Commentary on Calderón's paper 28, On an Inverse Boundary Value Problem

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1 Introduction

In the paper [U] we surveyed some of the most important developments motivated by Calderón's beautiful paper [C1] up to Calderón's 75th anniversary conference. As we describe in this note, his only article on inverse problems has continued to have a crucial impact in the field.

In this section we recall the problem which was considered by Calderón in the 40's when he was an engineer working for the Argentinian state oil company "Yacimientos Petrolíferos Fiscales" (YPF). Parenthetically Calderón said in his speech accepting the "Doctor Honoris Causa" of the Universidad Autónoma de Madrid that his work at YPF had been very interesting but he was not well treated there; he would have stayed at YPF otherwise ([C2]). It goes without saying that the bad treatment of Calderón by YPF was very fortunate for Mathematics!

Calderón's motivation was geophysical prospection, in particular oil exploration, using electrical methods. The question is whether one can determine the conductivity of the subsurface of the Earth by making voltage and current measurements at the surface. The problem of determining the electrical properties of a medium by making voltage and current measurements at the boundary has also raised the interest of the medical imaging community and is known as Electrical Impedance Tomography (EIT). One exciting potential application is the early diagnosis of breast cancer. The conductivity of a malignant breast tumor is typically 0.2 mho which is significantly higher than normal tissue which has been typically measured at 0.03 mho. See the book [Ho] and the issue of Physiological Measurement [HIMS] for applications of EIT to medical imaging and other fields.

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary (many of the results we will describe are valid for domains with Lipschitz boundaries). The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the potential $u \in H^1(\Omega)$ with given boundary voltage potential $f \in H^{\frac{1}{2}}(\partial\Omega)$ is a solution of the Dirichlet problem

$$\begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0 \text{ in } \Omega, \\ u\big|_{\partial\Omega} &= f. \end{aligned} \tag{1.1}$$

The Dirichlet to Neumann (DN) map, or voltage to current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega}, \tag{1.2}$$

*The research for this paper was partly supported by the NSF and by a John Simon Guggenheim fellowship

where ν denotes the unit outer normal to $\partial\Omega$.

The inverse problem is to determine γ knowing Λ_γ. Using the divergence theorem we have

$$Q_\gamma(f) := \int_\Omega \gamma|\nabla u|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) f\, dS \tag{1.3}$$

where dS denotes surface measure and u is the solution of (1.1). In other words $Q_\gamma(f)$ is the quadratic form associated to the linear map $\Lambda_\gamma(f)$, i.e., to know $\Lambda_\gamma(f)$ or $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ is equivalent. The form $Q_\gamma(f)$ measures the energy needed to maintain the potential f at the boundary. Calderón's point of view was to find enough solutions $u \in H^1(\Omega)$ of the conductivity equation $\mathrm{div}(\gamma\nabla u) = 0$ so that $|\nabla u|^2$ is dense in an appropriate topology in order to find γ in Ω. Notice that the DN map (or Q_γ) depends non-linearly on γ. Calderón considered the linearized problem at a constant conductivity. A crucial ingredient in his approach is the use of the harmonic complex exponential solutions:

$$u = e^{x\cdot\rho}, \text{ where } \rho \in \mathbb{C}^n \text{ with } \rho\cdot\rho = 0. \tag{1.4}$$

Sylvester and Uhlmann constructed in dimension $n \geq 2$ complex geometrical optics (CGO) solutions of the conductivity equation for C^2 conductivities similar to Calderón's [SU1,2]. This can be reduced to constructing solutions in the whole space (by extending $\gamma = 1$ outside a large ball containing Ω) for the Schrödinger equation with potential

$$q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{1.5}$$

Using this transformation one can reduce the inverse problem, for smooth enough conductivities, to study the DN map associated to the solution of the Dirichlet problem for the Schrödinger equation (assuming that zero is not a Dirichlet eigenvalue)

$$(\Delta - q)u = 0, \quad u|_{\partial\Omega} = f,$$

defined by

$$\Lambda_q(f) = \frac{\partial u}{\partial \nu}.$$

Let $-1 < \delta < 0$. Let ρ be as in (1.4) and $|\rho|$ large. The solutions of

$$(\Delta - q)u = 0$$

constructed in [SU1,2] have the form

$$u = e^{x\cdot\rho}(1 + \psi_q(x,\rho)) \tag{1.6}$$

with $\psi_q(\cdot,\rho) \in L^2_\delta(\mathbb{R}^n)$. Moreover $\psi_q(\cdot,\rho) \in H^2_\delta(\mathbb{R}^n)$ and for $0 \leq s \leq 2$ there exists $C = C(n,s,\delta) > 0$ such that

$$\|\psi_q(\cdot,\rho)\|_{H^s_\delta} \leq \frac{C}{|\rho|^{1-s}}. \tag{1.7}$$

Here L^2_δ is an appropriate weighted L^2 space and H^s_δ is the corresponding weighted Sobolev space.

These solutions were used in [SU1] to show in dimension $n \geq 3$ that the DN map Λ_γ determines uniquely the conductivity. This was the first breakthrough for the non-linear problem in dimension $n \geq 3$ and led to several other developments (again see [U]). In two dimensions A. Nachman showed in [N] that the solutions can be constructed for the Schrödinger equation with a potential of the form (1.5) for all non-zero complex frequencies. This combined with the $\overline{\partial}$ method allowed Nachman to prove that C^2 conductivities in two dimensions could be uniquely determined by the DN map. This was extended to Lipschitz conductivities in [BrU].

This summarizes the basic unique identifiability results for Calderón's problem. We now turn to recent developments.

2 Improvement in smoothness assumptions

Astala and Päivärinta [AP], in a seminal contribution, have recently extended the result of [N] and [BrU] significantly by proving that any L^∞ conductivity in two dimensions can be determined uniquely from the DN map. The proof again relies on CGO solutions. This is done by transforming the conductivity equation to a quasi-regular map. Let $\mathbb{D}$ be the unit disk in the plane. Then we have

Lemma 2.1 *Assume $u \in H^1(\mathbb{D})$ is real valued and satisfies the conductivity equation on $\mathbb{D}$. Then there exists a function $v \in H^1(\mathbb{D})$, unique up to a constant, such that $f = u + iv$ satisfies the Beltrami equation*

$$\overline{\partial} f = \mu \overline{\partial f}, \tag{2.1}$$

where $\mu = (1-\gamma)/(1+\gamma)$.

Conversely, if $f \in H^1(\mathbb{D})$ satisfies (2.1) with a real-valued μ, then $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ satisfy

$$\nabla \cdot \gamma \nabla u = 0 \quad \textit{and} \quad \nabla \cdot \frac{1}{\gamma} \nabla v = 0, \tag{2.2}$$

respectively, where $\gamma = (1-\mu)/(1+\mu)$.

Let us denote $\kappa = ||\mu||_{L^\infty} < 1$. Then (2.1) means that f is a quasi-regular map. The function v is called the γ-harmonic conjugate of u and it is unique up to a constant.

Astala and Päivärinta consider the μ-Hilbert transform $\mathcal{H}_\mu : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ that is defined by

$$\mathcal{H}_\mu : u\big|_{\partial\Omega} \mapsto v\big|_{\partial\Omega}$$

and show that the DN map Λ_γ determines $\mathcal{H}_\mu$ and vice versa.

Below we use the complex notation $z = x_1 + ix_2$. Moreover, for the equation (2.1), it is shown that for every $k \in \mathbf{C}$ there are complex geometrical optics solutions of the Beltrami equation that have the form

$$f_\mu(z,k) = e^{ikz} M_\mu(z,k), \tag{2.3}$$

where

$$M_\mu(z,k) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \to \infty. \tag{2.4}$$

More precisely, they prove that

Theorem 2.1 *For each $k \in \mathbf{C}$ and for each $2 < p < 1 + 1/\kappa$ the equation (2.1) admits a unique solution $f \in W^{1,p}_{loc}(\mathbf{C})$ of the form (2.3) such that the asymptotic formula (2.4) holds true.*

In the case of non-smooth coefficients the function M_μ growths sub-exponentially in k. Astala and Päivärinta introduce the "transport matrix" to deal with this problem. Using a result of Bers connecting pseudoanalytic functions with quasi-regular maps they show that this matrix is determined by the Hilbert transform H_μ and therefore the DN map. Then they use the transport matrix to show that Λ_γ determines uniquely γ. See [AP] for more details.

It is conjectured that in three dimensions or higher the best result on smoothness for unique identifiability from the DN map should be for Lipschitz conductivities. At the writing of this paper the best regularity result assumes 3/2 derivatives on the conductivity ([PPU], [BrT]). These results were proven by also constructing CGO solutions similar to (1.6).

In [GLU1] it is shown by also constructing CGO solutions that conormal potentials which are more singular than $L^{\frac{n}{2}}$ are uniquely determined by the DN map. As a corollary the determination of the conductivity from the DN map result of [GLU1] holds for $C^{1+\epsilon}, \epsilon > 0$ conormal conductivities.

In dimension $n \geq 3$ the smoothness assumptions for the identifiability results for first order perturbations of the Laplacian [To], including the magnetic Schrödinger operator, were improved considerably in [Sa].

3 The local problem in dimension $n \geq 3$

Substantial progress has been made on the *local problem*, that is whether from measuring the DN map on an open subset of the boundary one can determine the conductivity in the interior.

The paper [BuU] used the method of Carleman estimates with a linear weight to prove that, roughly speaking, knowledge of the DN map in "half" of the boundary is enough to determine uniquely a C^2 conductivity. In [K] the regularity assumption on the conductivity was relaxed to $C^{1+\epsilon}, \epsilon > 0$. Stability estimates for the uniqueness result of [BuU] were given in [HW]. Stability estimates for the magnetic Schrödinger operator with partial data in the setting of [BuU] can be found in [Tz].

The [BuU] result was substantially improved in [KSU]. The latter paper contains a global identifiability result where it is assumed that the DN map is measured on any open subset of the boundary for all functions supported, roughly, on the complement. We state below more precisely the result.

Let $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$, where $\mathrm{ch}\,(\Omega)$ denotes the convex hull of Ω. Define the front and the back faces of $\partial\Omega$ by

$$F(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu \leq 0\},\ B(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu > 0\}.$$

The main result of [KSU] is the following:

Theorem 3.1 *With Ω, x_0, $F(x_0)$, $B(x_0)$ defined as above, let $q_1, q_2 \in L^\infty(\Omega)$ be two potentials and assume that there exist open neighborhoods $\widetilde{F}$, $\widetilde{B} \subset \partial\Omega$ of $F(x_0)$ and $B(x_0) \cup \{x \in \partial\Omega; (x - x_0) \cdot \nu = 0\}$ respectively, such that*

$$\Lambda_{q_1} u = \Lambda_{q_2} u \text{ in } \widetilde{F},\ \text{for all } u \in H^{\frac{1}{2}}(\partial\Omega) \cap \mathcal{E}'(\widetilde{B}). \tag{3.1}$$

Then $q_1 = q_2$.

The proof of this result uses Carleman estimates for the Laplacian with limiting Carleman weights (LCW). The Carleman estimates allow one to construct, for large τ, a larger class of CGO solutions for the Schrödinger equation than previously used. These have the form

$$u = e^{\tau(\phi + i\psi)}(a + r), \tag{3.2}$$

where $\nabla\phi \cdot \nabla\psi = 0, |\nabla\phi|^2 = |\nabla\psi|^2$ and ϕ is the LCW. Moreover a is smooth and non-vanishing and $\|r\|_{L^2(\Omega)} = O(\frac{1}{\tau})$, $\|r\|_{H^1(\Omega)} = O(1)$. Examples of LCW are the linear phase $\phi(x) = x \cdot \omega, \omega \in S^{n-1}$, used previously, and the non-linear phase $\phi(x) = \ln|x - x_0|$, where $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$ which was used in [KSU]. Any conformal transformation of these would also be a LCW.

The local DN map for the magnetic Schrödinger operator was studied in [DKSU] and [KS]. We also mention that in [GrU] (resp. [IU]) CGO approximate solutions are concentrated near planes (resp. spheres) and obtained some local results related to the local DN map. For further application of these solutions see the next section.

4 Determination of cavities and inclusions

The CGO solutions have the property that they grow exponentially in a direction where the inner product of the real part of the complex phase with the direction is strictly positive, they are exponentially decaying if this inner product is negative and oscillatory if the inner product is zero. This was exploited by Ikehata in [I] to give a reconstruction procedure from the DN map of a cavity D with strongly convex C^2 boundary ∂D inside a conductive medium Ω with conductivity 1 such that $\Omega \setminus \overline{D}$ is connected. We sketch some of the details here. We define the DN map Λ_D by

$$\Lambda_D(f) := \frac{\partial u(f)}{\partial \nu}|_{\partial\Omega}, \tag{4.1}$$

where $u(f) \in H^2(\Omega)$ is the solution to

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \frac{\partial u}{\partial \nu}|_{\partial D} = 0, \\ u|_{\partial\Omega} = f \in H^{3/2}(\partial\Omega) \end{cases} \tag{4.2}$$

and ν is the unit normal of ∂D. If $D = \emptyset$, we denote Λ_D by Λ_0. Let $\omega, \omega^\perp$ be unit real vectors perpendicular to each other. For $\tau > 0$, consider the Calderón harmonic functions

$$v(x, \tau, \omega, \omega^\perp) = e^{-t\tau} e^{\tau x \cdot (\omega + i\omega^\perp)}. \tag{4.3}$$

Note that this function grows exponentially in the half space $x \cdot \omega > t$ and decays exponentially in the half space $x \cdot \omega < t$. For $t \in \mathbf{R}$, define the indicator function by

$$I_{\omega,\omega^\perp}(\tau, t) := \int_{\partial\Omega} ((\Lambda_D - \Lambda_0) v|_{\partial\Omega}) \overline{v|_{\partial\Omega}} \, dS. \tag{4.4}$$

We also define the support function $h_D(\omega)$ of D by

$$h_D(\omega) := \sup_{x \in D} x \cdot \omega. \tag{4.5}$$

Ikehata characterizes the support function in terms of the indicator function. More precisely we have

$$h_D(\omega) - t = \lim_{\tau \to \infty} \frac{I_{\omega,\omega^\perp}(\tau, t)}{2\tau}. \tag{4.6}$$

Hence, by taking many ω's, we can recover the shape of D. See [I], [IS] for more details and references, including numerical implementation of this method.

Using methods of hyperbolic geometry similar to [IU] it is shown in [IIsNSU] that one can reconstruct inclusions from the *local* DN map using CGO solutions that decay exponentially inside ball and grow exponentially outside, these are called *complex spherical waves*. A numerical implementation of this method has been done in [IIsNSU]. The construction of complex spherical waves can also be done using the CGO solutions constructed in [KSU]. This was done in [UW1] in order to detect elastic inclusions, and in [UW2] to detect inclusions in the two dimensional case for a large class of systems with inhomogeneous background.

5 Anisotropic conductivities

Anisotropic conductivities depend on direction. Muscle tissue in the human body is an important example of an anisotropic conductor. For instance cardiac muscle has a conductivity of 2.3 mho in the transverse direction and 6.3 in the longitudinal direction. The conductivity in this case is represented by a positive definite, smooth, symmetric matrix $\gamma = (\gamma^{ij}(x))$ on $\overline{\Omega}$.

Under the assumption of no sources or sinks of current in Ω, the potential u in Ω, given a voltage potential f on $\partial\Omega$, solves the Dirichlet problem

$$\begin{cases} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij} \frac{\partial u}{\partial x_j}\right) & = & 0 \text{ on } \Omega \\ u|_{\partial\Omega} & = & f. \end{cases} \tag{5.1}$$

The DN map is defined by

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega} \tag{5.2}$$

where $\nu = (\nu^1, \dots, \nu^n)$ denotes the unit outer normal to $\partial\Omega$ and u is the solution of (5.1). The inverse problem is whether one can determine γ by knowing Λ_γ. Unfortunately, Λ_γ doesn't determine γ uniquely. This observation is due to L. Tartar (see [KV] for an account).

Let $\psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism with $\psi\Big|_{\partial\Omega} = Id$ where Id denotes the identity map. We have

$$\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma \tag{5.3}$$

where

$$\widetilde{\gamma} = \left(\frac{(D\psi)^T \circ \gamma \circ (D\psi)}{|\det D\psi|}\right) \circ \psi^{-1}. \tag{5.4}$$

Here $D\psi$ denotes the (matrix) differential of ψ, $(D\psi)^T$ its transpose and the composition in (5.4) is to be interpreted as multiplication of matrices.

We have then a large number of conductivities with the same DN map: any change of variables of Ω that leaves the boundary fixed gives rise to a new conductivity with the same electrical boundary measurements. The question is then whether this is the only obstruction to unique identifiability of the conductivity. It is known that this is the case in two dimensions for C^3 conductivities. This is done by reducing the anisotropic problem to the isotropic one by using isothermal coordinates [Sy] and using Nachman's result [N]. The regularity was improved in [SuU] to Lipschitz conductivities using the techniques of [BrU] and to L^∞ conductivities in [ALP] using the results of [AP].

In the case of dimension $n \geq 3$, as was pointed out in [LeU], this is a problem of geometrical nature and makes sense for general compact Riemannian manifolds with boundary.

Let (M, g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric g is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right) \tag{5.5}$$

where (g^{ij}) is the matrix inverse of the matrix (g_{ij}). Let us consider the Dirichlet problem associated to (5.5)

$$\Delta_g u = 0 \text{ on } \Omega, \quad u\Big|_{\partial\Omega} = f. \tag{5.6}$$

We define the DN map in this case by

$$\Lambda_g(f) = \sum_{i,j=1}^n \nu^i g^{ij} \frac{\partial u}{\partial x_j} \sqrt{\det g}\Big|_{\partial\Omega} \tag{5.7}$$

The inverse problem is to recover g from Λ_g.

We have that

$$\Lambda_{\psi^* g} = \Lambda_g \tag{5.8}$$

where ψ is a C^∞ diffeomorphism of $\overline{M}$ which is the identity on the boundary. As usual $\psi^* g$ denotes the pull back of the metric g by the diffeomorphism ψ.

In the case that M is an open, bounded subset of $\mathbf{R}^n$ with smooth boundary, it is easy to see ([LeU]) that for $n \geq 3$

$$\Lambda_g = \Lambda_\gamma \tag{5.9}$$

where

$$(g_{ij}) = (\det \gamma^{kl})^{\frac{1}{n-2}} (\gamma^{ij})^{-1}, \quad (\gamma^{ij}) = (\det g_{kl})^{\frac{1}{2}} (g_{ij})^{-1}. \tag{5.10}$$

In the two dimensional case there is an additional obstruction since the Laplace-Beltrami operator is conformally invariant. More precisely we have

$$\Delta_{\alpha g} = \frac{1}{\alpha}\Delta_g$$

for any function α, $\alpha \neq 0$. Therefore we have that for $n = 2$

$$\Lambda_{\alpha(\psi^* g)} = \Lambda_g \tag{5.11}$$

for any smooth function $\alpha \neq 0$ so that $\alpha|_{\partial M} = 1$.

Lassas and Uhlmann ([LU]) proved that (5.8) is the only obstruction to unique identifiability of the conductivity for real-analytic manifolds in dimension $n \geq 3$. In the two dimensional case they showed that (5.11) is the only obstruction to unique identifiability for smooth Riemannian surfaces. Moreover these results assume that Λ_g is measured only on an open subset of the boundary. We state the two basic results.

Let Γ be an open subset of ∂M. We define for f, $\operatorname{supp} f \subseteq \Gamma$

$$\Lambda_{g,\Gamma}(f) = \Lambda_g(f)\Big|_{\Gamma}.$$

Theorem 5.1 ($n \geq 3$) *Let (M, g) be a real-analytic compact, connected Riemannian manifold with boundary. Let $\Gamma \subseteq \partial M$ be real-analytic and assume that g is real-analytic up to Γ. Then $(\Lambda_{g,\Gamma}, \partial M)$ determines uniquely (M, g).*

Theorem 5.2 ($n = 2$) *Let (M, g) be a compact Riemannian surface with boundary. Let $\Gamma \subseteq \partial M$ be an open subset. Then $(\Lambda_{g,\Gamma}, \partial M)$ determines uniquely the conformal class of (M, g).*

Notice that these two results don't assume any condition on the topology of the manifold except for connectedness. An earlier result of [LeU] assumed that (M, g) was strongly convex and simply connected and $\Gamma = \partial M$ in both results. Theorem 5.1 was extended in [LTU] to non-compact, connected real-analytic manifolds with boundary.

In two dimensions the invariant form of the conductivity equation is given by

$$\operatorname{div}_g(\beta \nabla_g)u = 0 \tag{5.12}$$

where β is the conductivity and div_g (resp. ∇_g) denotes divergence (resp. gradient) with respect to the Riemannian metric g. This includes the case considered by Calderón with g the Euclidean metric, the anisotropic case by taking ($g^{ij} = \gamma^{ij}$ and $\beta = \sqrt{\det g}$). It was shown in [SuU] for bounded domains of Euclidean space that the isometric class of (β, g) is determined uniquely by the DN map associated to (5.12).

6 Mathematics of invisibility

There has recently been considerable interest in the scientific community, and also the popular press, on the possibility of making objects "invisible", seemingly realizing science fiction dreams with a long history. In particular there were two recent articles in Science (Pendry, et al, [PSS] and Leonhardt[L]) which discussed theoretical "cloaking" devices. These would shield an enclosed object from detection by electromagnetic (EM) waves. In principle, such devices could be constructed using "metamaterials", a catchall phrase coined in the early 2000's, which refers to composites which have physical properties, especially those having to do with the propagation of EM radiation, very different from their constituent materials.

The prescriptions for cloaking devices made of such materials described by Pendry, et al, turn out to be special cases of mathematical constructions of anisotropic conductivities that were given in 2003 in [GLU2,3] for dimensions three and higher, using identical singular transformations - for example, the anisotropic conductivity that has the same boundary information as the homogeneous, isotropic conductivity 1 that is given in spherical coordinates (r, θ, ϕ) in [PSS] was described and rigorously justified earlier in [GLU2,3]. The anisotropic conductivities in these counterexamples are quite pathological - they exhibit perfect insulation in some directions and (in some cases) perfect conduction in others. The counterexamples of [GLU2,3], as well as the constructions of [PSS, L], involve conductivities that are not bounded below and/or above.

Also in 2003 in [GLU1] examples were constructed of very singular potentials that, when changed inside an interior surface, cannot be distinguished from the original by measurements at an outer boundary of solutions of the corresponding Schroedinger equations, interpreted necessarily in a weak, variational sense. Therefore, in theory, it is possible to construct a potential wall such that no particles can "tunnel" through it, using an analogy with quantum mechanics. Thus exterior observers can make no conclusion about the existence of objects or structures inside this wall. Making another analogy with quantum mechanics, in this nest the Schrödinger cat could live happily ever after.

We remark that the invisibility is actually on the level of the actual (distribution) solutions to the underlying PDEs, i.e., the conductivity or Schrödinger equations (at least at zero frequency), which [PSS, L] do not address. In fact, the constructions in [GLU2,3] work for more general classes of examples and can be constructed also for the vector Helmholtz equations at non-zero energy, and for Maxwell's equations [GKLU1]. The latter paper also includes an analysis of what happens inside the cloaked region in particular the case of internal currents. A construction of electromagnetic wormholes which act as invisible tunnels has been given in [GKLU2,3].

7 The boundary rigidity problem and the DN map

The boundary rigidity problem is that of determining the Riemannian metric of a compact Riemannian manifold with boundary (M, g) by measuring the lengths of geodesics joining points on the boundary. The information is encoded in the distance function d_g between boundary points. This problem also arose in geophysics in determining the substructure of the Earth by measuring the travel times of seismic waves. The Riemannian metric in the isotropic case is given by

$$ds^2 = \frac{1}{c^2(x)} dx^2 \tag{7.1}$$

where $c(x)$ denotes the wave speed.

Herglotz [H] considered the case where M is spherically symmetric and the sound speed is smooth and depends only on the radius. Under the condition that $\frac{d}{dr}(\frac{r}{c(r)}) > 0$, i.e. there are no regions of low velocities, they gave a formula to find $c(r)$ from the lengths of geodesics. The anisotropic case has also been of interest since it has been shown that the inner core of the Earth exhibits anisotropic behavior [Cre].

We have similarly to the invariance discussed in the previous section, that $d_{\psi^* g} = d_g$ for any diffeomorphism $\psi : M \to M$ that leaves the boundary pointwise fixed, i.e., $\psi|_{\partial M} = Id$ If this is the only obstruction the manifold is said to be boundary rigid.

It is easy to see that not all Riemannian manifolds are boundary rigid since the boundary distance function only takes into account the minimizing geodesics (first arrival time of waves) and not all geodesics. Some a-priori restriction is needed on the manifold. The most usual restriction assumed is simplicity of the manifold.

Definition 7.1 *A manifold (M,g) is simple if given any two points in M can be joined by a unique minimizing geodesic and the boundary is strictly convex.*

Michel conjectured that all simple manifolds are boundary rigid [Mi]. This has been proved by Gromov for simple subsets of Euclidean space [Gr], by Michel [Mi] for strict simple subsets of the hemisphere in two dimensions and is a consequence of the work of Besson, Courtois and Gallot for symmetric spaces of negative curvature [BCG]. These are all cases of constant curvature. For the case of variable curvature Croke [Cro] and independently Otal [O] proved that two dimensional surfaces of negative curvature are boundary rigid. Stefanov and Uhlmann showed that boundary rigidity holds for a generic class of simple manifolds [SteU]. Pestov and Uhlmann have shown the validity of the conjecture in general in the two dimensional case [PU1]. The result is:

Theorem 7.1 *Two dimensional compact, simple Riemannian manifolds with boundary are boundary rigid.*

The main lemma in the proof states that, under the assumptions of the theorem, the boundary distance function determines the DN map associated to the Laplace-Beltrami operator. In other words $d_{g_1} = d_{g_2}$ implies that $\Lambda_{g_1} = \Lambda_{g_2}$. By Theorem 5.2, there exists a diffeomorphism $\psi : M \longrightarrow M$, $\psi|_{\partial M} = Id$ and a function $\beta \neq 0, \beta|_{\partial M} = 1$ such that $g_1 = \beta\psi^* g_2$. Mukhometov's theorem [Mu] implies that $\beta = 1$ showing that $g_1 = \psi^* g_2$ proving Theorem 7.1.

The proof of [PU1] also gives a constructive method to determine the sound speed in the case that the (simple) metric is of the form (7.1). It is shown in [PU2] that if we know d_g we can determine

$$S_{X,h}[\rho] = \int_M (X, \nabla h)\rho(x)dx \tag{7.2}$$

where $\rho(x) = \frac{1}{c^2(x)}$, h is a harmonic function, and the vector field X is an arbitrary Cauchy-Riemann vector field. More precisely its contravariant components satisfy the Cauchy-Riemann equations

$$\frac{\partial X^1}{\partial x^1} = \frac{\partial X^2}{\partial x^2}, \quad \frac{\partial X^1}{\partial x^2} + \frac{\partial X^2}{\partial x^1} = 0.$$

Finding ρ is then reduced to finding enough holomorphic vector fields u and harmonic functions h so that the product of the gradients is dense in an appropriate space. This is very similar to the question considered by Calderón originally. We use as in [C1] complex exponential solutions.

We choose

$$X^1 = \zeta_2 e^{x\cdot\zeta}, \quad X_2 = \zeta_1 e^{x\cdot\zeta}, \quad h = e^{x\cdot\sigma} \tag{7.3}$$

with complex vectors $\zeta = (\zeta_1, \zeta_2), \sigma = (\sigma_1, \sigma_2) \in C^2; \zeta \cdot \zeta = \sigma \cdot \sigma = 0$ with $\sigma \neq -\overline{\zeta}$. We remark that we can write any $\zeta \in C^2; \zeta \cdot \zeta = 0$, in the form

$$\zeta = \eta + ik, \text{ with } \eta, k \in \mathbf{R}^2 \text{ satisfying } |k| = |\eta|, k \cdot \eta = 0.$$

Substituting (7.3) into (7.2) we obtain:

$$S_{X,h}[\rho] = (\zeta_2\sigma_1 + \zeta_1\sigma_2) \int_M \rho(x)e^{x\cdot(\zeta+\sigma)}dx. \tag{7.4}$$

Therefore we get

$$\frac{S_{X,h}[\rho]}{(\zeta_2\sigma_1 + \zeta_1\sigma_2)} = \int_M \rho(x)e^{x\cdot(\zeta+\sigma)}dx. \tag{7.5}$$

Now by taking the limit

$$\lim_{\sigma\to-\overline{\zeta}} \frac{S_{X,h}[\rho]}{(\zeta_2\sigma_1 + \zeta_1\sigma_2)} = \int_M \rho(x)e^{2ix\cdot k}dx \tag{7.6}$$

we recover the Fourier transform of ρ and therefore ρ.

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