GLOBAL UNIQUENESS FROM PARTIAL CAUCHY DATA IN TWO DIMENSIONS

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Abstract. We prove for a two dimensional bounded domain that the Cauchy data for the Schrödinger equation measured on an arbitrary open subset of the boundary determines uniquely the potential. This implies, for the conductivity equation, that if we measure the current fluxes at the boundary on an arbitrary open subset of the boundary produced by voltage potentials supported in the same subset, we can determine uniquely the conductivity. We use Carleman estimates with degenerate weight functions to construct appropriate complex geometrical optics solutions to prove the results.

1. Introduction

We consider the problem of determining a complex-valued potential \( q \) in a bounded two dimensional domain from the Cauchy data measured on an arbitrary open subset of the boundary for the associated Schrödinger equation \( \Delta + q \). A motivation comes from the classical inverse problem of electrical impedance tomography problem. In this inverse problem one attempts to determine the electrical conductivity of a body by measurements of voltage and current on the boundary of the body. This problem was proposed by Calderón [9] and is also known as Calderón’s problem. In dimensions \( n \geq 3 \), the first global uniqueness result for \( C^2 \)-conductivities was proven in [28]. In [25], [5] the global uniqueness result was extended to less regular conductivities. Also see [14] as for the determination of more singular conormal conductivities. In dimension \( n \geq 3 \) global uniqueness was shown for the Schrödinger equation with bounded potentials in [28]. The case of more singular conormal potentials was studied in [14].

In two dimensions the first global uniqueness result for Calderón’s problem was obtained in [24] for \( C^2 \)-conductivities. Later the regularity assumptions were relaxed in [6], and [2]. In particular, the paper [2] proves uniqueness for \( L^\infty \)-conductivities. In two dimensions a recent result of Bukgheim [7] gives unique identifiability of the potential from Cauchy data measured on the whole boundary for the associated Schrödinger equation. As for the uniqueness in determining two coefficients, see [10], [19].

In all the above mentioned articles, the measurements are made on the whole boundary. The purpose of this paper is to show the global uniqueness in two dimensions, both for the Schrödinger and conductivity equation, by measuring all the Neumann data on an arbitrary open subset \( \tilde{\Gamma} \) of the boundary produced by inputs of Dirichlet data supported on \( \tilde{\Gamma} \). We formulate this inverse problem more precisely below. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain
with smooth boundary, and let \( \nu \) be the unit outward normal vector to \( \partial \Omega \). We denote \( \frac{\partial u}{\partial \nu} = \nabla u \cdot \nu \). A bounded and non-zero function \( \gamma(x) \) (possibly complex-valued) models the electrical conductivity of \( \Omega \). Then a potential \( u \in H^1(\Omega) \) satisfies the Dirichlet problem

\[
\begin{align*}
\text{div}(\gamma \nabla u) &= 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} &= f,
\end{align*}
\]

where \( f \in H^{\frac{1}{2}}(\partial \Omega) \) is a given boundary voltage potential. The Dirichlet-to-Neumann (DN) map is defined by

\[
(1.2) \quad \Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega}.
\]

This problem can be reduced to studying the set of Cauchy data for the Schrödinger equation with the potential \( q \) given by:

\[
(1.3) \quad q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}.
\]

More generally we define the set of Cauchy data for a bounded potential \( q \) by:

\[
(1.4) \quad \hat{C}_q = \left\{ \left( u\big|_{\partial \Omega}, \frac{\partial u}{\partial \nu}\big|_{\partial \Omega} \right) \mid (\Delta + q)u = 0 \text{ on } \Omega, \quad u \in H^1(\Omega) \right\}.
\]

We have \( \hat{C}_q \subset H^{\frac{1}{2}}(\partial \Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \).

Let \( \Gamma \subset \partial \Omega \) be a non-empty open subset of the boundary. Denote \( \Gamma_0 = \partial \Omega \setminus \Gamma \).

Our main result gives global uniqueness by measuring the Cauchy data on \( \Gamma \). Let \( q_j \in C^{1+\alpha}(\overline{\Omega}), \ j = 1, 2 \) for some \( \alpha > 0 \) and let \( q_j \) be complex-valued. Consider the following sets of Cauchy data on an \( \Gamma_0 \):

\[
(1.5) \quad C_{q_j} = \left\{ \left( u\big|_{\Gamma_0}, \frac{\partial u}{\partial \nu}\big|_{\Gamma_0} \right) \mid (\Delta + q_j)u = 0 \text{ on } \Omega, \quad u\big|_{\partial D \cup \partial \Omega \setminus V} = 0, \quad u \in H^1(\Omega) \right\}, \quad j = 1, 2.
\]

**Theorem 1.1.** Assume \( C_{q_1} = C_{q_2} \). Then \( q_1 \equiv q_2 \).

Using Theorem 1.1 one concludes immediately as a corollary the following global identifiability result for the conductivity equation (1.1).

**Corollary 1.1.** With some \( \alpha > 0 \), let \( \gamma_j \in C^{3+\alpha}(\overline{\Omega}), \ j = 1, 2 \), be non-vanishing functions. Assume that

\[
\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \text{ in } \overline{\Gamma} \text{ for all } f \in H^{\frac{1}{2}}(\Gamma), \quad \text{supp} \ f \subset \overline{\Gamma}.
\]

Then \( \gamma_1 = \gamma_2 \).

Notice that Theorem 1.1 does not assume that \( \Omega \) is simply connected. An interesting inverse problem is where one can determine the potential or conductivity in a region of the plane with holes by measuring the Cauchy data only on the accessible boundary. Let \( \Omega, D \) be domains in \( \mathbb{R}^2 \) with smooth boundary such that \( \overline{D} \subset \Omega \). Let \( V \subset \partial \Omega \) be an open set. Let \( q_j \in C^{1+\alpha}(\overline{\Omega \setminus D}), \ j = 1, 2 \), for some \( \alpha > 0 \) and \( j = 1, 2 \). Let us consider the following set of partial Cauchy data

\[
(1.6) \quad \hat{C}_{q_j} = \{(u|_V, \frac{\partial u}{\partial \nu}|_V)|(\Delta + q_j)u = 0 \text{ in } \Omega \setminus \overline{D}, \ u|_{\partial D \cup \partial \Omega \setminus V} = 0, \ u \in H^1(\Omega)\}.
\]
Corollary 1.2. Assume $\tilde{q}_1 = \tilde{q}_2$. Then $q_1 = q_2$.

A similar result holds for the conductivity equation.

Corollary 1.3. Let $\gamma_j \in C^{3+\alpha}(\Omega \setminus D)$ $j = 1, 2$ be non vanishing functions. Assume $\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f)$ in $V$ $\forall f \in H^{1/2}(\partial \Omega \setminus D)$, $\text{supp } f \subset V$.

Then $\gamma_1 = \gamma_2$.

Another application of Theorem 1.1 is to the anisotropic conductivity problem. In this case the conductivity depends on direction and is represented by a positive definite symmetric matrix $\sigma = \{\sigma^{ij}\}$ on $\Omega$.

The conductivity equation with voltage potential $g$ on $\partial \Omega$ is given by

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} (\sigma^{ij} \frac{\partial u}{\partial x_j}) = 0 \quad \text{in } \Omega,$$

$$u|_{\partial \Omega} = g.$$

The Dirichlet-to-Neumann map is defined by

$$\Lambda_\sigma(g) = \sum_{i,j=1}^{2} \sigma^{ij} \nu_i \frac{\partial u}{\partial x_j} |_{\partial \Omega}.$$

It has been known for a long time that $\Lambda_\sigma$ does not determine $\sigma$ uniquely in the anisotropic case [23]. Let $F : \overline{\Omega} \to \overline{\Omega}$ be a diffeomorphism such that $F(x) = x$ for and $x$ from $\partial \Omega$. Then

$$\Lambda_{F_*\sigma} = \Lambda_\sigma,$$

where

$$F_*\sigma = \frac{(DF) \cdot \sigma \cdot (DF)^T \cdot F^{-1}}{|\text{det}DF|}.$$

Here $DF$ denotes the differential of $F$, $(DF)^T$ its transpose and the composition in (1.6) is matrix composition. The question of whether one can determine the conductivity up to the obstruction (1.6) has been solved for $C^2$ conductivities in [24], Lipschitz conductivities in [26] and merely $L^\infty$ conductivities in [3]. The method of proof in all these papers is the reduction to the isotropic case performed using isothermal coordinates [27]. Using the same method and Corollary 1.1 we obtain the following result:

Theorem 1.2. Let $\sigma_k = \{\sigma_k^{ij}\} \in C^{3+\alpha}(\overline{\Omega})$ for $k = 1, 2$ and some positive $\alpha$. Suppose that $\sigma_k$ are positive definite symmetric matrices on $\overline{\Omega}$. Let $\Gamma \subset \partial \Omega$ be some open set. Assume

$$\Lambda_{\sigma_1}(g)|_{\Gamma} = \Lambda_{\sigma_2}(g)|_{\Gamma} \quad \forall g \in H^{1/2}(\partial \Omega), \text{supp } g \subset \Gamma.$$

Then there exists a diffeomorphism

$$F : \overline{\Omega} \to \overline{\Omega}, \quad F|_{\partial \Omega} = \text{Identity}, \quad F \in C^{4+\alpha}(\overline{\Omega}), \alpha > 0$$

such that

$$F_*\sigma_1 = \sigma_2.$$
To the authors’ knowledge, there are no uniqueness results similar to Theorem 1.1 with Dirichlet data supported and Neumann data measured on the same arbitrary open subset of the boundary, even for smooth potentials or conductivities. In dimension $n \geq 3$, Isakov [18] proved global uniqueness assuming that $\Gamma_0$ is a subset of a plane or a sphere. In dimensions $n \geq 3$, [8] proves global uniqueness in determining a bounded potential for the Schrödinger equation with Dirichlet data supported on the whole boundary and Neumann data measured in roughly half the boundary. The proof relies on a Carleman estimate with a linear weight function. This implies a similar result for the conductivity equation with $C^2$ conductivities.

In [21], the regularity assumption on the conductivity was relaxed to $C^{3/2+\alpha}$ with some $\alpha > 0$. The corresponding stability estimates are proved in [15]. As for the stability estimates for the magnetic Schrödinger equation with partial data, see [29]. In [20], the result in [8] was generalized to show that by all possible pairs of Dirichlet data on an arbitrary open subset $\Gamma_0$ of the boundary and Neumann data on a slightly larger open domain than $\partial \Omega \setminus \Gamma_+$, one can uniquely determine the potential. The method of the proof uses Carleman estimates with non-linear weights. The case of the magnetic Schrödinger equation was considered in [11] and an improvement on the regularity of the coefficients is done in [22]. Stability estimates for the magnetic Schrödinger equation with partial data were proven in [29].

In two dimensions, the first general result was given by the authors in [17]. It is shown that the same global uniqueness result as [20] holds in this case. The two-dimensional case has special features since one can construct a much larger set of complex geometrical optics solutions than in higher dimensions. On the other hand, the problem is formally determined in two dimensions and therefore more difficult. The proof of our main result [17] is based on the construction of appropriate complex geometrical optics solutions by Carleman estimates with degenerate weight functions.

This paper is composed of four sections. In Section 2, we establish our key Carleman estimates, and in Section 3, we construct complex geometrical optics solutions. In Section 4, we complete the proof of Theorem 1.1. In the Appendix we consider the solvability of the Cauchy Riemann equations with Cauchy data on a subset of the boundary. We also establish a Carleman estimate for Laplace’s equation with degenerate harmonic weights.

2. Carleman estimates with degenerate weights

Throughout the paper we use the following notations:

Notations

$i = \sqrt{-1}$, $x_1, x_2, \xi_1, \xi_2 \in \mathbb{R}$, $z = x_1 + ix_2$, $\zeta = \xi_1 + i\xi_2$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$. $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $H^1(\Omega)$ denotes the space $H^1(\Omega)$ with norm $\|v\|^2_{H^1(\Omega)} = \|v\|^2_{H^1(\Omega)} + \tau^2\|v\|^2_{L^2(\Omega)}$. The tangential derivative on the boundary is given by $\partial_T = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$, with $\nu = (\nu_1, \nu_2)$ the unit outer normal to $\partial \Omega$, $B(\widehat{x}, \delta) = \{x \in \mathbb{R}^2 | |x - \widehat{x}| < \delta\}$, $f(x) : \mathbb{R}^2 \to \mathbb{R}^1$, $f''$ is the Hessian matrix with entries $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $\mathcal{L}(X,Y)$ denotes the Banach space of all bounded linear operators from a Banach space $X$ to another Banach space $Y$. 
Let $\Phi(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2) \in C^2(\Omega)$ be a holomorphic function in $\Omega$ with real-valued $\varphi$ and $\psi$:

\[(2.1) \quad \partial_z \Phi(z) = 0 \quad \text{in } \Omega.\]

Denote by $\mathcal{H}$ the set of critical points of a function $\Phi$

\[\mathcal{H} = \{z \in \Omega | \partial_z \Phi(z) = 0\}.\]

Assume that $\Phi$ has no critical points on the boundary, and that all the critical points are nondegenerate:

\[(2.2) \quad \mathcal{H} \cap \partial \Omega = \{\emptyset\}, \quad \partial_z^2 \Phi(z) \neq 0, \quad \forall z \in \mathcal{H}.\]

Then we know that $\Phi$ has only a finite number of critical points and we can set:

\[(2.3) \quad \mathcal{H} = \{\tilde{x}_1, ..., \tilde{x}_\ell\}.\]

Consider the following problem

\[(2.4) \quad \Delta u + q_0 u = f \quad \text{in } \Omega, \quad u|_{\Gamma_0} = g,\]

where $\nu$ is the unit outward normal vector to $\partial \Omega$ and $\Gamma_0 = \{x \in \partial \Omega | (\nu, \nabla \varphi) = 0\}$.

We have

**Proposition 2.1.** Let $q_0 \in L^\infty(\Omega)$. There exists $\tau_0 > 0$ such that for all $|\tau| > \tau_0$ there exists a solution to problem (2.4) such that

\[(2.5) \quad \|u e^{-\tau \varphi}\|_{L^2(\Omega)} \leq C(\|f e^{-\tau \varphi}\|_{L^2(\Omega)} \sqrt{|\tau|} + \|g e^{-\tau \varphi}\|_{L^2(\Gamma_0)}).\]

For the proof, see Proposition 2.2 in [17] and Proposition 5.2 in appendix.

Let us introduce the operators:

\[
\partial_z^{-1} g = \frac{1}{2\pi i} \int_{\Omega} \frac{g(\zeta, \overline{\zeta})}{\zeta - z} d\zeta \wedge d\overline{\zeta} = -\frac{1}{\pi} \int_{\Omega} \frac{g(\zeta, \overline{\zeta})}{\zeta - z} d\xi_2 d\xi_1,
\]

\[
\partial_{\overline{z}}^{-1} g = -\frac{1}{2\pi i} \int_{\Omega} \frac{\overline{g}(\zeta, \overline{\zeta})}{\zeta - z} d\zeta \wedge d\overline{\zeta} = -\frac{1}{\pi} \int_{\Omega} \frac{\overline{g}(\zeta, \overline{\zeta})}{\overline{\zeta} - \overline{z}} d\xi_2 d\xi_1 = \overline{\partial_z^{-1} g}.
\]

See e.g., pp.28-31 in [31] where $\partial_z^{-1}$ and $\partial_{\overline{z}}^{-1}$ are denoted by $T$ and $\overline{T}$ respectively. Then we know (e.g., p.47 and p.56 in [31]):

**Proposition 2.2.** A) Let $m \geq 0$ be an integer number and $\alpha \in (0, 1)$. The operators $\partial_z^{-1}, \partial_{\overline{z}}^{-1} \in \mathcal{L}(C^{m+\alpha} (\Omega), C^{m+\alpha+1}(\overline{\Omega}))$.

B) Let $1 \leq p \leq 2$ and $1 < \gamma < \frac{2p}{2-p}$. Then $\partial_z^{-1}, \partial_{\overline{z}}^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega))$.

We define two other operators:

\[(2.6) \quad R_{\Phi, \tau} g = e^{\tau (\Phi(z) - \overline{\Phi(z)})} \partial_z^{-1} (ge^{\tau (\Phi(z) - \overline{\Phi(z)})}), \quad \overline{R}_{\Phi, \tau} g = e^{\tau (\Phi(z) - \overline{\Phi(z)})} \partial_{\overline{z}}^{-1} (ge^{\tau (\Phi(z) - \overline{\Phi(z)})}).\]
Proposition 2.3. Let \( g \in C^\alpha(\Omega) \) for some positive \( \alpha \). The function \( R_{\Phi,\tau}g \) is a solution to
\[
\partial_x R_{\Phi,\tau}g - \tau(\partial_x \Phi(z)) R_{\Phi,\tau}g = g \quad \text{in } \Omega.
\]
The function \( \tilde{R}_{\Phi,\tau}g \) solves
\[
\partial_x \tilde{R}_{\Phi,\tau}g + \tau(\partial_x \Phi(z)) \tilde{R}_{\Phi,\tau}g = g \quad \text{in } \Omega.
\]

The proof is done by direct computations (see the proof of Proposition 3.3 in [17]).

Denote
\[
\mathcal{O}_\epsilon = \{ x \in \Omega | \text{dist}(x, \partial\Omega) \leq \epsilon \}.
\]

Proposition 2.4. Let \( g \in C^1(\Omega) \) and \( g|_{\mathcal{O}_\epsilon} = 0 \), \( g(x) \neq 0 \) for all \( x \in \mathcal{H} \). Then
\[
|R_{\Phi,\tau}g(x)| + |\tilde{R}_{\Phi,\tau}g(x)| \leq C \max_{x \in \mathcal{H}} |g(x)|/\tau
\]
for all \( x \in \mathcal{O}_{\epsilon/2} \). If \( g \in C^2(\Omega) \) and \( g|_{\mathcal{H}} = 0 \), then
\[
|R_{\Phi,\tau}g(x)| + |\tilde{R}_{\Phi,\tau}g(x)| \leq C/\tau^2
\]
for all \( x \in \mathcal{O}_{\epsilon/2} \).

The proof uses the Cauchy-Riemann equations and stationary phase (e.g., Section 4.5.3 in [13], Chapter VII, §7.7 in [16]). See also the proof of Proposition 3.4 in [17].

Denote
\[
r(z) = \Pi_{k=1}^n(z - \bar{z}_k) \text{ where } \mathcal{H} = \{ \vec{x}_1, \ldots, \vec{x}_\ell \}, \ \bar{z}_k = \vec{x}_{1,k} + i\vec{x}_{2,k}.
\]

The following proposition can be proved similarly to Proposition 3.5 in [17]:

Proposition 2.5. Let \( g \in C^1(\Omega) \) and \( g|_{\mathcal{O}_\epsilon} = 0 \). Then for each \( \delta \in (0,1) \), there exists a constant \( C(\delta) > 0 \) such that
\[
\|R_{\Phi,\tau}(r(z)g)\|_{L^2(\Omega)} \leq C(\delta)\|g\|_{C^1(\Omega)}/|\tau|^{1-\delta}, \quad \|\tilde{R}_{\Phi,\tau}(r(z)g)\|_{L^2(\Omega)} \leq C(\delta)\|g\|_{C^1(\Omega)}/|\tau|^{1-\delta}.
\]

Henceforth we set \( \psi_1 \equiv \text{Re}\partial_x \Phi = \partial_{x_1} \varphi \) and \( \psi_2 \equiv \text{Im}\partial_x \Phi = \partial_{x_2} \psi \). We also need the following result, which we can be proven in the same way as Proposition 2.1 in [17]. Note that
\[
\partial_{x_1}(e^{-ir\psi\bar{v}}e^{ir\psi}) = \partial_{x_1} \bar{v} - i\tau \psi_2 \bar{v}
\]
and
\[
\partial_{x_2}(e^{-ir\psi\bar{v}}e^{ir\psi}) = \partial_{x_2} \bar{v} - i\tau \psi_1 \bar{v},
\]
etc. which follow from the Cauchy-Riemann equations.

Proposition 2.6. Let \( \Phi \) satisfy (2.1) and (2.2). Let \( \bar{f} \in L^2(\Omega) \) and \( \bar{v} \) be solution to
\[
2\partial_{\bar{z}} \bar{v} - \tau(\partial_{\bar{z}} \Phi)\bar{v} = \bar{f} \quad \text{in } \Omega
\]
or \( \bar{v} \) be solution to
\[
2\partial_{\bar{z}}v - \tau(\partial_{\bar{z}} \Phi)v = \bar{f} \quad \text{in } \Omega.
\]
In the case (2.12) we have
\[
\|\partial_x(e^{-ir\Phi} \bar{v})\|^2_{L^2(\Omega)} - \tau \int_{\partial \Omega} (\nabla \varphi, \nu)|\bar{v}|^2 \, d\sigma
\]
\[= \left( \int_{\partial \Omega} i \left( \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \bar{v} \right) \bar{v} \, d\sigma + \|\partial_x(e^{-ir\Phi} \bar{v})\|^2_{L^2(\Omega)} \right) = \|\tilde{f}\|^2_{L^2(\Omega)}.
\]

In the case that \(\bar{v}\) solves (2.13) we have
\[
\|\partial_x(e^{ir\Phi} \bar{v})\|^2_{L^2(\Omega)} - \tau \int_{\partial \Omega} (\nabla \varphi, \nu)|\bar{v}|^2 \, d\sigma + \text{Re} \int_{\partial \Omega} i \left( \left( -\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2} \right) \bar{v} \right) \bar{v} \, d\sigma
\]
\[+ \|\partial_x(e^{ir\Phi} \bar{v})\|^2_{L^2(\Omega)} = \|\tilde{f}\|^2_{L^2(\Omega)}.
\]

We have

**Proposition 2.7.** Let \(g \in C^2(\Omega), g|_{\partial} = 0\) and \(g|_{\mathcal{H}} = 0\). Then
\[
\left\| R_{\Phi, \tau} g + \frac{g}{\tau \partial_z \Phi} \right\|_{L^2(\Omega)} + \left\| \tilde{R}_{\Phi, \tau} g - \frac{g}{\tau \partial_z \Phi} \right\|_{L^2(\Omega)} = o \left( \frac{1}{\tau} \right) \quad \text{as } |\tau| \to +\infty.
\]

**Proof.** By (2.2) and Proposition 2.4
\[
\left\| \tilde{R}_{\Phi, \tau} g \right\|_{C(\overline{\Omega})} + \left\| R_{\Phi, \tau} g \right\|_{C(\overline{\Omega})} = o \left( \frac{1}{\tau} \right).
\]
Therefore instead of (2.16) it suffices to prove
\[
\left\| \chi_1 R_{\Phi, \tau} g + \frac{g}{\tau \partial_z \Phi} \right\|_{L^2(\Omega)} + \left\| \chi_1 \tilde{R}_{\Phi, \tau} g - \frac{g}{\tau \partial_z \Phi} \right\|_{L^2(\Omega)} = o \left( \frac{1}{\tau} \right) \quad \text{as } |\tau| \to +\infty,
\]
where \(\chi_1 \in C_0^\infty(\Omega)\) and \(\chi_1|_{\partial} \equiv 1\). Denote \(w = \chi_1 \tilde{R}_{\Phi, \tau} g - \frac{g}{\tau \partial_z \Phi}\). Here we note that \(\frac{g}{\tau \partial_z \Phi} \in L^\infty(\Omega)\). This follows from (2.2), \(g \in C^1(\Omega)\) and \(g|_{\mathcal{H}} = 0\). Then (2.8) and \(g|_{\partial} = 0\) yield
\[
\partial_z w + \tau \partial_z \phi w = -\partial_z \left( \frac{g}{\tau \partial_z \Phi} \right) + (\partial_z \chi_1) \tilde{R}_{\Phi, \tau} g \quad \text{in } \Omega, \quad w|_{\partial} = 0.
\]

Note that by (2.2) and the fact that \(g|_{\mathcal{H}} = 0\), we have:
\[
\left| \partial_z \left( \frac{g}{\partial_z \Phi} \right) \right| = \left| \frac{\partial_z g}{\partial_z \Phi} - \frac{g}{\partial_z \Phi} \frac{\partial^2 \Phi}{\partial_z \Phi} \right| \leq \frac{C}{\Pi_{k=1}^l |x - \bar{x}_k|}.
\]
Consider the cut off function \(\chi \in C_0^\infty(\Omega)\) such that
\[
\chi \geq 0, \quad \chi|_{B(0, \frac{1}{2})} = 1.
\]
By (2.20) and Proposition 2.2 B),
\[
\tilde{R}_{\Phi, \tau} \left( \sum_{k=1}^l \chi((x - \bar{x}_k) \ln |\tau|) \partial_z \left( \frac{g}{\partial_z \Phi} \right) \right) \to 0 \quad \text{in } L^2(\Omega) \text{ as } |\tau| \to +\infty.
\]
In fact, fixing large $|\tau|$, small $\delta > 0$ and $p > 1$ such that $p - 1$ is sufficiently small, we apply Proposition 2.2 B) and (2.20) to conclude

$$
\left\| \tilde{R}_{\Phi,\tau} \left( \sum_{k=1}^{\ell} \chi((x - \tilde{x}_k) \ln |\tau|) \partial_z \left( \frac{g}{\partial_z \Phi} \right) \right) \right\|_{L^2(\Omega)}^2
\leq C \sum_{k=1}^{\ell} \int_{B(\bar{x}_k,\delta)} |\chi((x - \bar{x}_k) \ln |\tau|)|^p \left| \partial_z \left( \frac{g}{\partial_z \Phi} \right) \right|^p dx
\leq C' \sum_{k=1}^{\ell} \int_{B(\bar{x}_k,\delta)} |\chi((x - \bar{x}_k) \ln |\tau|)|^p \frac{1}{|x - \bar{x}_k|^p} dx
\leq C'' \int_0^\delta |\chi(\rho \ln |\tau|)|^p \rho^{1-p} d\rho.
$$

Thus we get (2.21) by Lebesgue’s theorem.

By Proposition 2.5, we obtain

$$(2.22) \quad \tilde{R}_{\Phi,\tau} \left( \left( 1 - \sum_{k=1}^{\ell} \chi((x - \bar{x}_k) \ln |\tau|) \right) \partial_z \left( \frac{g}{\partial_z \Phi} \right) \right) \rightarrow 0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad |\tau| \rightarrow +\infty.
$$

Therefore (2.21) and (2.22) yield

$$(2.23) \quad \left\| \tilde{R}_{\Phi,\tau} \left( \partial_z \left( \frac{g}{\partial_z \Phi} \right) \right) \right\|_{L^2(\Omega)} = o(1) \quad \text{as} \quad |\tau| \rightarrow +\infty.
$$

Denote $\tilde{w} = w + \frac{1}{\tau} \chi_1 \tilde{R}_{\Phi,\tau}(\partial_z(\frac{g}{\partial_z \Phi})).$

By (2.23), it suffices to prove

$$(2.24) \quad \| \tilde{w} \|_{L^2(\Omega)} = o \left( \frac{1}{\tau} \right) \quad \text{as} \quad |\tau| \rightarrow +\infty.
$$

In terms of (2.19) and (2.7), observe that

$$(2.25) \quad \partial_z \tilde{w} + \tau \partial_z \Phi \tilde{w} = f \quad \text{in} \quad \Omega, \quad \tilde{w}|_{\partial\Omega} = 0,
$$

where $f = \frac{1}{\tau}(\partial_z \chi_1) \tilde{R}_{\Phi,\tau}(\partial_z(\frac{g}{\partial_z \Phi})) + (\partial_z \chi_1) \tilde{R}_{\Phi,\tau} g$. By (2.23) and (2.17),

$$(2.26) \quad \| f \|_{L^2(\Omega)} = o \left( \frac{1}{\tau} \right) \quad \text{as} \quad |\tau| \rightarrow +\infty.
$$

Noting $\tilde{w}|_{\partial\Omega} = 0$, applying Proposition 2.6 to equation (2.25) and using (2.26), we obtain (2.24). As for the first term in (2.18), we can argue similarly. The proof of the proposition is completed.

3. Complex geometrical optics solutions

In this section, we construct complex geometrical optics solutions for the Schrödinger equation $\Delta + q_1$ with $q_1$ satisfying the conditions of Theorem 1.1. Consider

$$(3.1) \quad L_1 u = \Delta u + q_1 u = 0 \quad \text{in} \quad \Omega.
$$

We will construct solutions to (3.1) of the form

$$(3.2) \quad u_1(x) = e^{\tau \Phi(z)}(a(z) + a_0(z)/\tau) + e^{\tau \Phi(z)}(a(z) + a_1(z)/\tau) + e^{\tau \varphi} u_{11} + e^{\tau \varphi} u_{12}, \quad u_1|_{\Gamma_0} = 0.
$$
The function $\Phi$ satisfies (2.1), (2.2) and

$$\text{(3.3)} \quad \text{Im } \Phi|_{\Gamma_0} = 0.$$  

The amplitude function $a(z)$ is not identically zero on $\overline{\Omega}$ and has the following properties:

$$\text{(3.4)} \quad a \in C^2(\overline{\Omega}), \quad \partial_\nu a \equiv 0, \quad \text{Re } a|_{\Gamma_0} = 0.$$  

The function $u_{11}$ is given by

$$\text{(3.5)} \quad u_{11} = \frac{1}{4} e^{i\nu} R_{\Phi,1}(e_1(\partial_{\nu}^{-1}(a q_1) - M_1(z))) - \frac{1}{4} e^{-i\nu} R_{\Phi,1}(e_1(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z})))$$

$$- e^{i\nu} \frac{e_2(\partial_{\nu}^{-1}(a q_1) - M_1(z))}{4\partial_\nu \Phi} - e^{-i\nu} \frac{e_2(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z}))}{4\partial_\nu \Phi}$$

$$= w_1 e^{-\nu} + w_2 e^{-\nu},$$

where the polynomials $M_1(z)$ and $M_3(\overline{z})$ satisfy

$$\text{(3.6)} \quad \partial_{\nu}^j(\partial_{\nu}^{-1}(a q_1) - M_1(z)) = 0, \quad x \in \mathcal{H}, \ j = 0, 1, 2,$$

$$\text{(3.7)} \quad \partial_{\nu}^j(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z})) = 0, \quad x \in \mathcal{H}, \ j = 0, 1, 2.$$

The functions $e_1, e_2 \in C^\infty(\Omega)$ are constructed so that $e_1 + e_2 \equiv 1$ on $\overline{\Omega}$, $e_2$ vanishes in some neighborhood of $\mathcal{H}$ and $e_1$ vanishes in a neighborhood of $\partial \Omega$ and we set

$$w_1 = \frac{1}{4} e^{\nu} R_{\Phi,1}(e_1(\partial_{\nu}^{-1}(a q_1) - M_1(z))) - \frac{1}{4} e^{-\nu} R_{\Phi,1}(e_1(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z})))$$

and

$$w_2 = - e^{\nu} \frac{e_2(\partial_{\nu}^{-1}(a q_1) - M_1(z))}{4\partial_\nu \Phi} - e^{-\nu} \frac{e_2(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z}))}{4\partial_\nu \Phi}.$$ 

Finally $a_0, a_1$ are holomorphic functions such that

$$(a_0(z) + a_1(z))|_{\Gamma_0} = \left(\frac{(\partial_{\nu}^{-1}(a q_1) - M_1(z))}{4\partial_\nu \Phi} + \frac{(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z}))}{4\partial_\nu \Phi}\right)$$

Then, noting $\partial_{\nu}^{-1} \Phi = \partial_{\nu} \Phi$, (2.7) and (2.8), we have

$$\Delta w_1 = 4\partial_\nu \partial_\nu w_1$$

$$= - \partial_{\nu} (e^{\nu} \partial_{\nu} \tilde{R}_{\Phi,1}(e_1(\partial_{\nu}^{-1}(a q_1) - M_1(z))) + (\tau \partial_\nu \Phi) e^{\nu} \tilde{R}_{\Phi,1}(e_1(\partial_{\nu}^{-1}(a q_1) - M_1(z)))$$

$$- \partial_\nu (e^{-\nu} \partial_{\nu} \tilde{R}_{\Phi,1}(e_1(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z}))) + (\tau \partial_\nu \Phi) e^{-\nu} \tilde{R}_{\Phi,1}(e_1(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z})))$$

$$= - \partial_\nu (e^{\nu} e_1(\partial_{\nu}^{-1}(a q_1) - M_1(z))) - \partial_\nu (e^{-\nu} e_1(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z}))).$$

Moreover

$$\Delta w_2 = 4\partial_\nu \partial_\nu w_2$$

$$= - \partial_\nu (e^{\nu} e_2(\partial_{\nu}^{-1}(a q_1) - M_1(z))) - \partial_\nu (e^{-\nu} e_2(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z})))$$

$$- e^{\nu} \Delta \left(\frac{e_2(\partial_{\nu}^{-1}(a q_1) - M_1(z))}{4\tau \partial_\nu \Phi} \right) - e^{-\nu} \Delta \left(\frac{e_2(\partial_{\nu}^{-1}(a(z) q_1) - M_3(\overline{z}))}{4\tau \partial_\nu \Phi} \right).$$
Therefore
\begin{equation}
\Delta (u_{11} e^{\tau \varphi}) = \Delta (w_1 + w_2) = -aq_1 e^{\tau \Phi} - \overline{aq_1} e^{\tau \overline{\Phi}} - e^{\tau \Phi} \Delta \left( \frac{e_2 (\partial_z^{-1}(aq_1) - M_1(z))}{4\tau \partial_z \Phi} \right) - e^{\tau \overline{\Phi}} \Delta \left( \frac{e_2 (\partial_z^{-1}(a(z)q_1) - M_3(z))}{4\tau \partial_z \Phi} \right).
\end{equation}

By (3.4) and (3.3), observe that
\begin{equation}
(e^{\tau \Phi(z)}a(z) + e^{\tau \overline{\Phi(z)}}a(z)) |_{\Gamma_0} = 0.
\end{equation}

By Proposition 2.1, the inhomogeneous problem
\begin{equation}
\Delta (u_{12} e^{\tau \varphi}) + q_1 u_{12} e^{\tau \varphi} = -q_1 u_{11} e^{\tau \varphi} + h_1 e^{\tau \varphi} \quad \text{in } \Omega,
\end{equation}

\begin{equation}
u_{12} = -u_{11} \quad \text{on } \Gamma_0,
\end{equation}
has a solution where
\begin{equation}
h_1 = e^{\tau i \psi} \Delta \left( \frac{e_2 (\partial_z^{-1}(aq_1) - M_1(z))}{4\tau \partial_z \Phi} \right) + e^{-\tau i \psi} \Delta \left( \frac{e_2 (\partial_z^{-1}(a(z)q_1) - M_3(z))}{4\tau \partial_z \Phi} \right) - a_0 q_1 e^{\tau \Phi} / \tau - \overline{a_1} q_1 e^{\tau \overline{\Phi}} / \tau.
\end{equation}

Then, by (3.4) and (3.8) - (3.12), we see that (3.1) is satisfied.

By Proposition 2.1 there exists a positive \( \tau_0 \) such that for all \(|\tau| > \tau_0 \) there exists a solution to (3.10), (3.11) satisfying
\begin{equation}
\|u_{12}\|_{L^2(\Omega)} = o(\frac{1}{\tau}).
\end{equation}

This can be done because
\begin{equation}
\|q_1 u_{11} + h_1\|_{L^2(\Omega)} \leq C(\delta)/\tau^{1-\delta} \quad \forall \delta \in (0,1); \|u_{11}\|_{L^2(\partial\Omega)} = o(\frac{1}{\tau})
\end{equation}

and \((\nabla \varphi, \nu) = 0\) on \(\Gamma_0\). The latter fact can be seen as follows: On \(\partial\Omega\), the Cauchy-Riemann equations imply
\begin{equation}
(\nabla \varphi, \nu) = \nu_1 \partial_{x_1} \varphi + \nu_2 \partial_{x_2} \varphi = \nu_1 \partial_{x_2} \psi - \nu_2 \partial_{x_1} \psi = \partial_{\overline{\tau}} \psi,
\end{equation}

which is the tangential derivative of \(\psi = \text{Im} \Phi\) on \(\partial\Omega\). By (3.3) we see that the tangential derivative of \(\psi\) vanishes on \(\Gamma_0\).

Consider the Schrödinger equation
\begin{equation}
L_2 v = \Delta v + q_2 v = 0 \quad \text{in } \Omega.
\end{equation}

We will construct solutions to (3.14) of the form
\begin{equation}
v(x) = e^{-\tau \Phi(z)}(a(z) + b_0(z) / \tau) + e^{-\tau \overline{\Phi(z)}}(a(z) + b_1(z) / \tau) + e^{-\tau \varphi} v_{11} + e^{-\tau \varphi} v_{12}, \quad v |_{\Gamma_0} = 0.
\end{equation}
The construction of \( v \) repeats the corresponding steps of the construction of \( u_1 \). The only difference is that instead of \( q_1 \) and \( \tau \), we use \( q_2 \) and \( -\tau \) respectively. We provide the details for the sake of completeness. The function \( v_{11} \) is given by

\begin{equation}
(3.16) \quad v_{11} = -\frac{1}{4} e^{-i\tau \psi} \bar{R}_{\Phi,-\tau}(e_1(\partial_{\bar{z}}^{-1}(q_2a(z)) - M_2(z))) - \frac{1}{4} e^{i\tau \psi} R_{\Phi,\tau}(e_1(\partial_{\bar{z}}^{-1}(q_2a(z)) - M_4(\bar{z})))
+ \frac{e^{-i\tau \psi}}{\tau} \frac{e_2(\partial_{\bar{z}}^{-1}(aq_2) - M_2(z))}{4\partial_z \Phi} + \frac{e^{i\tau \psi}}{\tau} \frac{e_2(\partial_{\bar{z}}^{-1}(a(z)q_2) - M_4(\bar{z}))}{4\partial_z \Phi},
\end{equation}

where

\begin{equation}
(3.17) \quad \partial_{\bar{z}}^j(\partial_{\bar{z}}^{-1}(aq_2) - M_2(z)) = 0, \quad x \in \mathcal{H}, \ j = 0, 1, 2,
\end{equation}

\begin{equation}
(3.18) \quad \partial_{\bar{z}}^j(\partial_{\bar{z}}^{-1}(a(z)q_2) - M_4(\bar{z})) = 0, \quad x \in \mathcal{H}, \ j = 0, 1, 2.
\end{equation}

Finally \( b_0, b_1 \) are holomorphic functions such that

\[ (b_0 + b_1)|_{\Gamma_0} = -\frac{(\partial_{\bar{z}}^{-1}(aq_2) - M_2(z))}{4\partial_z \Phi} - \frac{(\partial_{\bar{z}}^{-1}(a(z)q_2) - M_4(\bar{z}))}{4\partial_z \Phi}. \]

Denote

\[ h_2 = e^{-\tau \psi} \Delta \left( \frac{e_2(\partial_{\bar{z}}^{-1}(a(z)q_2) - M_2(z))}{4\partial_z \Phi} \right) + e^{\tau \psi} \Delta \left( -\frac{b_0(z)}{\tau} e^{-\tau \psi(z)} - \frac{b_1(z)}{\tau} e^{-\tau \psi(z)} \right). \]

The function \( v_{12} \) is a solution to the problem:

\begin{equation}
\Delta(v_{12}e^{-\tau \varphi}) + q_2 v_{12}e^{-\tau \varphi} = -q_2 v_{11}e^{-\tau \varphi} - h_2 e^{-\tau \varphi} \quad \text{in } \Omega,
\end{equation}

\begin{equation}
(3.20) \quad v_{12}|_{\Gamma_0} = -v_{11}|_{\Gamma_0}, \quad \text{such that}
\end{equation}

\begin{equation}
(3.21) \quad \|v_{12}\|_{L^2(\Omega)} = o(\frac{1}{\tau}).
\end{equation}

4. Proof of the theorem.

We first apply stationary phase with a general phase function \( \Phi \) and then we construct an appropriate weight function.

**Proposition 4.1.** Suppose that \( \Phi \) satisfies (2.1), (2.2) and (3.3). Let \( \{\bar{x}_1, \ldots, \bar{x}_L\} \) be the set of critical points of the function \( \text{Im} \Phi \). Then for any potentials \( q_1, q_2 \in C^{1,\alpha}(\overline{\Omega}) \), \( \alpha > 0 \) with the same Dirichlet-to-Neumann maps and for any holomorphic function \( a \) satisfying (3.4), we have

\begin{equation}
(4.1) \quad 2 \sum_{k=1}^L \frac{\pi |q| |a|^2(\bar{x}_k)}{|(\text{det } \text{Im} \Phi')(\bar{x}_k)|^{\frac{1}{2}}} + \int_{\Omega} q(a_0 b_0 + \bar{a}_1 \bar{b}_1) dx
\end{equation}
\[ + \frac{1}{4} \int_{\Omega} \left( qa \frac{\partial^{-1}(aq_2) - M_2(z)}{\partial_2 \Phi} + q \frac{\partial^{-1}(aq_2) - M_4(\bar{z})}{\partial_4 \Phi} \right) dx \]

\[- \frac{1}{4} \int_{\Omega} \left( qa \frac{\partial^{-1}(aq_1) - M_1(z)}{\partial_2 \Phi} + q \frac{\partial^{-1}(aq_1) - M_3(\bar{z})}{\partial_4 \Phi} \right) dx = 0, \quad \tau > 0, \]

where we set \( q = q_1 - q_2 \).

**Proof.** We note by the Cauchy-Riemann equations that \( \{ \tilde{x}_{1,1} + i \tilde{x}_{1,2}, ..., \tilde{x}_{\ell,1} + i \tilde{x}_{\ell,2} \} = \{ z \in \overline{\Omega} | \partial_z \text{Im} \Phi(z) = 0 \} \). Let \( u_1 \) be a solution to (3.1) and satisfy (3.2), and \( u_2 \) be a solution to the following equation

\[ \Delta u_2 + q_2 u_2 = 0 \text{ in } \Omega, \quad u_2|_{\partial \Omega} = u_1|_{\partial \Omega}. \]

Since the Dirichlet-to-Neumann maps are equal, we have

\[ \nabla u_2 = \nabla u_1 \text{ on } \tilde{\Gamma}. \]

Denoting \( u = u_1 - u_2 \), we obtain

\[ \Delta u + q_2 u = -q_1 \text{ in } \Omega, \quad u|_{\partial \Omega} = \frac{\partial u}{\partial \nu} |_{\tilde{\Gamma}} = 0. \]

Let \( v \) satisfy (3.14) and (3.15). We multiply (4.2) by \( v \), integrate over \( \Omega \) and use \( v|_{\Gamma_0} = 0 \) and \( \partial u/\partial \nu = 0 \) on \( \tilde{\Gamma} \) to obtain \( \int_{\Omega} qu_1 v dx = 0 \). By (3.2), (3.13), (3.15) and (3.21), we have

\[ 0 = \int_{\Omega} qu_1 v dx = \int_{\Omega} q(a^2 + \bar{a}^2 + |a|^2 e^{\tau(\Phi - \bar{\Phi})} + |a|^2 e^{-\tau(\Phi - \bar{\Phi})} + \frac{a_0 b_0}{\tau} + \frac{a_1 b_1}{\tau} + u_{11} e^{\tau(\Phi - \bar{\Phi})} + \overline{a} e^{-\tau(\Phi - \bar{\Phi})} + (a e^{\tau(\Phi - \bar{\Phi})} + \overline{a} e^{-\tau(\Phi - \bar{\Phi})}) v_{11} e^{-\tau(\Phi - \bar{\Phi})} dx + o \left( \frac{1}{\tau} \right), \quad \tau > 0. \]

The first and second terms in the asymptotic expansion of (4.3) are independent of \( \tau \), so that

\[ \int_{\Omega} q(a^2 + \bar{a}^2) dx = 0. \]

Using stationary phase (see p.215 in [13], cf. [16]), we obtain

\[ \int_{\Omega} q(|a|^2 e^{\tau(\Phi - \bar{\Phi})} + |a|^2 e^{-\tau(\Phi - \bar{\Phi})}) dx = 2 \sum_{k=1}^{\ell} \frac{\pi q |a|^2(\tilde{e}_k)}{\tau |(\text{det Im } \Phi''(\tilde{e}_k))|^{1/2}} + o \left( \frac{1}{\tau} \right). \]

Here by the Cauchy-Riemann equations, we see that \( \text{sgn}(\text{Im } \Phi''(\tilde{e}_k)) = 0 \), where \( \text{sgn} A \) denotes the signature of the matrix \( A \), that is the number of positive eigenvalues of \( A \) minus the number of negative eigenvalues (e.g., [13], p.210). Moreover we use (2.2) and the Cauchy-Riemann equations to see that

\[ \text{det Im } \Phi''(z) = -(\partial_{x_1} \partial_{x_2} \varphi)^2 - (\partial_{x_1}^2 \varphi)^2 \neq 0. \]
since $\partial_{x}^{2}\Phi = -\frac{1}{2}\partial_{x}^{2}\varphi - \frac{1}{2}i\partial_{x_{1}}\partial_{x_{2}}\varphi \neq 0$ in $H$. We get:

$$I(4.9)$$

Using stationary phase again and (3.6) we conclude that

$$I(4.8)$$

$p.215$ in [13]), we get

By Proposition 2.7, we obtain

We compute

$$I_1 + I_2 + I_3 + I_4.$$
Similarly

\begin{equation}
\int_\Omega qv_1 e^{-\tau \varphi}(ae^{-\varphi} + \overline{a}e^{-\overline{\varphi}}) dx
\end{equation}

\begin{align*}
&= -\frac{1}{4} \int_\Omega q \left\{ e^{-\varphi} \tilde{R}_{\Phi,-\tau}(e_1(\partial_\tau^{-1}(aq_2) - M_2(z))) + e^{-\overline{\varphi}} R_{\Phi,\tau}(e_1(\partial_\tau^{-1}(\overline{a}q_2) - M_4(\overline{z}))) \right\} (ae^{-\varphi} + \overline{a}e^{-\overline{\varphi}}) dx \\
&+ \int_\Omega \left( \frac{e^{-\varphi}}{\tau} e_2(\partial_\tau^{-1}(aq_2) - M_2(z)) + \frac{e^{-\overline{\varphi}}}{\tau} e_2(\partial_\tau^{-1}(\overline{a}q_2) - M_4(\overline{z})) \right) (ae^{-\varphi} + \overline{a}e^{-\overline{\varphi}}) dx \\
&= -\frac{1}{4} \int_\Omega qa \tilde{R}_{\Phi,-\tau}(e_1(\partial_\tau^{-1}(aq_2) - M_2(z))) + q\overline{a} R_{\Phi,\tau}(e_1(\partial_\tau^{-1}(\overline{a}q_2) - M_4(\overline{z}))) dx \\
&- \frac{1}{4} \int_\Omega [q\overline{a}e^{\tau(\Phi - \overline{\Phi})}(\tilde{R}_{\Phi,-\tau}(e_1(\partial_\tau^{-1}(aq_2) - M_2(z))) + qae^{\tau(\Phi - \overline{\Phi})} R_{\Phi,\tau}(e_1(\partial_\tau^{-1}(\overline{a}q_2) - M_4(\overline{z}))) dx \\
&+ \int_\Omega \left( \frac{e^{-\tau(\Phi - \overline{\varphi})}}{\tau} \overline{a}e_2(\partial_\tau^{-1}(aq_2) - M_2(z)) + \frac{e^{\tau(\Phi - \overline{\varphi})}}{\tau} a e_2(\partial_\tau^{-1}(\overline{a}q_2) - M_4(\overline{z})) \right) dx \\
&= J_1 + J_2 + J_3 + J_4.
\end{align*}

By (3.17) and Proposition 2.7, we have

\begin{equation}
J_1 = \frac{1}{4\tau} \int_\Omega e_1 \left( qa \frac{\partial_\tau^{-1}(aq_2) - M_2(z)}{\partial_\Phi} + q\overline{a} \frac{\partial_\tau^{-1}(\overline{a}q_2) - M_4(\overline{z})}{\partial_\Phi} \right) dx + o\left( \frac{1}{\tau} \right).
\end{equation}

The stationary phase argument, (3.17) and Proposition 2.7 yield

\begin{equation}
J_2 = -\frac{1}{4} \int_\Omega [q\overline{a}e^{\tau(\Phi - \overline{\varphi})}\tilde{R}_{\Phi,-\tau}(e_1(\partial_\tau^{-1}(aq_2) - M_2(z))) + qae^{\tau(\Phi - \overline{\varphi})} R_{\Phi,\tau}(e_1(\partial_\tau^{-1}(\overline{a}q_2) - M_4(\overline{z}))) dx = o\left( \frac{1}{\tau} \right).
\end{equation}

By the stationary phase argument and (3.17), we see that

\begin{equation}
J_3 = o\left( \frac{1}{\tau} \right).
\end{equation}

Therefore, applying (4.5), (4.7), (4.11), (4.12), (4.9) and (4.13) in (4.3), we conclude that

\begin{equation}
2 \sum_{k=1}^{\ell} \frac{\pi |q|\alpha^2(\tilde{x}_k) \text{Re} e^{2\pi i \text{Im} \Phi(\tilde{x}_k)}}{|(\det \text{Im} \Phi')(\tilde{x}_k)|^2} + \frac{1}{4} \int_\Omega \left( qa \frac{\partial_\tau^{-1}(aq_2) - M_2(z)}{\partial_\Phi} + q\overline{a} \frac{\partial_\tau^{-1}(\overline{a}q_2) - M_4(\overline{z})}{\partial_\Phi} \right) dx
\end{equation}

\begin{equation}
-\frac{1}{4} \int_\Omega \left( qa \frac{\partial_\tau^{-1}(q_1a) - M_1(z)}{\partial_\Phi} + q\overline{a} \frac{\partial_\tau^{-1}(q_1\overline{a}) - M_3(\overline{z})}{\partial_\Phi} \right) dx = o(1).
\end{equation}

As \( \tau \to +\infty \). Passing to the limit in this equality and applying Bohr’s theorem (e.g., [4], p.393), we finish the proof of the proposition.
We need the following proposition in the construction of the phase function $\Phi$. Let $\tilde{y}_1, \ldots, \tilde{y}_m \in \Omega$. Denote by $\mathcal{R} = (\mathcal{R}(\tilde{y}_1), \ldots, \mathcal{R}(\tilde{y}_m))$ the following operator:

$$\mathcal{R}(\tilde{y}_k)g = (u(\tilde{y}_k), \partial_z u(\tilde{y}_k), \partial^2_z u(\tilde{y}_k)),$$

where

$$\partial_z u = 0 \text{ in } \Omega, \quad \text{Im } u|_{\Gamma_0} = 0, \quad \text{Im } u|_{\Gamma} = g.$$  

We have

**Proposition 4.2.** The operator $\mathcal{R} : C^\infty_0(\tilde{\Gamma}) \to \mathbb{C}^{3m}$ satisfies $\text{Im } \mathcal{R} = \mathbb{C}^{3m}$.

**Proof.** We note that $\text{Im } \mathcal{R} = \mathbb{C}^{3m}$ if and only if the closure of $\text{Im } \mathcal{R}$ is equal to $\mathbb{C}^{3m}$. Our proof is by contradiction. Assume that

$$\text{Im } \mathcal{R} \neq \mathbb{C}^{3m},$$

then there exists a nonzero vector

$$\check{A} = (A_0^1, A_1^1, A_2^1, \ldots, A_0^m, A_1^m, A_2^m) \in \mathbb{C}^{3m} \in (\text{Im } \mathcal{R})^\perp.$$

It is known that the problem (4.15) for a fixed $g$ has solution if and only if

$$\int_{\Gamma_0} g w dz = 0$$

for all $w$ such that

$$\partial_z w = 0 \text{ in } \Omega, \quad w|_{\partial \Omega} = \overline{z'(s)} \gamma(s).$$

Here $\gamma(s)$ is a real-valued function, $z(s)$ is the parametrization of $\partial \Omega$.

Let the function $p$ be a solution to the boundary value problem

$$\partial_z p = \sum_{k=1}^m (A_0^k \delta(x - \tilde{y}_k) - A_1^k \partial_z \delta(x - \tilde{y}_k) + A_2^k \partial^2_z \delta(x - \tilde{y}_k)),$$

$$\text{Re}[iz'(s)p]|_{\partial \Omega} = 0.$$  

By (4.16) a solution to (4.18), (4.19) exists. Let the function $u$ be a solution to problem (4.15). Since

$$\int_{\partial \Omega} pg dz = (\mathcal{R} g, \check{A}) = 0,$$

using the boundary condition (4.19) and Holmgren’s theorem, we have $p = \tilde{p} + w$ with $w$ solving (4.17) and $\tilde{p}$ such that supp $\tilde{p} \subset \{\tilde{y}_1, \ldots, \tilde{y}_m\}$. Since $p$ is a distribution we have that that $p = \sum_{k=1}^m \sum_{|\alpha| \leq j(k)} C_{k,\alpha} D^\alpha \delta(x - \tilde{y}_k)$.

This implies that (4.18) is possible only if $\check{A} = 0$ which is a contradiction. \qed

**End of proof of Theorem 1.1**
Proof. We will construct a complex geometrical optics solution of the form (3.2) where $\Phi$ and $a$ satisfy (2.1), (2.2), (3.3) and (3.4). Let $a(z)$ be a solution to the Riemann-Hilbert problem

$$\partial_z a = 0 \quad \text{in } \Omega, \quad \text{Re } a|_{\Gamma_0} = 0$$

which is not identically zero in $\Omega$. Let $\tilde{x}$ be an arbitrary point from $\Omega$ such that $a(\tilde{x}) \neq 0$.

Next we construct a holomorphic function $\Phi$ such that $\tilde{x} \in \mathcal{G} \equiv \{ x \in \overline{\Omega} | \partial_z \Phi(x) = 0 \}$, $\text{Im } \Phi(\tilde{x}) \neq \text{Im } \Phi(x)$ if $x \in \mathcal{G}$ and $x \neq \tilde{x}$.

Now we construct the function $\Phi$. Let $\tilde{\Omega}$ be a bounded domain in $\mathbb{R}^2$ such that $\overline{\Omega} \subset \tilde{\Omega}$, $\Gamma_0 \subset \partial \tilde{\Omega}$, $\partial \tilde{\Omega} \cap (\partial \tilde{\Omega} \setminus \Gamma_0) = \emptyset$. By Proposition 4.2 there exists a holomorphic function $u$ in $\tilde{\Omega}$ such that $\text{Im } u|_{\Gamma_0} = 0$ and $u(\tilde{x}) = \partial_z u(\tilde{x}) = 0$, and $\partial^2_{zz} u(\tilde{x}) \neq 0$. In general the function $u$ may have critical points on the boundary. Let $\Gamma_* \subset \partial \tilde{\Omega} \setminus \Gamma_0$ such that $u$ does not have any critical points on $\Gamma_*$. We will construct a complex geometrical optics solution of the form (3.2) where $\Phi = \hat{\Phi}$.

Proof. Let $\hat{\Omega}$ be a bounded domain in $\mathbb{R}^2$ such that $\overline{\Omega} \subset \hat{\Omega}$, $\Gamma_0 \subset \partial \hat{\Omega}$, $\partial \hat{\Omega} \cap (\partial \hat{\Omega} \setminus \Gamma_0) = \emptyset$. By Proposition 4.2 there exists a holomorphic function $u$ in $\hat{\Omega}$ such that $\text{Im } u|_{\Gamma_0} = 0$ and $u(\tilde{x}) = \partial_z u(\tilde{x}) = 0$, and $\partial^2_{zz} u(\tilde{x}) \neq 0$. In general the function $u$ may have critical points on the boundary. Let $\Gamma_* \subset \partial \tilde{\Omega} \setminus \Gamma_0$ such that $u$ does not have any critical points on $\Gamma_*$. We will construct a complex geometrical optics solution of the form (3.2) where $\Phi = \hat{\Phi}$.

Next we construct a holomorphic function $u + \epsilon p$ does not have a critical points on $\partial \tilde{\Omega}$ for all sufficiently small positive $\epsilon$ and $\text{Re } p|_{\Gamma_0} = 0$. In order to do this we use Proposition 5.1 proven in the Appendix. We set up appropriate Cauchy data for the Cauchy-Riemann equations (5.1) (see Appendix). On $\Gamma_0$ we set $\text{Re } p|_{\Gamma_0} = 0$ and $\frac{\partial p}{\partial \nu} = \frac{\partial \text{Im } p}{\partial \nu} < 0$ on $\Gamma_0$. Let $p$ be a holomorphic function in $\hat{\Omega}$ such that $\text{Re } p|_{\Gamma_0} = 0$ and $\frac{\partial p}{\partial \nu} = \frac{\partial \text{Im } p}{\partial \nu} < 0$ on $\Gamma_0$. Obviously the function $u + \epsilon p$ does not have any critical points on $\Gamma_0$ for all nonzero $\epsilon$.

On the other hand it might have a critical points on the remaining part of the boundary $\partial \tilde{\Omega} \setminus \Gamma_0$. The number of such a critical points is finite and the function $|\nabla u|^2$ has a zero of finite order at these points. By using a conformal transformation if necessary, we may assume that $\partial \tilde{\Omega} \setminus \Gamma_0$ is a segment on the line $\{x_2 = 0\}$. Let $\{(y_k, 0)\}_{k=1}^N$ be the set of critical points of the function $u$ on the boundary $\Gamma_0$.

We divide the set $\{y_k\}_{k=1}^N$ into two sets $\Omega_1$ and $\Omega_2$. Let us fix some point $y_k$. By Taylor’s formula $\frac{\partial u}{\partial x_1}(x_1, 0) = c_1(x_1 - y_k)^{\kappa_1 + 1} + o((x_1 - y_k)^{\kappa_1 + 1})$ and $\frac{\partial u}{\partial x_2}(x_1, 0) = c_2(x_1 - y_k)^{\kappa_2 + 1} + o((x_1 - y_k)^{\kappa_2 + 1})$ with some $(c_1, c_2) \neq 0$. If $c_2 \neq 0$ and $\kappa_2 \leq \kappa_1$ we say that $y_k \in \Omega_1$. If $c_1 \neq 0$ and $\kappa_2 > \kappa_1$ we say that $y_k \in \Omega_2$.

Let us consider two cases. Let $y_k \in \Omega_1$. Then if $\kappa_2$ is odd we take the Cauchy data for a holomorphic function $p_k$ be such that $\text{Re } p_k = 0$ and $\frac{\partial \text{Im } p}{\partial \nu}$ is positive near $y_k$ if $c_2$ is positive , $\frac{\partial \text{Im } p}{\partial \nu}$ is negative near $y_k$ if $c_2$ is negative and small on $\partial \tilde{\Omega} \setminus \Gamma_*$. If $\kappa_2$ is even and $\kappa_1 \neq \kappa_2$ we take the Cauchy data such that $\frac{\partial \text{Im } p}{\partial \nu}(y_k) - 1$, $\frac{\partial \text{Re } p}{\partial \nu}(y_k) - 1$ are small otherwise $\frac{1}{c_2} \frac{\partial \text{Im } p}{\partial \nu}(y_k) \neq \frac{1}{c_1} \frac{\partial \text{Re } p}{\partial \nu}(y_k)$.

Let $y_k \in \Omega_2$. Then if $\kappa_1$ is odd we take the holomorphic function $p_k$ such that $\frac{\partial \text{Re } p}{\partial \nu}$ is positive near $y_k$ if $c_1$ is positive , $\frac{\partial \text{Re } p}{\partial \nu}$ is negative near $y_k$ if $c_1$ is negative and small on $\partial \tilde{\Omega} \setminus \Gamma_*$. If $\kappa_1$ is even we take $\frac{\partial \text{Re } p}{\partial \nu}(y_k) - 1$, $\frac{\partial \text{Re } p}{\partial \nu}(y_k) - 1$ to be small. Now we have finished the construction of a Cauchy data on $\Gamma_0$ and in a neighborhood $\mathcal{U}$ of the set $\{(y_k, 0)\}_{k=1}^N$. On the part of the boundary $\partial \Omega \setminus (\Gamma_0 \cup \mathcal{U} \cup \Gamma_*)$ we continue $\text{Im } p$, $\text{Re } p$ up to smooth functions. By Proposition 5.1 and general results on a solvability of the boundary problem for $\partial_z$ operator there exists a holomorphic function $p$ which satisfies the above choice of the Cauchy data.

Denote by $\mathcal{H}$ the set of critical points of the function $u + \epsilon p$ on $\overline{\Omega}$. By the implicit function theorem, there exists a neighborhood of $\tilde{x}$ such that for all small $\epsilon$ in this neighborhood the
function \( u + \varepsilon p \) has only one critical point \( \hat{x}(\varepsilon) \), this critical point is nondegenerate and
\[ 4.20 \]
\[ \hat{x}(\varepsilon) \to \hat{x}, \quad \text{as} \quad \varepsilon \to 0. \]

Let us fix a sufficiently small \( \varepsilon \). Let \( \mathcal{H}_\varepsilon = \{ x_k, \varepsilon \}_{1 \leq k \leq N(\varepsilon)} \). By Proposition 4.2, there exists a holomorphic function \( w \) such that
\[ 4.21 \]
\[ w|_{\Gamma_0} = 0, \quad w(x_k, \varepsilon) \neq w(x_j, \varepsilon) \quad \text{for} \quad k \neq j, \quad \partial_z w|_{\mathcal{H}_\varepsilon} = 0, \quad \partial_z^2 w|_{\mathcal{H}_\varepsilon} \neq 0. \]

Denote \( \Phi_\delta = u + \varepsilon p + \delta w \). For all sufficiently small positive \( \delta \)
\[ \mathcal{H}_\varepsilon \subset \mathcal{G}_\delta = \{ x \in \Omega | \partial_z \Phi_\delta(x) = 0 \} \]
and
\[ 4.22 \]
\[ \inf_{\forall y \in \mathcal{H}_\varepsilon, x(\varepsilon) \neq y} |\Phi_\delta(\hat{x}(\varepsilon)) - \Phi_\delta(y)| > \hat{C}(\varepsilon) > 0, \quad C(\varepsilon) = O(\delta). \]

We show now that for all small positive \( \delta \) the critical points of the function \( \Phi_\delta \) are nondegenerate. Let \( \tilde{x} \) be a critical point of the function \( u + \varepsilon p \). If \( \tilde{x} \) is a nondegenerate critical point, by the implicit function theorem, there exists a ball \( B(\tilde{x}, \delta_1) \) such that the function \( \Phi_\delta \) in this ball has only one nondegenerate critical point for all small \( \delta \). Let \( \tilde{x} \) be a degenerate critical point of \( u + \varepsilon p \). Without loss of generality we may assume that \( \tilde{x} = 0 \). In some neighborhood of \( 0 \), we have
\[ \partial_\delta \Phi_\delta = \sum_{k=1}^\infty c_k z^{k+1} - \delta \sum_{k=1}^\infty b_k z^k \]
for some integer positive \( k \), some \( c_1 \neq 0 \) and thanks to (4.21) some \( b_1 \neq 0 \). Let \( z_\delta \in \mathcal{G}_\delta \) and \( z_\delta \to 0 \). Then either
\[ 4.23 \]
\[ z_\delta = 0 \quad \text{or} \quad z_\delta^k = \delta b_1 / c_1 + o(\delta). \]

Therefore \( \partial_z^2 \Phi(z_\delta) \neq 0 \) for all small \( \delta \). Hence we can apply Proposition 4.1 to conclude
\[ \sum_{x \in \mathcal{G}_\delta} q(x)c(x)e^{2\pi i \text{Im} \Phi_\delta(x)} = 0. \]

By (4.1) \( c(\hat{x}(\varepsilon)) \) is not equal to zero. Also we claim that for all small positive \( \delta \)
\[ 4.24 \]
\[ \text{Im} \Phi_\delta(\hat{x}(\varepsilon)) \neq \text{Im} \Phi_\delta(x) \quad \forall x \in \mathcal{G}_\delta \]
such that \( \hat{x}(\varepsilon) \neq x \).

Really, suppose that there exists a sequence \( \tilde{x}_\delta \in \mathcal{G}_\delta \) such that \( \text{Im} \Phi_\delta(\hat{x}(\varepsilon)) = \text{Im} \Phi_\delta(\tilde{x}_\delta) \) as \( \delta \to +0 \). Then taking if it is necessary a subsequence we have that \( \tilde{x}_\delta \to \tilde{x} \in \mathcal{H}_\varepsilon \) and \( \hat{x}(\varepsilon) \neq \tilde{x} \). In that case \( \text{Im} \Phi_\delta(\tilde{x}) = \text{Im} \Phi_\delta(\hat{x}(\varepsilon)) \). On the other hand since,
\[ \Phi_\delta(\hat{x}(\varepsilon)) + \sum_{k=1}^\infty \frac{c_k}{k + k + 1} (z - \tilde{z})^{k+1} - \delta \sum_{k=1}^\infty \frac{b_k}{k + 1} (z - \tilde{z})^{k+1} \]
we have from (4.23)
\[ \Phi_\delta(\hat{x}(\varepsilon)) = \Phi_\delta(\tilde{x}) - \Phi_\delta(\hat{x}(\varepsilon)) = \sum_{k=1}^\infty \frac{c_k}{k + k + 1} (z_\delta - \tilde{z})^{k+1} - \delta \sum_{k=1}^\infty \frac{b_k}{k + 1} (z_\delta - \tilde{z})^{k+1} \]
\[ = \sum_{k=1}^\infty \frac{c_k}{k + k + 1} (z_\delta - \tilde{z})^{k+1}(\delta b_1 / c_1 + o(\delta)) - \delta \sum_{k=1}^\infty \frac{b_k}{k + 1} (z_\delta - \tilde{z})^{k+1} \neq 0 \]
for all sufficiently small positive \( \delta \).

Since the exponents are linearly independent functions of \( \tau \), thanks to (4.24) we have \( q(\hat{x}(\varepsilon)) = 0 \). Thus (4.20) implies \( q(\tilde{x}) = 0 \), finishing the proof.
\[ \square \]
5. Appendix.

Consider the Cauchy problem for the Cauchy-Riemann equations

\[(5.1) \quad L(\phi, \psi) = \left( \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi}{\partial x_1} \right) = 0 \quad \text{in} \quad \Omega, \quad (\phi, \psi) |_{\Gamma_0} = B = (b_1(x), b_2(x)).\]

The following proposition establishes the solvability of (5.1) for a dense set of Cauchy data.

**Proposition 5.1.** There exist a set \( O \subset C^1(\Gamma_0) \) such that for each \( B \in O \) problem (5.1) has at least one solution \((\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) \in C^2(\Omega) \) and \( \overline{O} = C^1(\Gamma_0) \).

**Proof.** Consider the following extremal problem

\[(5.2) \quad J(\phi, \psi) = \|((\phi, \psi) - B)\|^2_{H^2(\Gamma_0)} + \epsilon \|((\phi, \psi))\|^2_{H^2(\partial\Omega)} + \frac{1}{\epsilon} \|\Delta L(\phi, \psi)\|^2_{L^2(\Omega)} \rightarrow \inf,\]

\[(5.3) \quad (\phi, \psi) \in X.\]

Here \( X = \{ \delta(x) = (\delta_1, \delta_2) | \delta \in H^2(\Omega), \Delta L\delta \in L^2(\Omega), L\delta|_{\partial\Omega} = 0, \delta|_{\partial\Omega} \in H^2(\partial\Omega) \}. \)

For each \( \epsilon > 0 \) there exists a unique solution to (5.2), (5.3) which we denote as \((\hat{\phi}_\epsilon, \hat{\psi}_\epsilon)\).

By Fermat’s theorem (see e.g. [1] p. 155) we have

\[J'(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon)[\delta] = 0, \quad \forall \delta \in X.\]

This equality can be written in the form

\[\left( (\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) - B, \delta \right)_{H^2(\Gamma_0)} + \epsilon \left( (\hat{\phi}_\epsilon, \hat{\psi}_\epsilon), \delta \right)_{H^2(\partial\Omega)} + \frac{1}{\epsilon} \left( \Delta L(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon), \Delta L\delta \right)_{L^2(\Omega)} = 0.\]

This equality implies that the sequence \( \{(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon)\} \) is bounded in \( H^2(\Gamma_0) \), the sequence \( \{\epsilon(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon)\} \) converges to zero in \( H^2(\partial\Omega) \) and \( \left\{\frac{1}{\epsilon} \Delta L(\hat{\phi}_\epsilon, \hat{\psi}_\epsilon)\right\} \) is bounded in \( L^2(\Omega) \).

Therefore there exist \( q \in H^2(\Gamma_0) \) and \( p \in L^2(\Omega) \) such that

\[ (\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) - B \rightharpoonup q \quad \text{weakly in} \quad H^2(\Gamma_0) \]

and

\[ (q, \delta)_{H^2(\Gamma_0)} + (p, \Delta L\delta)_{L^2(\Omega)} = 0 \quad \forall \delta \in X.\]

Next we claim that

\[ (5.6) \quad \Delta p = 0 \quad \text{in} \quad \Omega \]

in the sense of distributions. Suppose that (5.6) is already proved. This implies

\[ (p, \Delta L\delta)_{L^2(\Omega)} = 0 \quad \forall \delta \in H^4(\Omega), \quad L\delta|_{\partial\Omega} = \frac{\partial L\delta}{\partial \nu}|_{\partial\Omega} = 0.\]

This equality and (5.5) yield

\[ (5.7) \quad (q, \delta)_{H^2(\Gamma_0)} = 0 \quad \forall \delta \in H^4(\Omega), \quad L\delta|_{\partial\Omega} = \frac{\partial L\delta}{\partial \nu}|_{\partial\Omega} = 0.\]

Then using the trace theorem we conclude that \( q = 0 \) and (5.4) implies that

\[ (\hat{\phi}_\epsilon, \hat{\psi}_\epsilon) - B \rightharpoonup 0 \quad \text{weakly in} \quad H^2(\Gamma_0).\]
By the Sobolev embedding theorem
\[ (\hat{\phi}_{\epsilon_k}, \hat{\psi}_{\epsilon_k}) - B \to 0 \quad \text{in} \quad C^1(\overline{\Gamma_0}). \]
Therefore the sequence \( \{(\hat{\phi}_{\epsilon_k}, \hat{\psi}_{\epsilon_k}) - (\hat{\phi}_{\epsilon_k}, \hat{\psi}_{\epsilon_k})\} \), with
\[ L(\hat{\phi}_{\epsilon_k}, \hat{\psi}_{\epsilon_k}) = L(\hat{\phi}_{\epsilon_k}, \hat{\psi}_{\epsilon_k}) \quad \text{in} \quad \Omega, \quad \hat{\psi}_{\epsilon_k}|_{\Gamma_0} = 0 \]
represents the desired approximation for the solution of the Cauchy problem (5.1).

Now we prove (5.6). Let \( \tilde{x} \) be an arbitrary point in \( \Omega \) and let \( \tilde{\chi} \) be a smooth cut off function such that it is zero in some neighborhood of \( \partial \Omega \setminus \Gamma_0 \) and the set \( \mathcal{B} = \{ x \in \Omega | \tilde{\chi}(x) = 1 \} \) contains an open connected subset \( \mathcal{F} \) such that \( \tilde{x} \in \mathcal{F} \) and \( \Gamma_0 \cap \mathcal{F} \) is an open set in \( \partial \Omega \). By (5.5)
\[ 0 = (p, \Delta L(\tilde{\chi})_2(\Omega)) = (\tilde{\chi} p, \Delta L_2(\Omega)) + (p, [\Delta L, \tilde{\chi}]_2(\Omega)). \]
That is,
\[ (\tilde{\chi} p, \Delta L_2(\Omega)) + ([\Delta L, \tilde{\chi}^*] p, \delta)_2(\Omega) = 0 \quad \forall \delta \in \mathcal{X}. \]
This equality implies that \( \tilde{\chi} p \in H^1(\Omega) \).

Next we take another smooth cut off function \( \chi_1 \) such that \( \text{supp} \chi_1 \subset \mathcal{B} \). A neighborhood of \( \tilde{x} \) belongs to \( \mathcal{B}_1 = \{ x | \chi_1 = 1 \} \), the interior of \( \mathcal{B}_1 \) is connected, and \( \text{Int} \mathcal{B}_1 \cap \mathcal{P}_\epsilon \) contains an open subset \( \mathcal{O} \) in \( \partial \Omega \). Similarly to (5.9) we have
\[ (\tilde{\chi}_1 p, \Delta L_2(\Omega)) + ([\Delta L, \tilde{\chi}_1]^* p, \delta)_2(\Omega) = 0. \]
This equality implies that \( \tilde{\chi}_1 p \in H^2(\Omega) \). Let \( \omega \) be a domain such that \( \omega \cap \Omega = \emptyset, \partial \omega \cap \partial \Omega \subset \mathcal{O} \) contains an open set in \( \partial \Omega \).

We extend \( p \) on \( \omega \) by zero. Then
\[ (\Delta(\tilde{\chi}_1 p), L_2(\Gamma_{\omega, \omega})) + ([\Delta L, \tilde{\chi}^*] p, \delta)_2(\Omega, \omega) = 0. \]
Hence
\[ L^* \Delta(\tilde{\chi}_1 p) = 0 \quad \text{in} \quad \text{Int} \mathcal{B}_1 \cup \omega, \quad p|_{\omega} = 0. \]
By Holmgren’s theorem \( \Delta(\tilde{\chi}_1 p)|_{\text{Int} \mathcal{B}_1} = 0 \), that is, \( (\Delta p)(\tilde{x}) = 0 \). \( \square \)

Now we prove a Carleman estimate for the Laplace operator.

**Proposition 5.2.** Suppose that \( \Phi \) satisfies (2.1), (2.2). Let \( u \in H^1_0(\Omega) \) is a real valued function. Then we have:
\begin{align*}
\tau \| u e^{\tau \varphi} \|^2_{L^2(\Omega)} + \| u e^{\tau \varphi} \|^2_{H^1(\Omega)} + \| \frac{\partial u}{\partial \nu} e^{\tau \varphi} \|^2_{L^2(\Gamma_0)} + \tau^2 \| \frac{\partial \Phi}{\partial z} |ue^{\tau \varphi}|^2_{L^2(\Omega)} \\
\leq C(\| (\Delta u) e^{\tau \varphi} \|^2_{L^2(\Omega)} + \tau \int_{\Gamma} |\frac{\partial u}{\partial \nu}|^2 e^{2\tau \varphi} d\sigma).
\end{align*}

**Proof.** Denote \( \tilde{v} = u e^{\tau \varphi} \), \( \Delta u = f \) and
\[ \Omega_+ = \{ x \in \partial \Omega | (\nabla \varphi, \nu) > 0 \}, \quad \Omega_- = \{ x \in \partial \Omega | (\nabla \varphi, \nu) < 0 \}. \]
Observe that \( \Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \) and \( \varphi(x_1, x_2) = \frac{1}{2}(\Phi(z) + \bar{\Phi}(\bar{z})) \). Therefore
\[ e^{\tau \varphi} \Delta e^{-\tau \varphi} \tilde{v} = (2 \frac{\partial}{\partial z} - \tau \frac{\partial \Phi}{\partial z})(\frac{\partial}{\partial \bar{z}} - \tau \frac{\partial \bar{\Phi}}{\partial \bar{z}}) \tilde{v} = (2 \frac{\partial}{\partial z} - \tau \frac{\partial \Phi}{\partial z})(\frac{\partial}{\partial \bar{z}} - \tau \frac{\partial \bar{\Phi}}{\partial \bar{z}}) \tilde{v} = fe^{\tau \varphi}. \]
Denote \( \hat{w}_1 = \overline{Q(z)}(2 \frac{\partial}{\partial x} - \tau \frac{\partial}{\partial z}) \hat{v}, \hat{w}_2 = Q(z)(2 \frac{\partial}{\partial x} - \tau \frac{\partial}{\partial z}) \hat{v} \), \( \hat{v}_1(x_1, x_2) + i \hat{v}_2(x_1, x_2) \). \( Q(z) \) is some holomorphic function in \( \Omega \) which does not have zeros in \( \overline{\Omega} \). Thanks to the zero Dirichlet boundary condition for \( u \) we have

\[
\hat{w}_1|_{\partial \Omega} = 2\overline{Q(z)}\partial_z \hat{v}|_{\partial \Omega} = (\nu_1 + i\nu_2)Q(z)\frac{\partial \hat{v}}{\partial \nu}|_{\partial \Omega}, \quad \hat{w}_2|_{\partial \Omega} = 2Q(z)\partial_x \hat{v}|_{\partial \Omega} = (\nu_1 - i\nu_2)Q(z)\frac{\partial \hat{v}}{\partial \nu}|_{\partial \Omega}.
\]

By Proposition 2.6

\[
\|\left( \frac{\partial}{\partial x_1} - i\nu_2 \tau \right) \hat{w}_1 \|^2_{L^2(\Omega)} - \tau \int_{\partial \Omega} (\nabla \varphi, \nu) |Q|^2 \left| \frac{\partial \hat{v}}{\partial \nu} \right|^2 \mathrm{d} \sigma + \Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}) \hat{w}_1) \overline{\hat{w}_1} \mathrm{d} \sigma + \|\left( \frac{\partial}{\partial x_2} + \psi_1 \tau \right) \hat{w}_2 \|^2_{L^2(\Omega)} = \|Qfe^{\nu \varphi}\|^2_{L^2(\Omega)}.
\]

and

\[
\|\left( \frac{\partial}{\partial x_1} + i\nu_2 \tau \right) \hat{w}_2 \|^2_{L^2(\Omega)} - \tau \int_{\partial \Omega} (\nabla \varphi, \nu) |Q|^2 \left| \frac{\partial \hat{v}}{\partial \nu} \right|^2 \mathrm{d} \sigma + \Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}) \hat{w}_2) \overline{\hat{w}_2} \mathrm{d} \sigma + \|\left( \frac{\partial}{\partial x_2} - \psi_1 \tau \right) \hat{w}_2 \|^2_{L^2(\Omega)} = \|Qfe^{\nu \varphi}\|^2_{L^2(\Omega)}.
\]

Let us simplify the integral \( \Re i \int_{\partial \Omega} ((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}) \hat{w}_1) \overline{\hat{w}_1} \mathrm{d} \sigma \). We recall that \( \hat{v} = ue^{\tau \varphi} \) and \( \hat{w}_1 = Q(z)(\nu_1 + i\nu_2)\frac{\partial \hat{v}}{\partial \nu} - e^{\tau \varphi} \). Denote \( A + iB = \overline{Q(z)}(\nu_1 + i\nu_2) \). We get

\[
\Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}) \hat{w}_1) \overline{\hat{w}_1} \mathrm{d} \sigma = \Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} - 2\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2}) (A + iB) \frac{\partial \hat{v}}{\partial \nu} e^{\tau \varphi})(A - iB) \frac{\partial \hat{v}}{\partial \nu} e^{-\tau \varphi} \mathrm{d} \sigma = \]

\[
\Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} - 2\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2}) (A + iB) \frac{\partial \hat{v}}{\partial \nu} e^{\tau \varphi})(A - iB) \frac{\partial \hat{v}}{\partial \nu} e^{-\tau \varphi} \mathrm{d} \sigma =
\]

\[
\int_{\partial \Omega} (\partial_x \partial_y - \partial_y \partial_x) \frac{\partial \hat{v}}{\partial \nu} \frac{\partial \hat{v}}{\partial \nu} \mathrm{d} \sigma.
\]

Now we simplify the integral \( \Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}) \hat{w}_2) \overline{\hat{w}_2} \mathrm{d} \sigma \). We recall that \( \hat{v} = ue^{\tau \varphi} \) and \( \hat{w}_2 = (\nu_1 - i\nu_2)Q(z)\frac{\partial \hat{v}}{\partial \nu} = 2(\nu_1 - i\nu_2)Q(z)\frac{\partial \hat{v}}{\partial \nu} e^{\tau \varphi} \). A straightforward computation gives

\[
\Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2}) \hat{w}_2) \overline{\hat{w}_2} \mathrm{d} \sigma =
\]

\[
\Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2}) \hat{w}_2) \overline{\hat{w}_2} \mathrm{d} \sigma =
\]

\[
\Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2}) \hat{w}_2) \overline{\hat{w}_2} \mathrm{d} \sigma =
\]

\[
\Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2}) \hat{w}_2) \overline{\hat{w}_2} \mathrm{d} \sigma =
\]

\[
\Re \int_{\partial \Omega} i((\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2}) \hat{w}_2) \overline{\hat{w}_2} \mathrm{d} \sigma =
\]

\[
\int_{\partial \Omega} (\partial_x \partial_y - \partial_y \partial_x) \frac{\partial \hat{v}}{\partial \nu} \frac{\partial \hat{v}}{\partial \nu} \mathrm{d} \sigma.
\]
Using the above formula we obtain

\[
\|(\frac{\partial}{\partial x_1} + i\psi_2\tau)\tilde{w}_2\|_{L^2(\Omega)}^2 + \|(i\frac{\partial}{\partial x_2} - \psi_1\tau)\tilde{w}_1\|_{L^2(\Omega)}^2 - 2\tau \int_{\partial\Omega} (\nu, \nabla \varphi) |Q|^2 |\frac{\partial \tilde{v}}{\partial \nu}|^2 d\sigma
\]

\[
\|(i\frac{\partial}{\partial x_1} - i\psi_2\tau)\tilde{w}_1\|_{L^2(\Omega)}^2 + \|(i\frac{\partial}{\partial x_2} + \psi_1\tau)\tilde{w}_2\|_{L^2(\Omega)}^2
\]

\[
(5.12)
\]

\[
+ 2 \int_{\partial\Omega} (\partial_x AB - \partial_y BA) \frac{\partial \tilde{v}}{\partial \nu}^2 d\sigma = 2 |Qf e^{\tau \varphi}|_{L^2(\Omega)}^2.
\]

Let \( \tilde{\psi}_k \) be functions such that

\[
\frac{\partial \tilde{\psi}_1}{\partial x_1} = \psi_2, \quad \frac{\partial \tilde{\psi}_2}{\partial x_2} = \psi_1 \quad \text{in } \Omega.
\]

We can rewrite equality (5.12) in the form

\[
\left\| \frac{\partial}{\partial x_1} (e^{i\psi_1\tau} \tilde{w}_1) \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial}{\partial x_2} (e^{i\psi_2\tau} \tilde{w}_2) \right\|_{L^2(\Omega)}^2 - 2\tau \int_{\partial\Omega} (\nu, \nabla \varphi) |Q|^2 |\frac{\partial \tilde{v}}{\partial \nu}|^2 d\sigma
\]

\[
\left\| \frac{\partial}{\partial x_1} (e^{-i\psi_1\tau} \tilde{w}_1) \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial}{\partial x_2} (e^{i\psi_2\tau} \tilde{w}_2) \right\|_{L^2(\Omega)}^2
\]

\[
+ 2 \int_{\partial\Omega} (\partial_x AB - \partial_y BA) \frac{\partial \tilde{v}}{\partial \nu}^2 d\sigma = 2 |Qf e^{\tau \varphi}|_{L^2(\Omega)}^2.
\]

(5.13)

Observe that there exists some positive constant \( C \), independent of \( \tau \), such that

\[
\frac{1}{C} \left( \| \tilde{w}_1 \|_{L^2(\Omega)}^2 + \| \tilde{w}_2 \|_{L^2(\Omega)}^2 \right) \leq \frac{1}{2} \left\| \frac{\partial}{\partial x_1} (e^{i\psi_1\tau} \tilde{w}_1) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{\partial}{\partial x_2} (e^{i\psi_2\tau} \tilde{w}_2) \right\|_{L^2(\Omega)}^2
\]

\[
- \tau \int_{\partial\Omega} (\nu, \nabla \varphi) |Q|^2 |\frac{\partial \tilde{v}}{\partial \nu}|^2 d\sigma
\]

\[
\frac{1}{2} \left\| \frac{\partial}{\partial x_1} (e^{-i\psi_1\tau} \tilde{w}_1) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{\partial}{\partial x_2} (e^{i\psi_2\tau} \tilde{w}_2) \right\|_{L^2(\Omega)}^2.
\]

(5.14)

Since \( \tilde{v} \) is a real-valued function we have

\[
\left\| 2 \frac{\partial \tilde{v}}{\partial x_1} + \tau \psi_1 \tilde{v} \right\|_{L^2(\Omega)}^2 + \left\| 2 \frac{\partial \tilde{v}}{\partial x_2} - \tau \psi_2 \tilde{v} \right\|_{L^2(\Omega)}^2 \leq C_0 (\| \tilde{w}_1 \|_{L^2(\Omega)}^2 + \| \tilde{w}_2 \|_{L^2(\Omega)}^2).
\]

Therefore

\[
4 \left\| \frac{\partial \tilde{v}}{\partial x_1} \right\|_{L^2(\Omega)}^2 - 2\tau \int_{\Omega} (\frac{\partial \psi_1}{\partial x_1} - \frac{\partial \psi_2}{\partial x_2}) \tilde{v}^2 d\Omega
\]

\[
(5.15) + \| \tau \psi_1 \tilde{v} \|_{L^2(\Omega)}^2 + 4 \left\| \frac{\partial \tilde{v}}{\partial x_2} \right\|_{L^2(\Omega)}^2 + \| \tau \psi_2 \tilde{v} \|_{L^2(\Omega)}^2 \leq C_1 (\| \tilde{w}_1 \|_{L^2(\Omega)}^2 + \| \tilde{w}_2 \|_{L^2(\Omega)}^2).
\]

By the Cauchy-Riemann equations the second integral is zero.

Now since by assumption (2.2) the function \( \Phi \) has zeros of at most rder two we have

\[
(5.16)
\]

\[
\tau \| \tilde{v} \|_{L^2(\Omega)}^2 \leq C (\| \tilde{v} \|_{H^2(\Omega)}^2 + \| \tilde{v} \|_{L^2(\Omega)}^2).
\]

By (5.15), (5.16)

\[
(5.17)
\]

\[
\tau \| \tilde{v} \|_{L^2(\Omega)}^2 + \| \tilde{v} \|_{H^2(\Omega)}^2 + \tau^2 \left\| \frac{\partial \Phi}{\partial z} \right\|_{L^2(\Omega)}^2 \leq C_1 (\| \tilde{w}_1 \|_{L^2(\Omega)}^2 + \| \tilde{w}_2 \|_{L^2(\Omega)}^2).
\]
By (5.17) we obtain from (5.13), (5.14)

\[
\frac{1}{C_5} \left( \tau \| \tilde{v} \|_{L^2(\Omega)} + \| \tilde{v} \|_{H^1(\Omega)}^2 + \tau^2 \| \frac{\partial \Phi}{\partial z} \|_{L^2(\Omega)}^2 \right) - \tau \int_{\partial \Omega} (\nu, \nabla \varphi) \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma
\]

\[
+ \int_{\partial \Omega} 2(\partial_{\nu} AB - \partial_{\nu} BA) \left| \frac{\partial \tilde{v}}{\partial \nu} \right|^2 d\sigma \leq \| \text{fe}_{\tau \varphi} \|_{L^2(\Omega)}^2 + \tau \int_{\tilde{\Gamma}} \left| (\nu, \nabla \varphi) \left| \frac{\partial \tilde{v}}{\partial \nu} \right| \right|_{L^2(\Omega)}^2 d\sigma.
\]

Using Proposition 5.1 we make a choice of \( Q(z) \) such that \((\partial_{\nu} AB - \partial_{\nu} BA)\) is positive on \( \tilde{\Gamma}_0 \).

This concludes the proof of the Proposition. \( \Box \)

References

PARTIAL DIRICHLET-TO-NEUMANN MAP


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