. Different media appear to be the same, and the apparent location of objects can change. However, this does not yet constitute real invisibility, as nothing has been hidden from view.

In invisibility cloaking the aim is to hide an object inside a domain by surrounding it with a material so that even the presence of this object can not be detected by measurements on the domain's boundary. This means that all boundary measurements for the domain with this cloaked object included would be the same as if the domain were filled with a homogeneous, isotropic material. Theoretical models for this have been found by applying diffeomorphisms having singularities. These were first introduced in the framework of electrostatics, yielding counterexamples to the anisotropic Calderón problem in the form of singular, anisotropic conductivities in $\mathbb{R}^n, n \geq 3$, indistinguishable from a constant isotropic conductivity in that they have the same Dirichlet-to-Neumann map [42, 43]. The same construction was rediscovered for electromagnetism in [107], with the intention of actually building such a device with appropriately designed metamaterials; a modified version of this was then experimentally demonstrated in [118]. (See also [86] for a somewhat different approach to cloaking in the high frequency limit.)

The first constructions in this direction were based on blowing up the metric around a point [84]. In this construction, let (M, g) be a compact 2-dimensional manifold with non-empty boundary, let $x_0 \in M$ and consider the manifold

$$\widetilde{M} = M \setminus \{x_0\}$$

with the metric

$$\widetilde{g}_{ij}(x) = \frac{1}{d_M(x, x_0)^2} g_{ij}(x),$$

where $d_M(x, x_0)$ is the distance between x and x_0 on (M, g). Then $(\widetilde{M}, \widetilde{g})$ is a complete, non-compact 2-dimensional Riemannian manifold with the boundary $\partial\widetilde{M} = \partial M$. Essentially, the point x_0 has been "pulled to infinity". On the manifolds M and $\widetilde{M}$ we consider the boundary value problems

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases} \quad \text{and} \quad \begin{cases} \Delta_{\widetilde{g}} \widetilde{u} = 0 & \text{in } \widetilde{M}, \\ \widetilde{u} = f & \text{on } \partial\widetilde{M}, \\ \widetilde{u} \in L^\infty(\widetilde{M}). \end{cases}$$

These boundary value problems are uniquely solvable and define the DN maps

$$\Lambda_{M,g} f = \partial_\nu u|_{\partial M}, \quad \Lambda_{\widetilde{M},\widetilde{g}} f = \partial_\nu \widetilde{u}|_{\partial \widetilde{M}}$$

where ∂_ν denotes the corresponding conormal derivatives. Since, in the two dimensional case, functions which are harmonic with respect to the metric g stay harmonic with respect to any metric which is conformal to g, one can see that $\Lambda_{M,g} = \Lambda_{\widetilde{M},\widetilde{g}}$. This can be seen using e.g. Brownian motion or capacity arguments. Thus, the boundary measurements for (M, g) and $(\widetilde{M}, \widetilde{g})$ coincide. This gives a counter example for the inverse electrostatic problem on Riemannian surfaces – even the topology of possibly non-compact Riemannian surfaces can not be determined using boundary measurements (see Fig. 1).

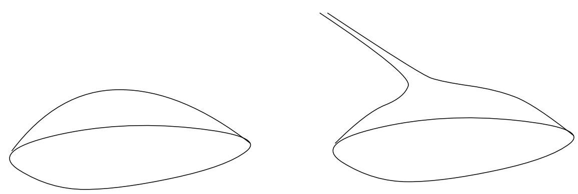

Figure 1. Blowing up a metric at a point, after [84]. The electrostatic boundary measurements on the boundary of the surfaces, one compact and the other noncompact but complete, coincide.

The above example can be thought as a "hole" in a Riemann surface that does not change the boundary measurements. Roughly speaking, mapping the manifold $\widetilde{M}$ smoothly to the set $M \setminus \overline{B}_M(x_0, \rho)$, where $B_M(x_0, \rho)$ is a metric ball of M, and by putting an object in the obtained hole $\overline{B}_M(x_0, \rho)$, one could hide it from detection at the boundary. This observation was used in [42, 43], where "undetectability" results were introduced in three dimensions, using degenerations of Riemannian metrics, whose singular limits can be considered as coming directly from singular changes of variables.

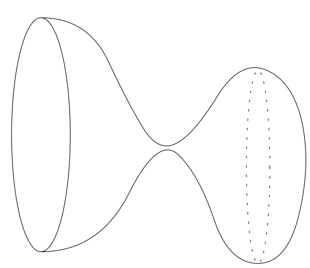

Figure 2. A typical member of a family of manifolds developing a singularity as the width of the neck connecting the two parts goes to zero.

The degeneration of the metric (see Fig. 2) can be obtained by considering surfaces (or manifolds in the higher dimensional cases) with a thin "neck" that is pinched. At the limit the manifold contains a pocket about which the boundary measurements do not give any information. If the collapsing of the manifold is done in an appropriate way, we have, in the limit, a singular Riemannian manifold which is indistinguishable in boundary measurements from a flat surface. Then the conductivity which corresponds to this metric is also singular at the pinched points, cf. the first formula in (114). The electrostatic measurements on the boundary for this singular conductivity will be the same as for the original regular conductivity corresponding to the metric g.

To give a precise, and concrete, realization of this idea, let $B(0,R) \subset \mathbb{R}^3$ denote the open ball with center 0 and radius R. We use in the sequel the set $N = B(0,2)$, the region at the boundary of which the electrostatic measurements will be made, decomposed into two parts, $N_1 = B(0,2) \setminus \overline{B}(0,1)$ and $N_2 = B(0,1)$. We call the interface $\Sigma = \partial N_2$ between N_1 and N_2 the *cloaking surface*.

We also use a "copy" of the ball $B(0,2)$, with the notation $M_1 = B(0,2)$, another ball $M_2 = B(0,1)$, and the disjoint union M of M_1 and M_2. (We will see the reason for distinguishing between N and M.) Let $g_{jk} = \delta_{jk}$ be the Euclidian metrics in M_1 and M_2 and let $\gamma = 1$ be the corresponding isotropic homogeneous conductivity. We define a singular transformation

$$F_1 : M_1 \setminus \{0\} \to N_1, \qquad F_1(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}, \quad 0 < |x| \leq 2. \tag{112}$$

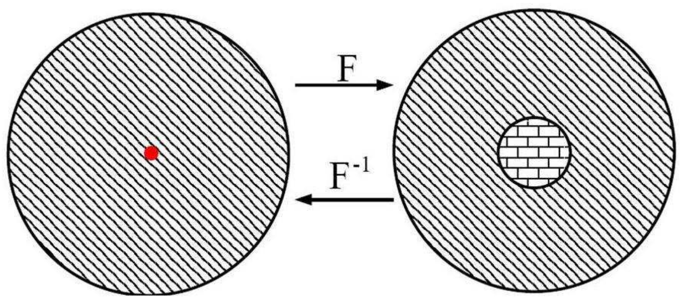

Figure 3. Map $F_1 : B(0,2) \setminus \{0\} \to B(0,2) \setminus \overline{B}(0,1)$

We also consider a regular transformation (diffeomorphism) $F_2 : M_2 \mapsto N_2$, which for simplicity we take to be the identity map $F_2 = Id$. Considering the maps F_1 and F_2 together, $F = (F_1, F_2)$, we define a map $F : M \setminus \{0\} = (M_1 \setminus \{0\}) \cup M_2 \to N \setminus \Sigma$.

The push-forward $\widetilde{g} = F_* g$ of the metric g in M by F is the metric in N given by

$$(F_* g)_{jk}(y) = \left. \sum_{p,q=1}^{n} \frac{\partial F^p}{\partial x^j}(x) \frac{\partial F^q}{\partial x^k}(x) g_{pq}(x) \right|_{x=F^{-1}(y)}. \tag{113}$$

This metric gives rise to a conductivity $\widetilde{\sigma}$ in N which is singular in N_1,

$$\widetilde{\sigma} = \begin{cases} |\widetilde{g}|^{1/2} \widetilde{g}^{jk} & \text{for } x \in N_1, \\ \delta^{jk} & \text{for } x \in N_2. \end{cases} \tag{114}$$

Thus, F forms an invisibility construction that we call "blowing up a point". Denoting by $(r,\phi,\theta) \mapsto (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ the spherical coordinates, we have

$$\widetilde{\sigma} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}, \quad 1 < |x| \leq 2. \tag{115}$$

Note that the anisotropic conductivity $\widetilde{\sigma}$ is singular degenerate on Σ in the sense that it is not bounded from below by any positive multiple of I. (See [79] for a similar calculation.)

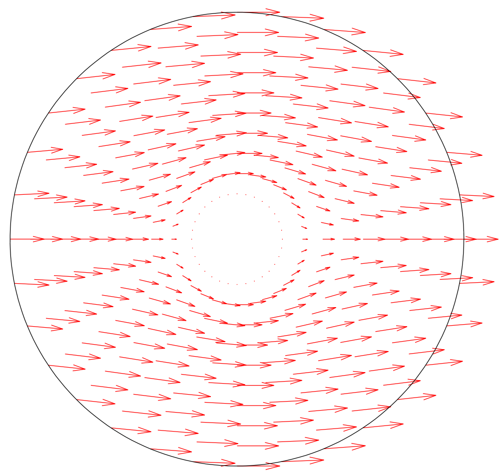

Figure 4. Analytic solutions for the currents

The Euclidian conductivity δ^{jk} in N_2 (114) could be replaced by any smooth conductivity bounded from below and above by positive constants. This would correspond to cloaking of a general object with non-homogeneous, anisotropic conductivity. Here, we use the Euclidian metric just for simplicity.

Consider now the *Cauchy data* of all solutions in the Sobolev space $H^1(N)$ of the conductivity equation corresponding to $\widetilde{\sigma}$, that is,

$$C_1(\widetilde{\sigma}) = \{(u|_{\partial N}, \nu\cdot \widetilde{\sigma}\nabla u|_{\partial N}) \ : \ u \in H^1(N),\ \nabla\cdot \widetilde{\sigma}\nabla u = 0\},$$

where ν is the Euclidian unit normal vector of ∂N.

Theorem 11.1 *([43]) The Cauchy data of all H^1-solutions for the conductivities $\widetilde{\sigma}$ and γ on N coincide, that is, $C_1(\widetilde{\sigma}) = C_1(\gamma)$.*

This means that all boundary measurements for the homogeneous conductivity $\gamma = 1$ and the degenerated conductivity $\widetilde{\sigma}$ are the same. The result above was proven in [42, 43] for the case of dimension $n \geq 3$. The same basic construction works in the two dimensional case [79].

Fig. 4 portrays an analytically obtained solution on a disc with conductivity $\widetilde{\sigma}$. As seen in the figure, no currents appear near the center of the disc, so that if the conductivity is changed near the center, the measurements on the boundary ∂N do not change.

The above invisibility result is valid for a more general class of singular cloaking transformations. A general class, sufficing at least for electrostatics, is given by the following result from [43]:

Theorem 11.2 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and $g = (g_{ij})$ a smooth metric on Ω bounded from above and below by positive constants. Let $D \subset\subset \Omega$ be such that there is a C^∞-diffeomorphism $F : \Omega \setminus \{y\} \to \Omega \setminus \overline{D}$ satisfying $F|_{\partial\Omega} = Id$ and such that*

$$dF(x) \geq c_0 I, \quad det\,(dF(x)) \geq c_1\, dist_{\mathbb{R}^n}\,(x, y)^{-1} \tag{116}$$

where dF is the Jacobian matrix in Euclidian coordinates on $\mathbb{R}^n$ and $c_0, c_1 > 0$. Let $\widehat{g}$ be a metric in Ω which coincides with $\widetilde{g} = F_* g$ in $\Omega \setminus \overline{D}$ and is an arbitrary regular positive definite metric in D^{int}. Finally, let σ and $\widehat{\sigma}$ be the conductivities corresponding to g and $\widehat{g}$, cf. (108). Then,

$$C_1(\widehat{\sigma}) = C_1(\sigma).$$

The key to the proof of Theorem 11.2 is a removable singularities theorem that implies that solutions of the conductivity equation in $\Omega \setminus \overline{D}$ pull back by this singular transformation to solutions of the conductivity equation in the whole Ω.

Returning to the case $\Omega = N$ and the conductivity given by (114), similar types of results are valid also for a more general class of solutions. Consider an unbounded quadratic form, A in $L^2(N, |\widetilde{g}|^{1/2}dx)$,

$$A_{\widetilde{\sigma}}[u, v] = \int_N \widetilde{\sigma} \nabla u \cdot \nabla v \, dx$$

defined for $u, v \in \mathcal{D}(A_{\widetilde{\sigma}}) = C_0^\infty(N)$. Let $\overline{A}_{\widetilde{\sigma}}$ be the closure of this quadratic form and say that

$$\nabla \cdot \widetilde{\sigma} \nabla u = 0 \quad \text{in } N$$

is satisfied in the finite energy sense if there is $u_0 \in H^1(N)$ supported in N_1 such that $u - u_0 \in \mathcal{D}(\overline{A}_{\widetilde{\sigma}})$ and

$$\overline{A}_{\widetilde{\sigma}}[u - u_0, v] = -\int_N \widetilde{\sigma} \nabla u_0 \cdot \nabla v \, dx, \quad \text{for all } v \in \mathcal{D}(\overline{A}_{\widetilde{\sigma}}).$$

Then the Cauchy data set of the finite energy solutions, denoted by

$$C_{f.e.}(\widetilde{\sigma}) = \Big\{ (u|_{\partial N}, \nu \cdot \widetilde{\sigma} \nabla u|_{\partial N}) \, | u \text{ is a finite energy solution of } \nabla \cdot \widetilde{\sigma} \nabla u = 0 \Big\},$$

coincides with the Cauchy data $C_{f.e.}(\gamma)$ corresponding to the homogeneous conductivity $\gamma = 1$, that is,

$$C_{f.e.}(\widetilde{\sigma}) = C_{f.e.}(\gamma). \tag{117}$$

Kohn, Shen, Vogelius and Weinstein [79] in an interesting article have considered the case when instead of blowing up a point one stretches a small ball into the cloaked region. In this case the conductivity is non-singular and one gets "almost" invisibility with a precise estimate in terms of the radius of the small ball.

11.1. Quantum Shielding

As mentioned in section 6, in [41], using CGO solutions, uniqueness was proven for the Calderón problem for Schrödinger operators having a more singular class of potentials, namely potentials conormal to submanifolds of $\mathbb{R}^n, n \geq 3$.

However, for more singular potentials, there are counterexamples to uniqueness. It was constructed in [41] a class of potentials that shield any information about the values of a potential on a region D contained in a domain Ω from measurements of solutions at $\partial\Omega$. In other words, the boundary information obtained outside the shielded region is independent of $q|_D$. On $\Omega \setminus D$, these potentials behave like $q(x) \sim -Cd(x, \partial D)^{-2-\epsilon}$ where d denotes the distance to ∂D and C is a positive constant. In D, Schrödinger's cat could live forever. From the point of view of quantum mechanics, q represents a potential barrier so steep that no tunneling can occur. From the point of view of optics and acoustics, no sound waves or electromagnetic waves will penetrate, or emanate from, D. However, this construction should be thought of as shielding, not cloaking, since the potential barrier that shields $q|_D$ from boundary observation is itself detectable.

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