UNIQUENESS FOR THE INVERSE BACKSCATTERING PROBLEM FOR ANGULARLY CONTROLLED POTENTIALS

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Abstract. We consider the problem of recovering a smooth, compactly supported potential on $\mathbb{R}^3$ from its backscattering data. We show that if two such potentials have the same backscattering data and the difference of the two potentials has controlled angular derivatives then the two potentials are identical. In particular, if two potentials differ by a finite linear combination of spherical harmonics and have the same backscattering data then the two potentials are identical.

Key words. backscattering, inverse problem, hyperbolic PDE

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1. Introduction.

1.1. Notation. Below $\preccurlyeq$ denotes 'less than or equal to a constant multiple' with the constant independent of the parameters. Also, B denotes the closed unit ball in $\mathbb{R}^3$, S denotes the unit sphere in $\mathbb{R}^3$ and $q(x)$ a smooth real valued function on $\mathbb{R}^3$ supported in B. We define the angular derivatives $\Omega_{ij} = x_i\partial_j - x_j\partial_i$ for $i, j = 1, 2, 3$, $i \neq j$ and note that

$$\Delta_S := \sum_{i<j} \Omega_{ij}^2$$

is the spherical laplacian, that is Δ_S is the Laplace-Beltrami operator on the unit sphere. Further

$$\int_S (\Omega_{ij} f)\, g\, dS = -\int_S f\,(\Omega_{ij} g)\, dS. \tag{1.1}$$

To any $\rho \geq 0$ and $\omega \in S$, we associate a unique $x = \rho\omega \in \mathbb{R}^3$. Let $\{\phi_n(\omega)\}_{n\geq 1}$ denote an orthonormal basis for $L^2(S)$ consisting of spherical harmonics. Each $\phi_n(\omega)$ is the restriction to S of a homogeneous harmonic polynomials $\phi_n(x)$, and the ϕ_n are indexed so that if $m < n$ then $\deg(\phi_m) \leq \deg(\phi_n)$; further

$$\Delta_S \phi_n = -d_n(d_n + 1)\phi_n \tag{1.2}$$

where $d_n = \deg(\phi_n)$ - see [Se66] and [SW71] for details.

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1.2. The problems and the results. Given a unit vector $\omega \in S$, let $U(x,t,\omega)$ be the solution of the IVP

$$U_{tt} - \Delta U - qU = 0, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.3}$$

$$U(x,t) = \delta(t - x\cdot\omega), \qquad x \in \mathbb{R}^3,\ t < -1. \tag{1.4}$$

We may express $U(x,t,\omega)$ in the form

$$U(x,t,\omega) = \delta(t - x\cdot\omega) + u(x,t,\omega)$$

where $u(x,t,\omega)$ is the solution of the IVP

$$u_{tt} - \Delta u - qu = q\,\delta(t - x\cdot\omega), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \tag{1.5}$$

$$u(x,t) = 0, \qquad x \in \mathbb{R}^3,\ t < -1. \tag{1.6}$$

Theorem 1.1 (Properties of the forward map). *Suppose $q(x)$ is a smooth function on $\mathbb{R}^3$ and supported in the unit ball B and ω, θ are arbitrary unit vectors in S.*

(a) $u(x,t,\omega)$ is supported in the region $t \geq x\cdot\omega$ and, in this region, u is the unique smooth solution of the characteristic IVP problem

$$u_{tt} - \Delta u - qu = 0, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \quad t \geq x\cdot\omega, \tag{1.7}$$

$$u(x, x\cdot\omega, \omega) = \frac{1}{2}\int_{-\infty}^{0} q(x + \sigma\omega)\,d\sigma, \qquad x \in \mathbb{R}^3 \tag{1.8}$$

$$u(x,t,\omega) = 0, \qquad x \in \mathbb{R}^3,\ t < -1. \tag{1.9}$$

(b) For any $\tau \in \mathbb{R}$ we have

$$\lim_{r\to\infty} r\,u(r\theta, r+\tau, \omega) = -\frac{1}{2\pi}\int_{x\cdot\theta=1} (\theta\cdot\nabla u)(x, \tau+1, \omega)\,dS_x. \tag{1.10}$$

(c) Further

$$\partial_t \int_{x\cdot\theta=1} u(x,t,\omega)\,dS_x = -\int_{x\cdot\theta=1} (\theta\cdot\nabla u)(x,t,\omega)\,dS_x \qquad \textit{for all } t \in \mathbb{R}, \tag{1.11}$$

and

$$\int_{x\cdot\theta=s} u(x,t,\omega)\,dS_x = \int_{x\cdot\theta=1} u(x, t-s+1, \omega)\,dS_x, \qquad \textit{for all } s \geq 1,\ t \in \mathbb{R}. \tag{1.12}$$

Actually something slightly stronger than (1.10) is true; if $x^* \in \mathbb{R}^3$ is orthogonal to θ then

$$lim_{r\to\infty} r\,u(x^* + r\theta, r+\tau, \omega) = -\frac{1}{2\pi}\int_{x\cdot\theta=1} (\theta\cdot\nabla u)(x, \tau+1, \omega)\,dS_x.$$

(Note that the RHS is independent of x^*.) This relation holds because the proof of (1.10) is valid if q is just compactly supported and supported in the region $x\cdot\theta \leq 1$. So if we take $v(x,t) = u(x^* + x, t, \omega)$ then v satisfies (1.7) - (1.9) except with $q(x)$ replaced by $q(x^* + x)$, hence the more general form of (1.10) follows from (1.10) applied to v.

The equation (1.10) gives a kind of Friedlander limit so

$$\int_{x\cdot\theta=1} (\theta\cdot\nabla u)(x, \tau+1, \omega)\,dS_x$$

is a good candidate for being called scattering data. This data is the Radon transform on the plane $x \cdot \theta = 1$ of the directional derivative of $u(x, \tau + 1, \omega)$ in the direction θ.

Since $u(x,t,\omega) = 0$ for $t < -1$, (1.11) shows that knowing the Radon transform of $(\theta \cdot \nabla u)(x,t,\omega)$ on the plane $x \cdot \theta = 1$ for all t for a fixed ω and θ is equivalent to knowing the Radon transform of $u(x,t,\omega)$ on the plane $x \cdot \theta = 1$ for all t. Hence we define

$$\alpha(\theta,\omega,\tau) := \int_{x\cdot\theta=1} u(x,\tau,\omega)\, dS_x, \qquad \theta,\omega \in S,\ \tau \in \mathbb{R}$$

as scattering data and our goal is the recovery of $q(\cdot)$ from the backscattering data

$$\beta(\omega,\tau) := \alpha(-\omega,\omega,\tau) = \int_{x\cdot\omega=-1} u(x,\tau,\omega)\, dS_x, \qquad \omega \in S,\ \tau \in \mathbb{R}.$$

Since $u(x,t,\omega) = 0$ for $t < x \cdot \omega$, we observe that the $\beta(\omega,\tau) = 0$ for $\tau \leq -1$.

We prove the following uniqueness results for the inverse backscattering problem.

Theorem 1.2 (Uniqueness for back-scattering data). *Suppose q_i, $i = 1,2$ are smooth functions on $\mathbb{R}^3$ with support in the unit ball B and $\beta_i(\cdot,\cdot)$ the corresponding back scattering data. If there is a constant C, independent of ρ and i,j such that*

$$\int_S |\Omega_{i,j}(q_1 - q_2)(\rho\omega)|^2\, d\omega \leq C \int_S |(q_1 - q_2)(\rho\omega)|^2\, d\omega, \qquad \forall \rho \in [0,1],\ \forall\ i,j = 1,2,3 \tag{1.13}$$

then $\beta_1(\cdot,\tau) = \beta_2(\cdot,\tau)$ for all $\tau \in [-1,0]$ implies $q_1 = q_2$.

If $p(x)$ is a smooth function on $\mathbb{R}^3$ then p has a spherical harmonic expansion $p(\rho\omega) = \sum_{n=1}^{\infty} p_n(\rho)\phi_n(\omega)$. One can show[1] that $p(x) = (q_1 - q_2)(x)$ satisfies the angular derivative condition (1.13) in Theorem 1.2 iff we can find C (independent of ρ) so that

$$\sum_{n=1}^{\infty} d_n(d_n+1)\, p_n(\rho)^2 \leq C \sum_{n=1}^{\infty} p_n(\rho)^2, \qquad \forall \rho \in [0,1]. \tag{1.14}$$

Clearly (1.14) holds if $p_n(\cdot) = 0$ for all $n \geq N$ for some N, but (1.14) also holds for some p with infinite spherical harmonic expansions. In fact, one may show that $d_n < \sqrt{n}$, so if we take $p_1(\rho)$ to be some non-zero function, and choose $p_n(\rho)$ so that

$$(\sqrt{n}+1)|p_n(\rho)| \leq |p_{n-1}(\rho)|, \qquad \forall n \geq 2,\ \rho \in [0,1]$$

[1] Since $(\Omega_{ij}p)(\rho\omega) = \sum_{n=1}^{\infty} p_n(\rho)\,(\Omega_{ij}\phi_n)(\omega)$ so

$$\begin{aligned}
\sum_{i<j}\int_S (\Omega_{ij}p)(\rho\omega)^2\, d\omega &= \sum_{m,n=1}^{\infty} p_n(\rho)\, p_m(\rho) \int_S \sum_{i<j} (\Omega_{ij}\phi_m)(\omega)(\Omega_{ij}\phi_n)(\omega)\, d\omega \\
&= -\sum_{m,n=1}^{\infty} p_n(\rho)\, p_m(\rho) \int_S \sum_{i<j} (\Omega_{ij}^2\phi_m)(\omega)\, \phi_n(\omega)\, d\omega = -\sum_{m,n=1}^{\infty} p_n(\rho)\, p_m(\rho) \int_S (\Delta_S\phi_m)(\omega)\, \phi_n(\omega)\, d\omega \\
&= \sum_{m,n=1}^{\infty} p_n(\rho)\, p_m(\rho)\, d_m(d_m+1) \int_S \phi_m(\omega)\, \phi_n(\omega)\, d\omega = \sum_{n=1}^{\infty} d_n(d_n+1)\, p_n(\rho)^2.
\end{aligned}$$

then p would satisfy (1.14) for some C.

The proof of Theorem 1.2 relies on two ideas. We use an identity obtained by using the solution of an adjoint problem, an idea used earlier by Sacks, Santosa and Symes in a different context, and by Stefanov in [St90]. Also, for functions f on $\mathbb{R}^3$, we estimate the L^2 norm of f on spheres by the Radon transform of f on planes outside the sphere using an idea motivated by the material on pages 185-190 in [LRS86]. The Radon transform estimate could also be obtained using Dean's theorem - see Chapter 7 in [Is06] - but we get stronger results using the idea in [LRS86].

The proof of Theorem 1.2 can be easily modified to give a stability inequality for q in the appropriate class of functions.

Next we give some elementary, known but interesting, results with proofs which are perhaps a little simpler than the original.

THEOREM 1.3. *(Elementary results) Suppose q_i, $i = 1, 2$ are smooth functions on $\mathbb{R}^3$ with support in the unit ball and $\beta_i(\cdot, \cdot)$ the corresponding backscattering data.*

(a) If $q_1 \geq q_2$ and $\beta_1(\omega, \tau) = \beta_2(\omega, \tau)$ for a fixed $\omega \in S$ and all $\tau \in [-1, 1]$, then $q_1 = q_2$.
(b) There is an $M > 0$ such that if $\|q_i\|_{C^2(\mathbb{R}^3)} \leq M$ for $i = 1, 2$ then $\beta_1(\cdot, \tau) = \beta_2(\cdot, \tau)$ for all $\tau \in [-1, 1]$ implies $q_1 = q_2$.

The result (a) was proved in [St90]. Lagergren in Chapter 8 of [La01] has a result much stronger than (b); he shows that $q_1 = q_2$ if $\sum_{i=1}^{3} \int_{\mathbb{R}^3} |\partial_i q(x)|\, dx < 1$. Of course, the proof in [La01] is quite long. Analogous to (b), the articles [SU97], [Wa98], [Wae98], [Wam98], [Wa00], study the inverse backscattering problem but for the acoustic equation, Maxwell's equation or the equations of elasticity and prove injectivity or stability for the problem when the coefficients are close to a constant.

The most basic question remains open - if for some smooth, compactly supported q, the backscattering data $\beta(\cdot, \cdot) = 0$ then is $q = 0$?

1.3. History. If $q(x)$ is a smooth compactly supported function in R^3, then for each real number k and unit vector $\theta \in S$, let $w(x, \theta, k)$ be the outgoing solution of the Helmholtz equation corresponding to the incoming wave $w_i(x, \theta, k) = e^{ikx\cdot\theta}$, that is w is the solution of (below $\rho = |x|$)

$$(-\Delta_x + q(x) - k^2) w(x, \theta, k) = 0, \qquad x \in R^3 \tag{1.15}$$

$$\lim_{\rho \to \infty} \rho \left(\frac{\partial u_s}{\partial \rho} - iku_s \right)(x, \theta, k) \;=\; 0, \tag{1.16}$$

where w_s is the scattered part of the solution

$$w_s(x, \theta, k) := (w - w_i)(x, \theta, k).$$

For large $|x|$

$$w_s(x, \theta, k) = \frac{k e^{ik|x|}}{|x|} w_\infty \left(\frac{x}{|x|}, \theta, k \right) + o\left(\frac{1}{|x|} \right) \tag{1.17}$$

and the function $w_\infty(\omega, \theta, k)$, $\omega, \theta \in S$ and $k \in \mathbb{R}$, is called the far field pattern associated to $q(.)$ and $w_\infty(-\omega, \omega, k)$ with $\omega \in S$ and $k \in \mathbb{R}$ is called the backscattering data - see [CK] for details. One may show that $w_\infty(-\omega, \omega, k)$ is a constant multiple of the Fourier transform of our time domain backscattering data $\beta(\omega, \tau)$ - see [Uh01].

The inverse backscattering problem consists of inverting the map $q(x) \mapsto w_\infty(-\omega, \omega, k)$. A formal computation of the derivative of this map around $q = 0$ shows that this derivative maps q to its Fourier transform, hence the inverse backscattering problem is usually studied as the question of inverting the map from q to the inverse Fourier transform of the backscattering data, that is, the study of the inversion of the map

$$\mathcal{F} : q(x) \mapsto q_b(x) := \int_0^\infty \int_S e^{-ikx\cdot\omega} k^2 w_\infty(-\omega, \omega, k/2)\, d\omega\, dk;$$

q_b is called the Born approximation to q.

In [ER92] it was shown that the map $q \to q_b$ is an analytic map in appropriate spaces and this map is an isomorphism on a dense open subset of this space (which includes a neighborhood of $q = 0$). The articles [St92], [Uh01] (the details of [Uh01] are given in [MU08]) give other proofs of this result, [La01] has related results and [Wa02] has similar results for the even dimension case. One may also study what can be recovered of q from the backscattering data for a single frequency. Many different q can result in the same backscattering data for a fixed frequency but in [HKS05] it was shown that a certain subset (possibly empty) determined by the data is guaranteed to be in the convex hull of the support of all such q.

If a plane wave impinges on a potential with a singularity across a surface, the transmitted wave is the same as the original but the reflected wave is one degree smoother. So, if a medium is probed by a plane wave then the resulting wave is a sum of the original wave plus a sum of waves which are the result of one or more reflections. The wave resulting from a single reflection will be the most singular, the result of two reflections will be one degree smoother, the result of three reflections will be two degrees smoother and so on. Since q_b is the result of applying the inverse of the single reflection process to the backscattering data, one expects q_b and q to the have the same principal singularity. This idea was implemented in [GU93] to show that if q is a conormal potential then one can recover the conormal singularities of q from the singularities of the backscattering amplitude. Using tools from Harmonic Analysis, in [OPS01] for two dimensions and then in [RV05] for three dimensions, it was shown that for *arbitrary* (not necessarily conormal) smooth enough q, $q_b - q$ is smoother than q, that is q_b captures the principle singularities of q. Please see [RR12] for an accurate statement of the most recent results - also see [BM09] for related results.

2. Proof of Theorem 1.1.

(a) We seek $U(x, t, \omega)$ in the form

$$U(x, t, \omega) = \delta(t - x\cdot\omega) + u(x, t, \omega)H(t - x\cdot\omega);$$

then

$$U_{tt} - \Delta U - qU = (u_{tt} - \Delta u - qu)H(t - x\cdot\omega) + 2(u_t + \omega\cdot\nabla u - q/2)\delta(t - x\cdot\omega).$$

So we need to choose a smooth $u(x, t)$ on the region $t \geq x\cdot\omega$ so that $u_{tt} - \Delta u - qu = 0$ on this region and $u_t + \omega\cdot\nabla u = q/2$ on the plane $x\cdot\omega = t$. The last relation is equivalent to

$$\frac{d}{d\sigma} u(x + \sigma\omega, x\cdot\omega + \sigma, \omega) = \frac{q(x + \sigma\omega)}{2}$$

and integrating it with respect to σ and noting that $u(x,t,\omega)=0$ for $t<-1$, we obtain

$$u(x, x\cdot\omega, \omega) = \frac{1}{2}\int_{-\infty}^{0} q(x+\sigma\omega)\, d\sigma.$$

So we need to show that the characteristic IVP (1.7)-(1.9) has a unique smooth solution. The uniqueness of the solution may be proved by standard energy estimates. The existence is proved by using a progressing wave expansion and converting the problem to the solution of an initial value problem.
We give an outline of the proof - the details can be filled in quite easily. Below, for any $j \geq 0$,

$$s_+^j = \begin{cases} s^j, & s \geq 0 \\ 0, & s < 0. \end{cases};$$

note that $s_+^0 = H(s)$. Pick any positive integer N; we seek a solution $U(x,t)$ of (1.3), (1.4) in the form of a progressing wave expansion

$$U(x,t) = \delta(t - x\cdot\omega) + \sum_{j=0}^{N} a_j(x)(t - x\cdot\omega)_+^j + R_N(x,t).$$

The $a_j(x)$ are constructed by solving the associated transport equations - see [CH89]. One may show that the $a_j(x)$ are smooth functions completely determined by q and its derivatives of order $j+2$ or less and $R_N(x,t)$ is the solution of the IVP

$$(\Box - q)R_N(x,t) = F(x)(t - x\cdot\omega)_+^N, \qquad (x,t) \in \mathbb{R}^3\times]R, \tag{2.1}$$

$$R_N(x,t) = 0, \qquad x\in\mathbb{R}^3,\ t<-1, \tag{2.2}$$

for some smooth function $F(x)$ completely determined by q and its derivatives up to order $2N$. Since the RHS of (2.1) is of class C^{N-1} on $\mathbb{R}^3\times\mathbb{R}$, by the well-posedness theory for hyperbolic PDEs (using integral equation arguments), the system (2.1), (2.2) has a unique solution of class C^{N-1}. Next one may check that

$$u(x,t) = \sum_{j=0}^{N} a_j(x)(t - x\cdot\omega)^j + R_N(x,t), \qquad t \geq x\cdot\omega$$

solves (1.7) - (1.9), so we have proved the existence of a u of class C^{N-1} for every N. The uniqueness of u allows us to claim that u is smooth on $t \geq x\cdot\omega$.

(c) We prove (c) before (b) because the proof of (b) is more complicated. We assume that $\theta = (0,0,1)$, write $x = (x', z)$ with $x'\in\mathbb{R}^2$, $z\in\mathbb{R}$, and define

$$v(z,t) := \int_{\mathbb{R}^2} u(x', z, t)\, dx'.$$

Since $u_{tt} - \Delta u = 0$ in the region $|x|\geq 1$ and $u(x,t)=0$ for $t\leq -1$, in the region $z\geq 1$ we have

$$v_{tt} - v_{zz} = \int_{\mathbb{R}^2} (u_{tt} - u_{zz})(x', z, t)\, dx' = -\int_{\mathbb{R}^2} (\Delta_{x'} u)(x', z, t)\, dx' = 0,$$

and $v(z,t) = 0$ for $t \leq -1$. Hence $v(z,t) = f(t-z)$, on the region $z\geq 1$, for some function f. Hence, for $z\geq 1$, we have

$$\int_{\mathbb{R}^2} u(x', z, t)\, dx' = v(z,t) = f(t-z) = v(1, t-z+1) = \int_{\mathbb{R}^2} u(x', 1, t-z+1)\, dx'$$

proving one part of (c). Next

$$\int_{x\cdot e=1}(e\cdot\nabla u)(x,t)\,dS_x=\int_{\mathbb{R}^2}u_z(x',1,t)\,dx'=v_z(1,t)=-f'(t-1);$$

and

$$\begin{aligned}\partial_t\left(\int_{x\cdot e=1}u(x,t)\,dS_x\right)&=\partial_t\left(\int_{\mathbb{R}^2}u(x',1,t)\,dx'\right)=\partial_t(v(1,t))=\partial_t(f(t-1))\\&=f'(t-1)=-\int_{-\infty}^t v_z(1,\tau)\,d\tau,\end{aligned}$$

proving the other part of (c).

(b) As before, we assume $\theta=e=(0,0,1)$ and write x as (x',z) where $x'\in\mathbb{R}^2$ and $z\in\mathbb{R}$. Let $f(x',t)=u_z(x',z=1,t)$, that is f is the value of u_z on the hyperplane $x\cdot e=1$. Then u is the solution of the IBVP

$$u_{tt}-\Delta u=0,\qquad (x,t)\in\mathbb{R}^3\times\mathbb{R},\quad z\geq 1, \tag{2.3}$$

$$u(x,t)=0,\qquad t<<0,) \tag{2.4}$$

$$u_z(x',1,t)=f(x',t),\qquad (x',t)\in\mathbb{R}^2\times\mathbb{R}. \tag{2.5}$$

We may show that (see [Rak03] for example)

$$\begin{aligned}u(x',z+1,t)&=-\frac{1}{2\pi}\frac{\delta(t-|(x',z)|)}{|(x',z)|}*f(x',t)=-\frac{1}{2\pi}\int_{R^2\times\mathbb{R}}f(y',\tau)\frac{\delta(t-\tau-|(x'-y',z)|)}{|(x'-y',z)|}\,dy'\,d\tau\\&=-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{f(x'+y',t-|(y',z)|)}{|(y',z)|}\,dy'\end{aligned}$$

Hence, for $z>0$,

$$z\,u(0,z+1,z+s)=-\frac{z}{2\pi}\int_{\mathbb{R}^2}\frac{f(y',z+s-|(y',z)|)}{|(y',z)|}\,dy'.$$

and we show that

$$\lim_{z\to\infty}z\int_{\mathbb{R}^2}\frac{f(y',z+s-|(y',z)|)}{|(y',z)|}\,dy'=\int_{\mathbb{R}^2}f(y',s)\,dy'. \tag{2.6}$$

This will imply that

$$\lim_{z\to\infty}zu(0,z+1,z+s=-\frac{1}{2\pi}\int_{\mathbb{R}^2}f(y',s)\,dy',\qquad \lim_{z\to\infty}u(0,z+1,z+s)=0$$

and hence

$$\lim_{z\to\infty}(z+1)\,u(0,z+1,z+1+s)=-\frac{1}{2\pi}\int_{\mathbb{R}^2}f(y',s+1)\,dy',$$

proving (b). So it remains to prove (2.6).
Since $u(x,t)$ is supported in the region $t\geq x\cdot\omega$ and smooth in that region, we define $a(x,t)$ to be a smooth function on $\mathbb{R}^3\times\mathbb{R}$ which equals $u(x,t)$ in the region $t\geq x\cdot\omega$ and is zero for $t<-1$. So $u(x,t)=a(x,t)H(t-x\cdot\omega)$ and

$$u_z(x',z,t)=a_z(x',z,t)H(t-x\cdot\omega)-\omega_3 a(x',z,t)\delta(t-x\cdot\omega).$$

Hence $f(x', t) = g(x', t) + h(x', t)$ where taking $\omega' = (\omega_1, \omega_2)$ if $\omega = (\omega_1, \omega_2, \omega_3)$, we define

$$g(x', t) := a_z(x', 0, t) H(t - x \cdot \omega), \qquad h(x', t) = -\omega_3 a(x', 0, x' \cdot \omega') \delta(t - x' \cdot \omega').$$

We prove (2.6) separately for the cases $f = g$ and $f = h$ because while g is a continuous function (almost), h is a distribution.
We first examine the g case. For a fixed s, in the integral on the LHS of (2.6), $g(x', t)$ is evaluated only at points with $t \leq s$ and since $u(x, t)$ has compact support when $t \leq s$, the region of integration (in the y' plane) on the LHS of (2.6) is a bounded subset of $\mathbb{R}^2$, independent of z. Since the integral uses the values of $g(y', t)$ only on a compact subset (independent of z) of the (y', t) space, g is a bounded function. Further $z/|(y', z)|$ is bounded above by 1, so the integrand of the LHS of (2.6) is dominated by a bounded function (independent of z) of compact support. Further, as $z \to \infty$, we have

$$\frac{z}{|(y', z)|} \to 1, \qquad z + s - |(y', z)| \to s,$$

and $g(y', t)$ is continuous everywhere except on $t = y' \cdot \omega'$. So by the dominated convergence theorem we have (2.6) in the g case.
Now we examine the h case. If $\omega_3 = 0$ there is nothing to prove, so we assume that $\omega_3 \neq 0$. We define $b(x') := -\omega_3 a(x', 0, x' \cdot \omega') s$ so we have

$$h(x', t) = b(x') \delta(t - x' \cdot \omega').$$

As before, the integral on the LHS of (2.6) is over a bounded region (independent of z) in the y' plane and we have to show that

$$\lim_{z \to \infty} \int_{\mathbb{R}^2} \frac{z}{|(y', z)|} b(y') \delta(z + s - |(y', z)| - y' \cdot \omega') \, dy' = \int_{\mathbb{R}} b(y') \delta(s - y' \cdot \omega') \, dy'$$

weakly as distributions in s - the two sides are distributions in s when $\omega' = 0$ which corresponds to the backscattering and the peak scattering cases. So for any compactly supported smooth function $\phi(s)$ on $\mathbb{R}$ we have to show that

$$\lim_{z \to \infty} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{z}{|(y', z)|} b(y') \, \delta(z + s - |(y', z)| - y' \cdot \omega') \, \phi(s) \, dy' \, ds = \int_{\mathbb{R}} \int_{\mathbb{R}} b(y') \, \delta(s - y' \cdot \omega') \, \phi(s) \, dy' \, ds,$$

that is

$$\lim_{z \to \infty} \int_{\mathbb{R}^2} \frac{z}{|(y', z)|} b(y') \, \phi(y' \cdot \omega' + |(y', z)| - z) \, dy' = \int_{\mathbb{R}} b(y') \, \phi(y' \cdot \omega') \, dy'.$$

This again follows from the Dominated Convergence Theorem because as $z \to \infty$, we have

$$\frac{z}{|(y', z)|} \to 1, \qquad z - |(y', z)| \to 0.$$

3. A useful identity. We derive an identity used in the proofs of Theorems 2 and 3.

Let U_i, $i = 1, 2$, be the solution of (1.3), (1.4) when $q = q_i$, and let $\beta_i(\cdot, \cdot)$ be the backscattering data. Define $v := U_1 - U_2 = u_1 - u_2$, $p := q_1 - q_2$ and $\beta := \beta_1 - \beta_2$; then

$$v_{tt} - \Delta u - q_1 v = p U_2, \qquad (x, t) \in \mathbb{R}^3, \tag{3.1}$$

$$v(x, t) = 0, \qquad x \in \mathbb{R}^3, \; t \leq -1. \tag{3.2}$$

We show that v and β satisfy the following identity.

Proposition 3.1. *For any $\tau \geq -1$ and all $\omega \in S$ we have*

$$4(\partial_\tau \beta)(\omega, 2\tau+1) = \int_{x\cdot\omega=\tau} p(x)\, dS_x + \int_{-1}^{\tau} \int_{x\cdot\omega=t} k(x,\omega,\tau)\, p(x)\, dS_x\, dt, \tag{3.3}$$

where

$$k(x,\omega,\tau) := (u_1+u_2)(x, 2\tau - x\cdot\omega, \omega) + \int_{x\cdot\omega}^{2\tau - x\cdot\omega} u_1(x,s,\omega)\, u_2(x, 2\tau - s, \omega)\, ds,$$

is smooth on the region $-1 \leq x\cdot\omega \leq \tau$ with $\tau \in \mathbb{R}$, $\omega \in S$ and $x \in \mathbb{R}^3$.

Proof. Choose any $\tau \geq -1$ and define $W_1(x,t) = U_1(x, 2\tau - t, \omega)$; then W_1 satisfies (1.3) with q replaced by q_1 and noting that $U_1(x,t,\omega) = \delta(t - x\cdot\omega)$ for $t \leq -1$ we have

$$W_1(x,t) = \delta(2\tau - t - x\cdot\omega) \qquad \text{for } t \geq 2\tau + 1.$$

Noting that $v(x,t)$ is compactly supported in the region $-1 \leq t \leq 2\tau+1$, working formally (which can be made rigorous by integrating $p(x)\, u_2(x,t,\omega)\, u_1(x, 2\tau - t, \omega)$ over the region $2\tau - x\cdot\omega \leq t \leq x\cdot\omega$) we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \int_{-1}^{2\tau+1} p(x)\, U_2(x,t,\omega)\, W_1(x,t)\, dt\, dx &= \int_{\mathbb{R}^3} \int_{-1}^{2\tau+1} (v_{tt} - \Delta v - q_1 v)(x,t)\, W_1(x,t)\, dt\, dx \\
&= \int_{\mathbb{R}^3} (v_t W_1 - W_{1t} v)(x, 2\tau+1)\, dx - \int_{\mathbb{R}^3} (v_t W_1 - W_{1t} v)(x,-1)\, dx \\
&= \int_{\mathbb{R}^3} v_t(x, 2\tau+1)\, \delta(-1 - x\cdot\omega)\, dx + \int_{\mathbb{R}^3} v(x, 2\tau+1)\, \delta'(-1 - x\cdot\omega)\, dx \\
&= \int_{x\cdot\omega=-1} v_t(x, 2\tau+1)\, dS_x - \int_{\mathbb{R}^3} v(x, 2\tau+1)\, (\omega\cdot\nabla)(\delta(-1 - x\cdot\omega))\, dx \\
&= \int_{x\cdot\omega=-1} v_t(x, 2\tau+1)\, dS_x + \int_{x\cdot\omega=-1} (\omega\cdot\nabla v)(x, 2\tau+1)\, dS_x \\
&= 2(\partial_\tau \beta)(\omega, 2\tau+1)
\end{aligned} \tag{3.4}$$

with the last step following from (1.11) of Theorem 1.1 with $\theta = -\omega$ and the definition of β. We now analyze the LHS of (3.4). For $\tau \geq -1$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \int_{-1}^{2\tau+1} p(x)\, U_2(x,t,\omega)\, W_1(x,t)\, dt\, dx &= \int_{\mathbb{R}^3} \int_{-1}^{2\tau+1} p(x)\, U_2(x,t,\omega)\, U_1(x, 2\tau - t, \omega)\, dt\, dx \\
&= \int_{\mathbb{R}^3} \int_{-1}^{2\tau+1} p(x)\, \delta(t - x\cdot\omega)\, \delta(2\tau - t - x\cdot\omega)\, dt\, dx \\
&\quad + \int_{\mathbb{R}^3} \int_{-1}^{2\tau+1} p(x)\, \delta(t - x\cdot\omega)\, u_1(x, 2\tau - t, \omega)\, dt\, dx \\
&\quad + \int_{\mathbb{R}^3} \int_{-1}^{2\tau+1} p(x)\, u_2(x,t,\omega)\, \delta(2\tau - t - x\cdot\omega)\, dt\, dx \\
&\quad + \int_{\mathbb{R}^3} \int_{-1}^{2\tau+1} p(x)\, u_1(x, 2\tau - t, \omega)\, u_2(x,t,\omega)\, dt\, dx.
\end{aligned}$$

The first of these four integrals on the RHS is the Radon transform of p. In the second integral, using the support of u_1, the region of integration is $x\cdot\omega \leq 2\tau - t$ where $x\cdot\omega = t$ and hence $t \leq \tau$. In the third integral the region of integration is $x \cdot \omega \le t$ where $x \cdot \omega = 2\tau - t$ and hence $\tau \le t$. In the fourth integral the region of integration is $x \cdot \omega \le t$ and $x \cdot \omega \le 2\tau - t$ so adding these two we get $x \cdot \omega \le \tau$. Hence

$$
\begin{aligned}
\int_{\mathbb{R}^3} \int_{-1}^{2\tau+1} p(x)\, U_2(x,t,\omega)\, W_1(x,t)\, dt\, dx &= \frac{1}{2} \int_{x\cdot\omega=\tau} p(x)\, dS_x + \int_{-1}^{\tau} \int_{x\cdot\omega=t} p(x) u_1(x, 2\tau - t, \omega)\, dS_x\, dt \\
&\quad + \int_{\tau}^{2\tau+1} \int_{x\cdot\omega=2\tau-t} p(x)\, u_2(x,t,\omega)\, dS_x\, dt \\
&\quad + \int_{-1\le x\cdot\omega\le\tau} \int_{x\cdot\omega}^{2\tau - x\cdot\omega} p(x)\, u_1(x,t,\omega)\, u_2(x, 2\tau - t, \omega)\, dt\, dx \\
&= \frac{1}{2} \int_{x\cdot\omega=\tau} p(x)\, dS_x + \int_{-1}^{\tau} \int_{x\cdot\omega=t} p(x)\, (u_1 + u_2)(x, 2\tau - t, \omega)\, dS_x\, dt \\
&\quad + \int_{-1\le x\cdot\omega\le\tau} \int_{x\cdot\omega}^{2\tau - x\cdot\omega} p(x)\, u_1(x,s,\omega)\, u_2(x, 2\tau - s, \omega)\, ds\, dx \\
&= \frac{1}{2} \int_{x\cdot\omega=\tau} p(x)\, dS_x + \int_{-1}^{\tau} \int_{x\cdot\omega=t} p(x)\, (u_1 + u_2)(x, 2\tau - t, \omega)\, dS_x\, dt \\
&\quad + \int_{-1}^{\tau} \int_{x\cdot\omega=t} \int_{t}^{2\tau - t} p(x)\, u_1(x,s,\omega)\, u_2(x, 2\tau - s, \omega)\, ds\, dS_x\, dt \\
&= \frac{1}{2} \int_{x\cdot\omega=\tau} p(x)\, dS_x + \int_{-1}^{\tau} \int_{x\cdot\omega=t} p(x)\, k(x,\omega,\tau)\, dS_x\, dt \qquad (3.5)
\end{aligned}
$$

for all unit vectors ω and all $\tau \ge -1$; here

$$
k(x,\omega,\tau) := (u_1 + u_2)(x, 2\tau - x\cdot\omega, \omega) + \int_{x\cdot\omega}^{2\tau - x\cdot\omega} u_1(x,s,\omega)\, u_2(x, 2\tau - s, \omega)\, ds
$$

in the region $\omega \in S$, $-1 \le \tau$ and $x \cdot \omega \le \tau$. Note that, in this region, $k(x,\omega,\tau)$ depends on the values of values of $u_1(\cdot,\cdot,\omega)$, and $u_2(\cdot,\cdot,\omega)$ at points (x', t') where $t' \ge x' \cdot \omega$ because, in this region, for the first term $2\tau - x\cdot\omega \ge x \cdot \omega$ and in the integral, $s \ge x \cdot \omega$ and $2\tau - s \ge x \cdot \omega$. Hence $k(x,\omega,\tau)$ is a smooth function on this region. Combining (3.4) and (3.5) we obtain

$$
2(\partial_\tau \beta)(\omega, 2\tau + 1) = \frac{1}{2} \int_{x\cdot\omega=\tau} p(x)\, dS_x + \int_{-1}^{\tau} \int_{x\cdot\omega=t} p(x)\, k(x,\omega,\tau)\, dS_x\, dt \qquad (3.6)
$$

which proves the proposition. □

4. Proof of Theorem 1.2.

4.1. An expansion.

For $x \in \mathbb{R}^n$, define the vectors

$$
T_{ij} = x_i e_j - x_j e_i, \qquad i, j = 1, \cdots, n,
$$

which are tangential, at x, to the origin centered sphere through x; here e_i is the unit vector along the x_i axis. Note that

$$
\Omega_{ij} := x_i \partial_j - x_j \partial_i = T_{ij} \cdot \nabla.
$$

For any vector v in $\mathbb{R}^n$, we express v in terms of x and the T_{ij}.

Proposition 4.1. *For any $x, v \in \mathbb{R}^n$, we have*

$$|x|^2 v = \sum_{i<j} (v \cdot T_{ij}) T_{ij} + (v \cdot x) x. \tag{4.1}$$

Proof. Let $v = (v_1, \cdots, v_n)$ and $x = (x_1, \cdots, x_n)$; then taking the dot product of the RHS (4.1) with e_k we obtain

$$\begin{aligned} e_k \cdot (\text{RHS of (4.1)}) &= \sum_{i<j} (v \cdot T_{ij})(T_{ij} \cdot e_k) + (v \cdot x) x_k \\ &= \sum_{i<k} (v \cdot T_{ik})(T_{ik} \cdot e_k) + \sum_{k<j} (v \cdot T_{kj})(T_{kj} \cdot e_k) + (v \cdot x) x_k \\ &= \sum_{i<k} (v \cdot T_{ik}) x_i - \sum_{k<j} (v \cdot T_{kj}) x_j + (v \cdot x) x_k \\ &= \sum_{i<k} (v \cdot T_{ik}) x_i + \sum_{k<j} (v \cdot T_{jk}) x_j + (v \cdot x) x_k \\ &= \sum_{i} (v \cdot T_{ik}) x_i + (v \cdot x) x_k \\ &= \sum_{i} (v_k x_i - v_i x_k) x_i + (v \cdot x) x_k \\ &= v_k |x|^2. \end{aligned}$$

□

4.2. A derivative of the Radon transform. For each $\tau \in \mathbb{R}$ and $\omega \in S$ and any smooth function $p(x)$ on $\mathbb{R}^3$ supported in B, we define the Radon transform

$$P(\tau, \omega) := \int_{x \cdot \omega = \tau} p(x)\, dS_x.$$

Hence, by the Divergence theorem

$$P(\tau, \omega) = \int_{x \cdot \omega \leq \tau} (\omega \cdot \nabla p)(x)\, dx = \int_{-\infty}^{\tau} \int_{x \cdot \omega = t} (\omega \cdot \nabla p)(x)\, dS_x\, dt$$

so

$$P_\tau(\tau, \omega) = \int_{x \cdot \omega = \tau} (\omega \cdot \nabla p)(x)\, dS_x. \tag{4.2}$$

Given $\omega \in S$ and $\tau \in [0, 1]$, to any point $x \in \mathbb{R}^3$ on the plane $x \cdot \omega = \tau$ we associate $\rho = |x|$, and (r, θ) the polar coordinates of x as points on the plane $x \cdot \omega = \tau$ - so r is the distance of x from the line through the origin in the direction ω; see Figure 4.1. On the line through the origin and ω we choose a point Q so that the vector xQ is orthogonal to the vector x. Let α denote the unit vector in the direction xQ; our goal is to express, at x, the vertical directional derivative $\omega \cdot \nabla p$ in terms of the radial derivative p_r and the (angular) derivative in the direction α. From the similar triangles OCx and xCQ we have

$$\frac{|xQ|}{|x|} = \frac{|CQ|}{|Cx|} = \frac{|Cx|}{|OC|}$$

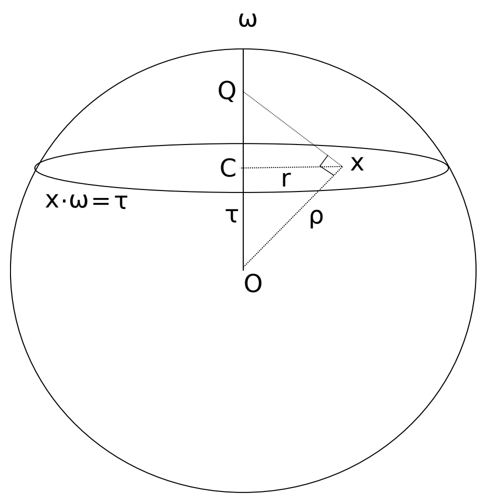

FIG. 4.1. *The decomposition of ω*

that is $|xQ| = \rho r/\tau$ and $|CQ| = r^2/\tau$. Now, as vectors we have $xQ = xC + CQ$ so

$$\frac{\rho r}{\tau}\alpha = -r\hat{r} + \frac{r^2}{\tau}\omega$$

where $\hat{r}$ is the unit vector in the radial direction at x, that is in the direction Cx. Hence

$$\omega = \frac{\rho}{r}\alpha + \frac{\tau}{r}\hat{r}$$

implying

$$(\omega \cdot \nabla p)(x) = \frac{\tau}{r}p_r(x) + \frac{\rho}{r}(\alpha \cdot \nabla p)(x).$$

Substituting this in (4.2) we obtain

$$\begin{aligned} P_\tau(\tau, \omega) &= \tau \int_0^{2\pi} \int_0^\infty p_r \, dr \, d\theta + \int_{x\cdot\omega=\tau} \frac{\rho}{r}(\alpha \cdot \nabla p)(x) \, dS_x \\ &= -2\pi\tau p(\tau\omega) + \int_{x\cdot\omega=\tau} \frac{\rho}{r}(\alpha \cdot \nabla p)(x) \, dS_x. \end{aligned} \tag{4.3}$$

For a fixed $\tau \in [0, 1]$, the plane $x \cdot \omega = \tau$ may be parametrized by ρ and θ and we note that

$$dS = r \, dr \, d\theta = \rho \, d\rho \, d\theta$$

because $\rho^2 = r^2 + \tau^2$ gives $r \, dr = \rho \, d\rho$. So using the support of p and (4.3) we obtain

$$P_\tau(\tau, \omega) = -2\pi\tau p(\tau\omega) + \int_\tau^1 \int_0^{2\pi} \frac{\rho^2}{\sqrt{\rho^2 - \tau^2}} (\alpha \cdot \nabla p)(x) \, d\theta \, d\rho. \tag{4.4}$$

Since $\alpha \perp x$, from Proposition 4.1

$$|x|^2(\alpha \cdot \nabla p)(x) = \sum_{i<j}(\alpha \cdot T_{ij})(T_{ij} \cdot \nabla p)(x) = \sum_{i<j}(\alpha \cdot T_{ij})(\Omega_{ij}p)(x).$$

Now $|\alpha| = 1$ and $|T_{ij}| \le 2|x|$, so $|\alpha \cdot T_{ij}| \le 2|x|$ and hence

$$|x|^2\,|(\alpha\cdot\nabla p)(x)| \le 2|x| \sum_{i<j} |(\Omega_{ij}p)(x)|.$$

So (4.4) leads to

$$\tau|p(\tau\omega)| \preccurlyeq |P_\tau(\tau,\omega)| + \sum_{i<j}\int_\tau^1\int_0^{2\pi} \frac{\rho}{\sqrt{\rho^2-\tau^2}} |(\Omega_{ij}p)(x)|\,d\theta\,d\rho;$$

note that the x in the above integral lies on the plane $x\cdot\omega = \tau$ and ρ, θ determine a unique x on this plane. Hence, using the Cauchy-Schwartz inequality,

$$\begin{aligned}\tau^2|p(\tau\omega)|^2 &\preccurlyeq |P_\tau(\tau,\omega)|^2 + \left(\int_\tau^1\int_0^{2\pi}\frac{\rho}{\sqrt{\rho^2-\tau^2}}\,d\theta\,d\rho\right)\sum_{i<j}\int_\tau^1\int_0^{2\pi}\frac{\rho}{\sqrt{\rho^2-\tau^2}}|(\Omega_{ij}p)(x)|^2\,d\theta\,d\rho\\ &\preccurlyeq |P_\tau(\tau,\omega)|^2 + \sum_{i<j}\int_\tau^1\int_0^{2\pi}\frac{\rho}{\sqrt{\rho^2-\tau^2}}|(\Omega_{ij}p)(x)|^2\,d\theta\,d\rho.\end{aligned}$$

Hence

$$\tau^2\int_S |p(\tau\omega)|^2\,d\omega \preccurlyeq \int_S |P_\tau(\tau,\omega)|^2\,d\omega + \sum_{i<j}\int_S\int_\tau^1\int_0^{2\pi}\frac{\rho}{\sqrt{\rho^2-\tau^2}}|(\Omega_{ij}p)(x)|^2\,d\theta\,d\rho\,d\omega. \tag{4.5}$$

If we define

$$f(x) := \frac{1}{\sqrt{\rho^2-\tau^2}}|(\Omega_{ij}p)(x)|^2$$

then the integral on the RHS of (4.5) is (below $e = (0,0,1)$)

$$\begin{aligned}\int_S\int_\tau^1\int_0^{2\pi} f(x)\,\rho\,d\theta\,d\rho\,d\omega &= \int_S\int_{x\cdot\omega=\tau} f(x)\,dS_x\,d\omega = \int_S\int_{\mathbb{R}^3} f(x)\,\delta(x\cdot\omega-\tau)\,dx\,d\omega\\ &= \int_{\mathbb{R}^3} f(x)\left(\int_S \delta(x\cdot\omega-\tau)\,d\omega\right)dx\\ &= \int_{\mathbb{R}^3} f(x)\left(\int_S \delta(|x|\,e\cdot\omega-\tau)\,d\omega\right)dx\\ &= 2\pi\int_{\mathbb{R}^3} f(x)\int_0^\pi \delta(|x|\,\cos u-\tau)\,\sin u\,du\,dx\\ &= 2\pi\int_{\mathbb{R}^3}\frac{f(x)}{|x|}H(|x|-\tau)\,dx.\end{aligned}$$

Hence (4.5) gives us, for all $\tau\in[0,1]$,

$$\begin{aligned}\tau^2\int_S|p(\tau\omega)|^2\,d\omega &\preccurlyeq \int_S|P_\tau(\tau,\omega)|^2\,d\omega + \sum_{i<j}\int_{|x|\ge\tau}\frac{1}{\rho\sqrt{\rho^2-\tau^2}}|(\Omega_{ij}p)(x)|^2\,dx\\ &\preccurlyeq \int_S|P_\tau(\tau,\omega)|^2\,d\omega + \int_\tau^1\frac{\rho}{\sqrt{\rho^2-\tau^2}}\sum_{i<j}\int_S|(\Omega_{ij}p)(\rho\omega)|^2\,dw\,d\rho. \qquad (4.6)\end{aligned}$$

4.3. The proof. We are given that $\beta_1(\omega,\tau) = \beta_2(\omega,\tau)$ for all $\omega \in S$ and all $\tau \in [-1,0]$. Hence, from Proposition 3.1

$$\int_{x\cdot\omega=\tau} p(x)\, dS_x = -\int_{-1}^{\tau}\int_{x\cdot\omega=t} p(x)\, k(x,\tau,\omega)\, dS_x\, dt, \qquad \forall\tau\in[-1,0],\ \forall\omega\in S. \tag{4.7}$$

It will be more convenient to deal with positive τ rather than negative τ, so in (4.7) we replace ω by $-\omega$, τ by $-\tau$ and t by $-t$. We obtain

$$\int_{x\cdot\omega=\tau} p(x)\, dS_x = \int_{\tau}^{1}\int_{x\cdot\omega=t} p(x)\, k'(x,\tau,\omega)\, dS_x\, dt, \qquad \forall\tau\in[0,1],\ \forall\omega\in S \tag{4.8}$$

where

$$k'(x,\tau,\omega) = -k'(x,-\tau,-\omega).$$

Differentiating (4.8) with respect to τ, we have, for all $\tau\in[0,1]$ and all $\omega\in S$,

$$P_\tau(\tau,\omega) = -\int_{x\cdot\omega=\tau} p(x)\, k'(x,\tau,\omega)\, dS_x + \int_{\tau}^{1}\int_{x\cdot\omega=t} p(x)\, k'_\tau(x,\tau,\omega)\, dS_x\, dt.$$

Noting that p is supported in the unit ball, we have for all $\tau\in[0,1]$

$$\begin{aligned}\int_S |P_\tau(\tau,\omega)|^2\, d\omega &\preccurlyeq \int_S\int_{x\cdot\omega=\tau} |p(x)|^2\, dS_x\, d\omega + \int_{\tau}^{1}\int_S\int_{x\cdot\omega=t} |p(x)|^2\, dS_x\, d\omega\, dt\\ &= \int_B |p(x)|^2 \int_S \delta(x\cdot\omega-\tau)\, d\omega\, dx + \int_{\tau}^{1}\int_B |p(x)|^2 \int_S \delta(x\cdot\omega - t)\, d\omega\, dx\, dt.\end{aligned}$$

These $d\omega$ integrals were computed earlier to be $2\pi|x|^{-1}H(|x|-\tau)$ and $2\pi|x|^{-1}H(|x|-t)$ respectively, so

$$\begin{aligned}\int_S |P_\tau(\tau,\omega)|^2\, d\omega &\preccurlyeq \int_{|x|\geq\tau} \frac{|p(x)|^2}{|x|}\, dx + \int_{\tau}^{1}\int_{|x|\geq t} \frac{|p(x)|^2}{|x|}\, dx\, dt \preccurlyeq \int_{|x|\geq\tau} \frac{|p(x)|^2}{|x|}\, dx\\ &\preccurlyeq \int_{\tau}^{1} \rho \int_S |p(\rho\omega)|^2\, d\omega.\end{aligned} \tag{4.9}$$

So using (4.6) we obtain, for all $\tau\in[0,1]$ and all $\omega\in S$,

$$\tau^2 \int_S |p(\tau\omega)|^2\, d\omega \preccurlyeq \int_{\tau}^{1}\rho\int_S |p(\rho\omega)|^2\, d\omega + \int_{\tau}^{1} \frac{\rho}{\sqrt{\rho^2-\tau^2}} \sum_{i<j}\int_S |(\Omega_{ij}p)(\rho\omega)|^2\, dw\, d\rho.$$

If we define

$$E(\rho) := \int_S |p(\rho\omega)|^2\, d\omega, \qquad \rho\in[0,1]$$

then using the angular derivative property (1.13) of p we obtain, for all $\tau\in[0,1]$,

$$\begin{aligned}\tau^2 E(\tau) &\preccurlyeq \int_{\tau}^{1} \rho E(\rho)\, d\rho + \int_{\tau}^{1} \frac{\rho}{\sqrt{\rho^2-\tau^2}} E(\rho)\, d\rho \preccurlyeq \int_{\tau}^{1} \frac{\rho}{\sqrt{\rho^2-\tau^2}} E(\rho)\, d\rho\\ &\preccurlyeq \int_{\tau}^{1} \frac{E(\rho)}{\sqrt{\rho-\tau}}\, d\rho.\end{aligned}$$

Pick any small $\epsilon > 0$. Then for all $\tau \in [\epsilon, 1]$ we have

$$E(\tau) \preccurlyeq \int_\tau^1 \frac{E(\rho)}{\sqrt{\rho - \tau}} \, d\rho.$$

Substituting this inequality back in itself we obtain, for all $\tau \in [\epsilon, 1]$,

$$E(\tau) \preccurlyeq \int_\tau^1 \int_\rho^1 \frac{E(s)}{\sqrt{\rho - \tau}\,\sqrt{s - \rho}} \, ds \, d\rho = \int_\tau^1 E(s) \int_\tau^s \frac{1}{\sqrt{\rho - \tau}\,\sqrt{s - \rho}} \, d\rho \, ds = \pi \int_\tau^1 E(s) \, ds.$$

Hence, by Gronwall's inequality, $E(\tau) = 0$ for all $\tau \in [\epsilon, 1]$ for all $\epsilon > 0$. So $p = 0$ and the theorem is proved.

5. Proof of Theorem 1.3.

(a) If $p = q_1 - q_2$ and $\beta = \beta_1 - \beta_2$ then, from Proposition 3.1, we have for the fixed ω and all $\tau \in [-1, 1]$ that

$$P(\tau, \omega) := \int_{x \cdot \omega} p(x) \, dS_x = -\int_{-1}^{\tau} \int_{x \cdot \omega = t} p(x) \, k(x, \omega, \tau) \, dS_x \, dt.$$

Since $p = q_1 - q_2 \geq 0$ we obtain

$$P(\tau, \omega) \leq \int_{-1}^{\tau} \int_{x \cdot \omega = t} p(x) \, |k(x, \omega, \tau)| \, dS_x \, dt \leq C \int_{-1}^{\tau} P(\tau, \omega) \, dt, \qquad \forall \tau \in [-1, 1].$$

Hence by Gronwall's inequality $P(\tau, \omega) = 0$ for all $\tau \in [-1, 1]$ for this fixed ω. Since $p \geq 0$ and is continuous, this implies $p(x) = 0$ on $x \cdot \omega = \tau$ for all $\tau \in [-1, 1]$. Since p is supported in the unit ball we obtain $p = 0$.

(b) Define

$$k_{max} = \max \ \{(|k| + |k_\tau|)(x, \omega, \tau) \ : \ x \in \mathbb{R}^3, \ \omega \in S, \ \tau \in [-1, 1], -1 \leq x \cdot \omega \leq \tau, \ |x| \leq 1\}.$$

Again, from the hypothesis and Proposition 3.1 we have

$$P(\tau, \omega) = -\int_{-1}^{\tau} \int_{x \cdot \omega = t} p(x) \, k(x, \omega, \tau) \, dS_x \, dt, \qquad \forall \omega \in S, \ \tau \in [-1, 1].$$

Hence, for all $\omega \in S, \ \tau \in [-1, 1]$ we have

$$\begin{aligned} |P_\tau(\tau, \omega)| &\leq \int_{x \cdot \omega = \tau} |p(x)| \, |k(x, \omega, \tau)| \, dS_x + \int_{-1}^{\tau} \int_{x \cdot \omega = t} |p(x)| \, |k_\tau(x, \omega, \tau)| \, dS_x \, dt \\ &\leq k_{max} \left(\int_{x \cdot \omega = \tau} |p(x)| \, dS_x + \int_{-1}^{\tau} \int_{x \cdot \omega = t} |p(x)| \, dS_x \, dt \right) \end{aligned}$$

so, since p is supported in the unit ball, from the Cauchy-Schwartz inequality

$$|P_\tau(\omega, \tau)|^2 \leq 3\pi k_{max}^2 \left(\int_{x \cdot \omega = \tau} |p(x)|^2 \, dS_x + \int_{-1}^{\tau} \int_{x \cdot \omega = t} |p(x)|^2 \, dS_x \, dt \right)$$

and hence

$$\begin{aligned} \int_{-1}^{1} |P_\tau(\omega, \tau)|^2 \, d\tau &\leq 3\pi k_{max}^2 \left(\int_{-1}^{1} \int_{x \cdot \omega = \tau} |p(x)|^2 \, dS_x \, d\tau + \int_{-1}^{1} \int_{-1}^{\tau} \int_{x \cdot \omega = t} |p(x)|^2 \, dS_x \, dt \, d\tau \right) \\ &\leq 12\pi k_{max}^2 \int_{\mathbb{R}^3} |p(x)|^2 \, dx. \end{aligned}$$

Hence, noting that $P(\tau, \cdot) = 0$ for $|\tau| \geq 1$, from the Plancherel formula we have

$$\int_{\mathbb{R}^3} |p(x)|^2 \, dx = \frac{1}{8\pi^2} \int_{-1}^{1} \int_S |P_\tau(\omega, \tau)|^2 \, d\omega \, d\tau \leq 6k_{max}^2 \int_{\mathbb{R}^3} |p(x)|^2 \, dx.$$

Hence $p = 0$ if we can find an $M > 0$ so that $6k_{max}^2 < 1$ if $\|q\|_{C^2(\mathbb{R}^3)} \leq M$.
The expression for $k(x, \omega, \tau)$ is given in Proposition (3.1). With that in mind, we note that as x, τ vary over the region region $-1 \leq x \cdot \omega \leq \tau \leq 1$, the point (x, s) with $s \in [x \cdot \omega, 2\tau - x \cdot \omega]$ will vary over the region (x, t) with $-1 \leq x \cdot \omega \leq t \leq 3$. Hence, if we define

$$\|u\|_* := \max\{|u(x, t)| + |u_t(x, t)| \,:\, |x| \leq 1, \ \ -1 \leq x \cdot \omega \leq t \leq 3\},$$

where $u(x, t)$ is the solution of (1.7) - (1.9) corresponding to $q = q_1$ or $q = q_2$, then from the expression for $k(x, \omega, \tau)$ in Proposition (3.1) we have

$$k_{max} \leq 2\|u\|_* + 3\|u\|_*^2 + 4\|u\|_* + 2\|u\|_*^2 + 6\|u\|_*^2 = 6\|u\|_* + 11\|u\|_*^2$$

So the proof of the proposition will be complete if we can show the following.
Proposition 5.1. *If $q(x)$ is a smooth function on $\mathbb{R}^3$ with support in the unit ball, $u(x, t, \omega)$ the solution of (1.7) - (1.9), and $\|q\|_{C^2}$ is small enough (independent of u) then $\|u\|_* \leq 8\|q\|_{C^2}$.* The proof of Proposition 5.1 is given in section 5.1.

5.1. Proof of Proposition 5.1. $u(x, t, \omega)$ is the solution of the characteristic initial value problem (1.7) - (1.9). Further, $u_t(x, t, \omega)$ is also the solution of (1.7) - (1.9) except that the characteristic condition (1.8) will have a different RHS. So Proposition 5.1 will follow from an estimate for a characteristic initial value problem if we can just determine $u_t(x, x \cdot \omega, \omega)$.

There is no loss of generality in assuming that $\omega = (0, 0, 1)$ and below u_i will denote the partial derivative of u with respect to x_i. Since

$$u(x_1, x_2, x_3, x_3) = \frac{1}{2} \int_{-\infty}^{0} q(x_1, x_2, x_3 + s) \, ds,$$

we have

$$(u_3 + u_t)(x, x_3) = \frac{q(x)}{2}.$$

Also, from (1.7)

$$\begin{aligned}\partial_3[(u_t - u_3)(x, x_3)] &= (u_{tt} - u_{33})(x, x_3) = (u_{11} + u_{22} + qu)(x, x_3) \\ &= \frac{1}{2} \int_{-\infty}^{0} (q_{11} + q_{22})(x_1, x_2, x_3 + s) \, ds + \frac{q(x)}{2} \int_{-\infty}^{0} q(x_1, x_2, x_3 + s) \, ds \\ &= (\text{call it}) \, Q(x).\end{aligned}$$

Hence

$$(u_t - u_3)(x, x_3) = \int_{-\infty}^{0} Q(x_1, x_2, x_3 + s) \, ds$$

so

$$u_t(x, x_3) = \frac{q(x)}{2} + \frac{1}{2} \int_{-\infty}^{0} Q(x_1, x_2, x_3 + s) \, ds.$$

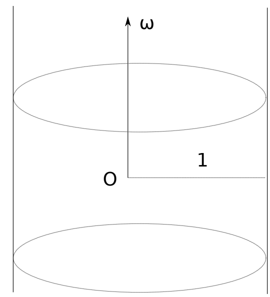

FIG. 5.1. *Support of f*

Since Q depends on the second order derivatives of q, Proposition 5.1 follows from the following result for solutions of characteristic initial boundary value problems. Suppose ω is a unit vector in $\mathbb{R}^3$, $q(x)$ a smooth function on $\mathbb{R}^3$ which is supported on the unit ball. Further, let $f(x)$ be a smooth function on $\mathbb{R}^3$ with $f(x)$ supported in the cylinder of radius 1 and axis the line through the origin parallel to ω, and $f(x)$ zero if $x \cdot \omega \leq -1$, that is

$$\text{supp } f \subseteq \{x \in \mathbb{R}^3 \, : \, \|x - \langle x, \omega \rangle \omega\| \leq 1, \; x \cdot \omega \geq -1\};$$

see Figure 5.1. Let $a(x,t)$ be the solution of the characteristic IVP

$$a_{tt} - \Delta a - qa = 0, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \quad t \geq x \cdot \omega \tag{5.1}$$

$$a(x, x \cdot \omega) = f(x), \quad x \in \mathbb{R}^3, \tag{5.2}$$

$$a(x,t) = 0 \qquad t < -1. \tag{5.3}$$

Define $\|a\|_\infty := \sup\{|a(x,t)| \, : \, \|x\| \leq 1, \; -1 \leq x \cdot \omega \leq t \leq 3\}$,

$$\|q\|_\infty := \sup_{x \in \mathbb{R}^3} |q(x)|, \qquad \|f\|_* := \sup\{|(\omega \cdot \nabla f)(x)| \, : \, x \in \mathbb{R}^3, \; x \cdot \omega \leq 3\}.$$

We show that if $\|q\|_\infty \leq 1/4$ then

$$\|a\|_\infty \leq 8 \|f\|_*.$$

We show this by expressing a as the solution of an integral equation. We obtain a very crude estimate but that will be enough for our purposes. A much sharper estimate may be obtained along with a proof of the existence of a using a Volterra argument as in [Ro74].

For a fixed $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ with $t > x \cdot \omega \geq -1$, define

$$G(y,s) = \frac{1}{4\pi} \frac{\delta(t - s - |x-y|)}{|x-y|};$$

then $G(y,s)$ is the solution of the backward IVP

$$\begin{aligned} \Box_{y,s} G(y,s) &= \delta(x-y, t-s), \qquad (y,s) \in \mathbb{R}^3 \times \mathbb{R} \\ G(y,s) &= 0, \quad s > t, \; y \in \mathbb{R}^3. \end{aligned}$$

Since, for a fixed y, $G(y,s)$ is zero for s large, and, for a fixed s, $G(y,s)$ is compactly supported in y, an application of the divergence theorem gives us

$$\begin{aligned}
a(x,t) &= \int_{s\geq y\cdot\omega} a(y,s)\,\delta(x-y,t-s)\,dy\,ds = \int_{s\geq y\cdot\omega} a(y,s)\,\Box_{y,s}G(y,s)\,dy\,ds \\
&= \int_{s\geq y\cdot\omega} \Box_{y,s}a\,G\,dy\,ds + \int_{s\geq y\cdot\omega} (aG_s - Ga_s)_s + \nabla_y\cdot(G\nabla_y a - a\nabla_y G)\,dy\,ds \\
&= \int_{s\geq y\cdot\omega} qaG\,dy\,ds + \frac{1}{\sqrt{2}}\int_{y\cdot\omega=s} G(a_s+\omega\cdot\nabla a) - a(G_s+\omega\cdot\nabla G)\,dS_{y,s} \\
&= \int_{s\geq y\cdot\omega} qaG\,dy\,ds + \int_{\mathbb{R}^3} G(y,\omega\cdot y)\,\omega\cdot\nabla_y(a(y,\omega\cdot y)) - a(y,\omega\cdot y)\,\omega\cdot\nabla_y(G(y,\omega\cdot y))\,dy \\
&= \int_{s\geq y\cdot\omega} qaG\,dy\,ds + 2\int_{\mathbb{R}^3} G(y,\omega\cdot y)\,\omega\cdot\nabla_y(a(y,\omega\cdot y))\,dy \\
&= \int_{s\geq y\cdot\omega} qaG\,dy\,ds + 2\int_{\mathbb{R}^3} G(y,\omega\cdot y)\,\omega\cdot\nabla_y f(y)\,dy. \qquad (5.4)
\end{aligned}$$

In the last step we used the divergence theorem on the plane $s = y\cdot\omega$ and note that $a(y, y\cdot\omega)$ is zero if $y\cdot\omega \leq -1$ and $G(y, y\cdot\omega) = 0$ if $y\cdot\omega \geq t$.

Let $g(y) := \omega\cdot\nabla_y f(y)$ and extend $a(y,s)$ to be zero for $s < y\cdot\omega$; then from (5.4)

$$\begin{aligned}
4\pi a(x,t) &= 2\int_{\mathbb{R}^3} \frac{g(y)\,\delta(t-y\cdot\omega-|x-y|)}{|x-y|}\,dy + \int_{\mathbb{R}}\int_{\mathbb{R}^3} \frac{q(y)\,a(y,s)\,\delta(t-s-|x-y|)}{|x-y|}\,dy\,ds \\
&= 2\int_{\mathbb{R}^3} \frac{g(y)}{|x-y|}\,\delta(t-y\cdot\omega-|x-y|)\,dy + \int_{\mathbb{R}^3} \frac{q(y)\,a(y,t-|x-y|)}{|x-y|}\,dy \\
&= 2\int_{\mathbb{R}^3} \frac{g(x+y)}{|y|}\,\delta(t-(x+y)\cdot\omega-|y|)\,dy + \int_{\mathbb{R}^3} \frac{q(x+y)\,a(x+y,t-|y|)}{|y|}\,dy
\end{aligned}$$

Without loss of generality we may assume that $\omega = (0,0,1)$ and also note that $a(x,t)$ is supported in the region $t \geq x\cdot\omega = x_3$. So for $t > x_3 \geq -1$, we have

$$4\pi a(x,t) = 2\int_{\mathbb{R}^3} \frac{g(x+y)}{|y|}\,\delta(t-x_3-y_3-|y|)\,dy + \int_{\mathbb{R}^3} \frac{q(x+y)\,a(x+y,t-|y|)}{|y|}\,dy \qquad (5.5)$$

The first integral in (5.5) is over the surface $t-|y| = x_3+y_3$ and the second integral is over the region $t-|y| \geq x_3+y_3$ because of the support of a. We denote $t-x_3$ by λ, note $\lambda > 0$, so the region of integration for the second integral is $\lambda - y_3 \geq |y|$, that is $y_3 \leq \lambda$ and $(\lambda-y_3)^2 \geq |y|^2$ which simplifies to

$$-\left(y_3 - \frac{\lambda}{2}\right) \geq \frac{1}{2\lambda}(y_1^2+y_2^2).$$

Since g is supported in the region $x_3+y_3 \geq -1$, the first integral in (5.5) is over the paraboloid

$$\Sigma := \{y\in\mathbb{R}^3 : -\left(y_3-\frac{\lambda}{2}\right) = \frac{1}{2\lambda}(y_1^2+y_2^2), \quad y_3 \geq -x_3-1\}$$

and the second integral is over the region

$$D := \{y\in\mathbb{R}^3 : -\left(y_3-\frac{\lambda}{2}\right) \geq \frac{1}{2\lambda}(y_1^2+y_2^2), \quad y_3 \geq -x_3-1\}.$$

Note that on Σ and on D, y_1, y_2 are restricted to the set where

$$1 + x_3 + \frac{\lambda}{2} \geq \frac{1}{2\lambda}(y_1^2+y_2^2)$$

that is where $y_1^2 + y_2^2 \leq R^2$ with $R^2 = \lambda(\lambda + 2(x_3 + 1))$.

Now, on Σ

$$\left|\frac{\partial}{\partial y_3}(t - x_3 - y_3 - |y|)\right| = \left|1 + \frac{y_3}{|y|}\right| = \frac{|y_3 + |y||}{|y|} = \frac{t - x_3}{|y|} = \frac{\lambda}{|y|}$$

so

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \frac{g(x+y)}{|y|}\,\delta(t - x_3 - y_3 - |y|)\,dy\right| &= \left|\frac{1}{\lambda}\int_{y_1^2+y_2^2 \leq R^2} \frac{g(x+y)}{|y|}\,dy_1\,dy_2\right| \\ &\leq \frac{1}{\lambda}\int_{y_1^2+y_2^2 \leq R^2} |g(x+y)|\,dy_1\,dy_2 \\ &\leq \pi(\lambda + 2(x_3+1))\|f\|_*.\end{aligned}$$

In the last inequality, we used that on Σ, we have $y_3 - \lambda/2 \leq 0$ so $y_3 + x_3 \leq (t + x_3)/2 \leq 2$ for (x,t) in the region $\|x\| \leq 1$, $t \leq 3$. Hence, for $|x| \leq 1$, $\; -1 \leq x_3 \leq t \leq 3$, from (5.5) we have

$$\begin{aligned}|a(x,t)| &\leq \frac{\lambda + 2(x_3+1)}{2}\,\|f\|_* + \frac{\|q\|_\infty\,\|a\|_\infty}{4\pi}\int_{|y|\leq 2} \frac{1}{|y|}\,dy \\ &\leq 4\|f\|_* + 2\|q\|_\infty\,\|a\|_\infty.\end{aligned}$$

Hence

$$\|a\|_\infty \leq 4\|f\|_* + 2\|q\|_\infty\,\|a\|_\infty$$

so if $\|q\|_\infty \leq 1/4$ then

$$\|a\|_\infty \leq 8\|f\|_*.$$

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