Two dimensional compact simple Riemannian manifolds are boundary distance rigid

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Abstract

We prove that knowing the lengths of geodesics joining points of the boundary of a two-dimensional, compact, simple Riemannian manifold with boundary, we can determine uniquely the Riemannian metric up to the natural obstruction.

1 Introduction and statement of the results

Let (M, g) be a compact Riemannian manifold with boundary ∂M. Let $d_g(x, y)$ denote the geodesic distance between x and y. The inverse problem we address in this paper is whether we can determine the Riemannian metric g knowing $d_g(x, y)$ for any $x \in \partial M$, $y \in \partial M$. This problem arose in rigidity questions in Riemannian geometry [M], [C], [Gr]. For the case in which M is a bounded domain of Euclidean space and the metric is conformal to the Euclidean one, this problem is known as the inverse kinematic problem which arose in Geophysics and has a long history (see for instance [R] and the references cited there).

The metric g cannot be determined from this information alone. We have $d_{\psi^* g} = d_g$ for any diffeomorphism $\psi : M \to M$ that leaves the boundary pointwise fixed, i.e., $\psi|_{\partial M} = Id$, where Id denotes the identity map and $\psi^* g$ is the pull-back of the metric g. The natural

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question is whether this is the only obstruction to unique identifiability of the metric. It is easy to see that this is not the case. Namely one can construct a metric g and find a point x_0 in M so that $d_g(x_0, \partial M) > \sup_{x,y \in \partial M} d_g(x, y)$. For such a metric, d_g is independent of a change of g in a neighborhood of x_0. The hemisphere of the round sphere is another example.

Therefore it is necessary to impose some a-priori restrictions on the metric. One such restriction is to assume that the Riemannian manifold is **simple**. A compact Riemannian manifold (M, g) with boundary is simple if it is is simply-connected, any geodesic has no conjugate points and ∂M is strictly convex, that is, the second fundamental form of the boundary is positive definite in every boundary point. Any two points of a simple manifold can be joined by a unique geodesic.

R. Michel conjectured in [M] that simple manifolds are boundary distance rigid that is d_g determines g uniquely up to an isometry which is the identity on the boundary. This is known for simple subspaces of Euclidean space (see [Gr]), simple subspaces of an open hemisphere in two dimensions (see [M]), simple subspaces of symmetric spaces of constant negative curvature [BCG], simple two dimensional spaces of negative curvature (see [C1] or [O]).

In this paper we prove that simple two dimensional compact Riemannian manifolds are boundary distance rigid. More precisely we show

Theorem 1.1. *Let $(M, g_i), i = 1, 2$ be two dimensional simple compact Riemannian manifolds with boundary. Assume*

$$d_{g_1}(x, y) = d_{g_2}(x, y) \quad \forall (x, y) \in \partial M \times \partial M.$$

Then there exists a diffeomorphism $\psi : M \to M$, $\psi|_{\partial M} = Id$, so that

$$g_2 = \psi^* g_1.$$

As it has been shown in [Sh], Theorem 1.1 follows from

Theorem 1.2. *Let $(M, g_i), i = 1, 2$ be two dimensional simple compact Riemannian manifolds with boundary. Assume*

$$d_{g_1}(x, y) = d_{g_2}(x, y) \quad \forall (x, y) \in \partial M \times \partial M$$

and $g_1(x) = g_2(x) \quad \forall x \in \partial M$. Then there exists a diffeomorphism $\psi : M \to M$, $\psi|_{\partial M} = Id$, so that

$$g_2 = \psi^* g_1.$$

We will prove Theorem 1.2. The function d_g measures the travel times of geodesics joining points of the boundary. In the case that both g_1 and g_2 are conformal to the Euclidean metric e (i.e., $(g_k)_{ij} = \alpha_k \delta_{ij}$, $k = 1, 2$ with δ_{ij} the Krönecker symbol), as mentioned earlier, the problem we are considering here is known in seismology as the inverse kinematic problem. In this case, it has been proven by Mukhometov in two dimensions [Mu] that if $(M, g_i), i = 1, 2$ are simple and $d_{g_1} = d_{g_2}$, then $g_1 = g_2$. More generally the same method of proof shows that if $(M, g_i), i = 1, 2$ are simple compact Riemannian manifolds with boundary and they are in the same conformal class, i.e. $g_1 = \alpha g_2$ for a positive function α and $d_{g_1} = d_{g_2}$ then $g_1 = g_2$ [Mu1]. In this case the diffeomorphism ψ must be the identity. For related results and generalizations see [B], [BG], [C], [GN], [MR].

We mention a closely related inverse problem. Suppose we have a Riemannian metric in Euclidean space which is the Euclidean metric outside a compact set. The inverse scattering problem for metrics is to determine the Riemannian metric by measuring the scattering operator (see [G]). A similar obstruction occurs in this case with ψ equal to the identity outside a compact set. It was proven in [G] that from the wave front set of the scattering operator one can determine, under some non-trapping assumptions on the metric, the **scattering relation** on the boundary of a large ball. We proceed to define in more detail the scattering relation and its relation with the boundary distance function.

Let ν denote the unit-inner normal to ∂M. We denote by $\Omega(M) \to M$ the unit-sphere bundle over M:

$$\Omega(M) = \bigcup_{x \in M} \Omega_x, \quad \Omega_x = \{\xi \in T_x(M) : |\xi|_g = 1\}.$$

$\Omega(M)$ is a $(2 \dim M - 1)$-dimensional compact manifold with boundary, which can be written as the union $\partial\Omega(M) = \partial_+\Omega(M) \cup \partial_-\Omega(M)$

$$\partial_\pm\Omega(M) = \{(x, \xi) \in \partial\Omega(M),\ \pm(\nu(x), \xi) \geq 0\}.$$

The manifold of inner vectors $\partial_+\Omega(M)$ and outer vectors $\partial_-\Omega(M)$ intersect at the set of tangent vectors

$$\partial_0\Omega(M) = \{(x, \xi) \in \partial\Omega(M), \quad (\nu(x), \xi) = 0\}.$$

Let (M, g) be an n-dimensional compact manifold with boundary. We say that (M, g) is **non-trapping** if each maximal geodesic is finite. Let (M, g) be non-trapping and the boundary ∂M strictly convex. Denote by $\tau(x, \xi)$ the length of the geodesic $\gamma(x, \xi, t), t \geq 0$, starting at the point x in the direction $\xi \in \Omega_x$. This function is smooth on $\Omega(M) \setminus \partial_0\Omega(M)$. The function $\tau^0 = \tau|_{\partial\Omega(M)}$ is equal zero on $\partial_-\Omega(M)$ and is smooth on $\partial_+\Omega(M)$. Its odd part with respect to ξ

$$\tau_-^0(x, \xi) = \frac{1}{2}\left(\tau^0(x, \xi) - \tau^0(x, -\xi)\right)$$

is a smooth function.

Definition 1.1. *Let (M, g) be non-trapping with strictly convex boundary. The scattering relation $\alpha : \partial\Omega(M) \to \partial\Omega(M)$ is defined by*

$$\alpha(x, \xi) = (\gamma(x, \xi, 2\tau_-^0(x, \xi)), \dot{\gamma}(x, \xi, 2\tau_-^0(x, \xi))).$$

The scattering relation is a diffeomorphism $\partial\Omega(M) \to \partial\Omega(M)$. Notice that $\alpha|_{\partial_+\Omega(M)}$: $\partial_+\Omega(M) \to \partial_-\Omega(M)$, $\alpha|_{\partial_-\Omega(M)} : \partial_-\Omega(M) \to \partial_+\Omega(M)$ are diffeomorphisms as well. Obviously, α is an involution, $\alpha^2 = id$ and $\partial_0\Omega(M)$ is the hypersurface of its fixed points, $\alpha(x,\xi) = (x,\xi),\ (x,\xi) \in \partial_0\Omega(M)$.

A natural inverse problem is whether the scattering relation determines the metric g up to an isometry which is the identity on the boundary. In the case that (M,g) is a simple manifold, and we know the metric at the boundary, knowing the scattering relation is equivalent to knowing the boundary distance function ([M]). We show in this paper that if we know the scattering relation we can determine the Dirichlet-to-Neumann (DN) map associated to the Laplace-Beltrami operator of the metric. We proceed to define the DN map.

Let (M,g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric g is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right)$$

where (g^{ij}) is the inverse of the metric g. Let us consider the Dirichlet problem

$$\Delta_g u = 0 \text{ on } M, \quad u\Big|_{\partial M} = f.$$

We define the DN map in this case by

$$\Lambda_g(f) = (\nu, \nabla u|_{\partial M})$$

The inverse problem is to recover g from Λ_g.

In the two dimensional case the Laplace-Beltrami operator is conformally invariant. More precisely

$$\Delta_{\beta g} = \frac{1}{\beta}\Delta_g$$

for any function β, $\beta \neq 0$. Therefore we have that for $n = 2$

$$\Lambda_{\beta(\psi^* g)} = \Lambda_g$$

for any non-zero β satisfying $\beta|_{\partial M} = 1$.

Therefore the best that one can do in two dimensions is to show that we can determine the conformal class of the metric g up to an isometry which is the identity on the boundary That this is the case is a result proven in [LeU] for simple metrics and for general connected two dimensional Riemannian manifolds with boundary in [LaU].

In this paper we prove:

Theorem 1.3. *Let $(M, g_i), i = 1,2$ be compact, simple two dimensional Riemannian manifolds with boundary. Assume that $\alpha_{g_1} = \alpha_{g_2}$. Then $\Lambda_{g_1} = \Lambda_{g_2}$.*

The proof of Theorem 1.2 is reduced then to the proof of Theorem 1.3. In fact from Theorem 1.3 and the result of [LaU] we get that we can determine the conformal class of the metric up to an isometry which is the identity on the boundary. Now by Mukhometov's result we have that the conformal factor must be one proving that the metrics are isometric via a diffeomorphism which is the identity at the boundary. In other words $d_{g_1} = d_{g_2}$ implies that $\alpha_{g_1} = \alpha_{g_2}$. By Theorem 1.3 $\Lambda_{g_1} = \Lambda_{g_2}$. By the result of [LeU], [LaU], there exists a diffeomorphism $\psi : M \longrightarrow M$, $\psi|_{\partial M} =$ Identity and a function $\beta \neq 0, \beta|_{\partial M} =$ identity such that $g_1 = \beta\psi^* g_2$. By Mukhometov's theorem $\beta = 1$ showing that $g_1 = \psi^* g_2$ proving Theorem 1.2. and Theorem 1.1.

The proof of Theorem 1.3 consists in showing that from the scattering relation we can determine the traces at the boundary of conjugate harmonic functions, which is equivalent information to knowing the DN map associated to the Laplace-Beltrami operator. The steps to accomplish this are outlined below. It relies in a connection between the Hilbert transform and geodesic flow.

We embed (M, g) into a compact Riemannian manifold (S, g) with no boundary. Let φ_t be the geodesic flow on $\Omega(S)$ and $\mathcal{H} = \frac{d}{dt}\varphi_t|_{t=0}$ be the geodesic vector field. Introduce the map $\psi : \Omega(M) \to \partial_-\Omega(M)$ defined by

$$\psi(x, \xi) = \varphi_{\tau(x,\xi)}(x, \xi), \quad (x, \xi) \in \Omega(M).$$

The solution of the boundary value problem for the transport equation

$$\mathcal{H}u = 0, \quad u|_{\partial_+\Omega(M)} = w$$

can be written in the form

$$u = w_\psi = w \circ \alpha \circ \psi.$$

Let u^f be the solution of the boundary value problem

$$\mathcal{H}u = -f, \quad u|_{\partial_-\Omega(M)} = 0,$$

which we can write as

$$u^f(x, \xi) = \int_0^{\tau(x,\xi)} f(\varphi_t(x, \xi))dt, \quad (x, \xi) \in \Omega(M).$$

In particular

$$\mathcal{H}\tau = -1.$$

The trace

$$If = u^f|_{\partial_+\Omega(M)}$$

is called **the geodesic X-ray transform** of the function f. By the fundamental theorem of calculus we have

$$I\mathcal{H}f = (f \circ \alpha - f)|_{\partial_+\Omega(M)}. \tag{1.1}$$

In what follows we will consider the operator I acting only on functions that do not depend on ξ, unless otherwise indicated. Let $L^2_\mu(\partial_+\Omega(M))$ is the real Hilbert space, with scalar product given by

$$(u,v)_{L^2_\mu(\partial_+\Omega(M))} = \int_{\partial_+\Omega(M)} \mu uv d\Sigma, \quad \mu = (\xi, \nu).$$

Here the measure $d\Sigma = d(\partial M) \wedge d\Omega_x$ where $d(\partial M)$ is the induced volume form on the boundary by the standard measure on M and

$$d\Omega_x = \sqrt{detg} \sum_{k=1}^{n} (-1)^{k+1} \xi^k d\xi^1 \wedge ... \wedge \hat{d\xi^k} \wedge ... d\xi^n.$$

As usual the scalar product in $L^2(M)$ is defined by

$$(u,v) = \int_M uv\sqrt{detg}dx.$$

The operator I is a bounded operator from $L^2(M)$ into $L^2_\mu(\partial_+\Omega(M))$. The adjoint I^* : $L^2_\mu(\partial_+\Omega(M)) \to L^2(M)$ is given by

$$I^* w(x) = \int_{\Omega_x} w_\psi(x,\xi) d\Omega_x,$$

We will study the solvability of equation $I^* w = h$ with smooth right hand side. Let $w \in C^\infty(\partial_+\Omega(M))$. Then the function w_ψ will not be smooth on $\Omega(M)$ in general. We have that $w_\psi \in C^\infty(\Omega(M) \setminus \partial_0\Omega(M))$. We give below necessary and sufficient conditions for smoothness of w_ψ on $\Omega(M)$.

We introduce the operators of even and odd continuation with respect to α:

$$A_\pm w(x,\xi) = w(x,\xi), \qquad (x,\xi) \in \partial_+\Omega(M),$$

$$A_\pm w(x,\xi) = \pm (\alpha^* w)(x,\xi), \quad (x,\xi) \in \partial_-\Omega(M).$$

The scattering relation preserves the measure $|(\xi,\nu)|d\Sigma$ and therefore the operators $A_\pm$: $L^2_\mu(\partial_+\Omega(M)) \to L^2_{|\mu|}(\partial\Omega(M))$ are bounded, where $L^2_{|\mu|}(\partial\Omega(M))$ is real Hilbert space with scalar product

$$(u,v)_{L^2_{|\mu|}(\partial\Omega(M))} = \int_{\partial\Omega(M)} |\mu| uv d\Sigma, \quad \mu = (\xi, \nu).$$

The adjoint of $A_\pm$ is a bounded operator $A^*_\pm : L^2_{|\mu|}(\partial\Omega(M)) \to L^2_\mu(\partial_+\Omega(M))$ given by

$$A^*_\pm u = (u \pm u \circ \alpha)|_{\partial_+\Omega(M)}.$$

Using A^*_- formula (1.1) can be written in the form

$$I\mathcal{H}f = -A^*_- f^0, \quad f^0 = f|_{\partial\Omega(M)}. \tag{1.2}$$

The space $C_{\alpha}^{\infty}\left(\partial_{+}\Omega\left(M\right)\right)$ is defined by

$$C_{\alpha}^{\infty}\left(\partial_{+}\Omega\left(M\right)\right)=\left\{w\in C^{\infty}\left(\partial_{+}\Omega\left(M\right)\right): w_{\psi}\in C^{\infty}\left(\Omega\left(M\right)\right)\right\}.$$

We have the following characterization of the space of smooth solutions of the transport equation

Lemma 1.1.

$$C_{\alpha}^{\infty}(\partial_{+}\Omega(M))=\{w\in C^{\infty}(\partial_{+}\Omega(M)) : A_{+}w\in C^{\infty}(\partial\Omega(M))\}.$$

Now we can state the main theorem for solvability for I^{*}.

Theorem 1.4. *Let (M,g) be a simple, compact two dimensional Riemannian manifold with boundary. Then the operator $I^{*}: C_{\alpha}^{\infty}(\partial_{+}\Omega(M))\to C^{\infty}(M)$ is onto.*

Now we define the Hilbert transform:

$$Hu(x,\xi)=\frac{1}{2\pi}\int_{\Omega_{x}}\frac{1+(\xi,\eta)}{(\xi_{\perp},\eta)}u(x,\eta)d\Omega_{x}(\eta),\quad \xi\in\Omega_{x}, \tag{1.3}$$

where the integral is understood as a principle value integral. Here $\perp$ means a 90^{o} degree rotation. In coordinates $(\xi_{\perp})_{i}=\varepsilon_{ij}\xi^{j}$, where

$$\varepsilon=\sqrt{\det g}\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

The Hilbert transform H transforms even (respectively odd) functions with respect to ξ to even (respectively odd) ones. If H_{+} (respectively H_{-}) is the even (respectively odd) part of the operator H:

$$H_{+}u(x,\xi)=\frac{1}{2\pi}\int_{\Omega_{x}}\frac{(\xi,\eta)}{(\xi_{\perp},\eta)}u(x,\eta)d\Omega_{x}(\eta),$$
$$Hu_{-}(x,\xi)=\frac{1}{2\pi}\int_{\Omega_{x}}\frac{1}{(\xi_{\perp},\eta)}u(x,\eta)d\Omega_{x}(\eta)$$

and u_{+},u_{-} are the even and odd parts of the function u, then $H_{+}u=Hu_{+}, H_{-}u=Hu_{-}$.

We introduce the notation $\mathcal{H}_{\perp}=(\xi_{\perp},\nabla)=-(\xi,\nabla_{\perp})$, where $\nabla_{\perp}=\varepsilon\nabla$ and ∇ is the covariant derivative with respect to the metric g. The following commutator formula for the geodesic vector field and the Hilbert transform is very important in our approach.

Theorem 1.5. *Let (M,g) be a two dimensional Riemannian manifold. For any smooth function u on $\Omega(M)$ we have the identity*

$$[H,\mathcal{H}]u=\mathcal{H}_{\perp}u_{0}+(\mathcal{H}_{\perp}u)_{0} \tag{1.4}$$

where

$$u_{0}(x)=\frac{1}{2\pi}\int_{\Omega_{x}}u(x,\xi)d\Omega_{x}$$

is the average value.

Now we can prove the Theorem 1.3.

Separating the odd and even parts with respect to ξ in (1.4) we obtain the identities:

$$H_+\mathcal{H}u - \mathcal{H}H_-u = (\mathcal{H}_\perp u)_0, \quad H_-\mathcal{H}u - \mathcal{H}H_+u = \mathcal{H}_\perp u_0.$$

Let (M,g) be a non-trapping strictly convex manifold. Take $u = w_\psi, w \in C_\alpha^\infty(\partial_+(\Omega))$. Then

$$2\pi\mathcal{H}H_+w_\psi = -\mathcal{H}_\perp I^*w$$

and using the formula (1.2) we conclude

$$2\pi A_-^* H_+ A_+ w = I\mathcal{H}_\perp I^* w, \tag{1.5}$$

since $w_\psi|_{\partial\Omega(M)} = A_+w$.

Let (h, h_*) be a pair of conjugate harmonic functions on M,

$$\nabla h = \nabla_\perp h_*, \quad \nabla h_* = -\nabla_\perp h.$$

Notice, that $\delta\nabla = \triangle$ is the Laplace-Beltrami operator and $\delta\nabla_\perp = 0$. Let $I^*w = h$. Since $I\mathcal{H}_\perp h = I\mathcal{H}h_* = -A_-^* h_*^0$, where $h_*^0 = h_*|_{\partial M}$, we obtain from (1.5)

$$2\pi A_-^* H_+ A_+ w = -A_-^* h_*^0. \tag{1.6}$$

The following theorem gives the key to obtain the DN map from the scattering relation.

Theorem 1.6. *Let M be a 2-dimensional simple manifold. Let $w \in C_\alpha^\infty(\partial_+\Omega(M))$ and h_* is harmonic continuation of function h_*^0. Then the equation (1.6) holds iff the functions $h = I^*w$ and h_* are conjugate harmonic functions.*

Proof. The necessity has already been established. Using (1.2) and (1.5) the equality (1.6) can be written in the form

$$I\mathcal{H}_\perp h = I\mathcal{H}q,$$

where q is an arbitrary smooth continuation onto M of the function h_*^0 and $h = I^*w$. Thus, the ray transform of the vector field $\nabla q + \nabla_\perp h$ equal 0. Consequently, this field is potential ([An]), that is, $\nabla q + \nabla_\perp h = \nabla p$ and $p|_{\partial M} = 0$. Then the functions h and $h_* = q - p$ are conjugate harmonic functions and $h_*|_{\partial M} = h_*^0$. We have finished the proof of the main theorem.

□

In summary we have the following procedure to obtain the DN map from the scattering relation. For an arbitrary given smooth function h_*^0 on ∂M we find a solution $w \in C_\alpha^\infty(\partial_+\Omega(M))$ of the equation (1.6). Then the functions $h^0 = 2\pi(A_+w)_0$ (notice, that

$2\pi(A_+w)_0 = I^*w|_{\partial M})$ and h_*^0 are the traces of conjugate harmonic functions. It gives the map

$$h_*^0 \to (\nu_\perp, \nabla h^0) = (\nu, \nabla h_*|_{\partial M}).$$

which is the DN map. This proves Theorem 1.3.

A brief outline of the paper is as follows. In section 2 we collect some facts and definition we need later. In section 3 we study the solvability of $I^*w = h$ on Sobolev spaces and prove Theorem 1.4. In section 4 we make a detailed study of the scattering relation and prove Lemma 1.1. In section 5 we prove Theorem 1.5.

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2 Preliminaries and Notation

Here we will give some definitions and formulas, that we need in what follows. For further references see [E], [J], [K], [Sh]. Let $\pi : T(M) \to M$ be the tangent bundle over an n-dimensional Riemannian manifold (M, g). We will denote points of the manifold $T(M)$ by pairs (x, ξ). The connection map $K : T(T(M)) \to T(M)$ is defined by its local representation

$$K(x,\xi,y,\eta) = (x, \eta + \Gamma(x)(y,\xi)), \quad (\Gamma(x)(y,\xi))^i = \Gamma^i_{jk}(x) y^j \xi^k, i = 1, ..., n,$$

where Γ^i_{jk} are the Christoffel symbols of the metric g

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}).$$

The linear map $K(x,\xi) = K|_{(x,\xi)} : T_{(x,\xi)}(T(M)) \to T_x M$ defines the horizontal subspace $H_{(x,\xi)} =$ Ker $K(x,\xi)$. It can be identified with the tangent space $T_x(M)$ using the isomorphism

$$J^h_{(x,\xi)} = (\pi'(x,\xi)|_{H_{(x,\xi)}})^{-1} : T_x(M) \to H_{(x,\xi)}.$$

The vertical space $V_{(x,\xi)} =$ Ker $\pi'(x,\xi)$ can also identified with $T_x(M)$ by using the isomorphism

$$J^v_{(x,\xi)} = (K(x,\xi)|_{V_{(x,\xi)}})^{-1} : T_x(M) \to V_{(x,\xi)}.$$

The tangent space $T_{(x,\xi)}(T(M))$ is the direct sum of the horizontal and vertical subspaces, $T_{(x,\xi)}(T(M)) = H_{(x,\xi)} \oplus V_{(x,\xi)}$. An arbitrary vector $X \in T_{(x,\xi)}(T(M))$ can be uniquely decomposed in the form

$$X = J^h_{(x,\xi)} X_h + J^v_{(x,\xi)} X_v,$$

where

$$X_h = \pi'(x,\xi)X, \quad X_v = K(x,\xi)X.$$

We will call X_h, X_v the horizontal and vertical components of the vector X and use the notation $X = (X_h, X_v)$. If in local coordinates $X = (X^1, ..., X^{2n})$ then X_h, X_v is given by

$$X_h^i = X^i, \quad X_v^i = X^{i+n} + \Gamma^i_{jk}(x)X^j\xi^k, \quad i = 1, ..., n.$$

Let N be a smooth manifold and $f : T(M) \to N$ a smooth map. Then the derivative $f'(x,\xi) : T_{(x,\xi)}(T(M)) \to T_{f(x,\xi)}(N)$ defines the horizontal $\nabla_h f(x,\xi)$ and vertical $\nabla_v f(x,\xi)$ derivatives:

$$\nabla_h f(x,\xi) = f'(x,\xi) \circ J^h_{(x,\xi)} : T_x(M) \to T_{f(x,\xi)}(N),$$
$$\nabla_v f(x,\xi) = f'(x,\xi) \circ J^v_{(x,\xi)} : T_x(M) \to T_{f(x,\xi)}(N).$$

We have that

$$f'(x,\xi)X = (\nabla_h f(x,\xi), X_h) + (\nabla_v f(x,\xi), X_v). \tag{2.1}$$

In local coordinates

$$\nabla_{hj} f^{(\alpha)}(x,\xi) = (\frac{\partial}{\partial x^j} - \Gamma^i_{jk}(x)\xi^k \frac{\partial}{\partial \xi^i}) f^{(\alpha)}(x,\xi),$$

$$\nabla_{vj} f^{(\alpha)}(x,\xi) = \frac{\partial}{\partial \xi^j} f^{(\alpha)}(x,\xi), \quad \alpha = 1, ..., \mathrm{dim}N.$$

We now state the definition of vertical and horizontal derivatives for semibasic tensor fields. We recommend chapter 3 of [Sh] for more details.

Let $T^r_s(M)$ denote the bundle of tensor fields of degree (r,s) on M. A section of this bundle is called a tensor field of degree (r,s). Let $\pi^r_s : T^r_s(M) \to M$ be the projection. A fiber map $u : T(M) \to T^r_s(TM)$, i.e. $\pi^r_s \circ u = \pi$ is called a semibasic tensor field of degree (r,s) on the manifold $T(M)$. Denote by ξ the semibasic vector field given by the identity map $T(M) \to T(M)$. An arbitrary tensor field u of degree (r,s) on the manifold M, i.e. section $u : M \to T^r_s(M)$ defines by the formula $u \circ \pi$ a semibasic tensor field (since $\pi^r_s \circ (u \circ \pi) = (\pi^r_s \circ u) \circ \pi = id \circ \pi = \pi$). The map $u \to u \circ \pi$ identifies tensor fields on M and ξ-constant semibasic tensor fields on $T(M)$. Using the metric g we can identify the bundle $T^r_s(M)$ with $T^{r+s}_0(M)$ and the bundle $T^0_{r+s}(M)$ with $T^*_{r+s}(M)$.

We can define invariantly horizontal ∇u and vertical $\nabla_\xi u$ derivatives of semibasic tensor field u ([Sh]). They are also semibasic tensor fields. In local coordinates the horizontal and vertical derivatives are given by

$$(\nabla u)_{i_1...i_{m+1}} = \tilde{\nabla}_{i_{m+1}} u_{i_1...i_m} - \Gamma^j_{i_{(m+1)}k}\xi^k \frac{\partial u_{i_1...i_m}}{\partial \xi^j}, \quad (\nabla_\xi u)_{i_1...i_{m+1}} = \frac{\partial u_{i_1...i_m}}{\partial \xi^{i_{m+1}}},$$

where $\tilde{\nabla}$ denotes the usual covariant derivative on the manifold (M,g). Notice, that for ξ-constant tensor fields, $\nabla u = \tilde{\nabla} u$ and since we identify ξ-constant semibasic tensor fields with tensor fields on M, we will use one notation ∇ for covariant and horizontal derivatives.

We define tangent derivatives of semibasic tensor fields on the submanifold of the unit sphere $\Omega(M)$ by

$$\nabla_\Omega u = \nabla(u \circ p)|_{\Omega(M)}, \quad \partial u = \nabla_\xi (u \circ p)|_{\Omega(M)},$$

where $p : T(M) \to \Omega(M)$ is the projection $p(x, \xi) = (x, \xi/|\xi|)$. Obviously (ξ, ∂)=0. Since $\nabla_\Omega |\xi| = 0$ we will use the notation ∇ instead of ∇_Ω. In addition we recall the following formulas (see [Sh])

$$\nabla g = 0, \quad \nabla \xi = 0, \quad \partial_j \xi^i = \delta^i_j - \xi^i \xi_j,$$

$$[\nabla, \partial] = 0, \quad [\partial_i, \partial_j] = \xi_i \partial_j - \xi_j \partial_i,$$

$$[\nabla_i, \nabla_j] u = -R^p_{qij} \partial_p u,$$

where R is the curvature tensor. In the last formula u is a scalar.

3 The geodesic X-ray transform

In this section we study the solvability of the equation $I^* w = h$ and prove Theorem 1.4.

Lemma 3.1. *Let V be an open set of a Riemannian manifold (M, g). We can define the ray transform as before. Then the normal operator $I^* I$ is an elliptic pseudodifferential operator of order -1 on V with principal symbol $c_n |\xi|^{-1}$ where c_n is a constant.*

Proof. It is easy to see, that

$$(I^* I f)(x) = \int\limits_{\Omega_x} d\Omega_x \int\limits_{-\tau(x,-\xi)}^{\tau(x,\xi)} f(\gamma(x, \xi, t))\, dt = 2 \int\limits_{\Omega_x} d\Omega_x \int\limits_{0}^{\tau(x,\xi)} f(\gamma(x, \xi, t))\, dt. \tag{3.1}$$

Before we continue we make a remark concerning notation. We have used up to now the notation $\gamma(x, \xi, t)$ for a geodesic. But it is known [J] , that a geodesic depends smoothly on the point x and vector $\xi t \in T_x(M)$. Therefore in what follows we will also use sometimes the notation $\gamma(x, \xi t)$ for a geodesic. Since the manifold M is simple and any small enough neighborhood U (in (S, g)) is also simple (an open domain is simple if its closure is simple). For any point $x \in U$ there is an open domain $D^U_x \subset T_x(U)$ such that the exponential map $exp_x : D^U_x \to U$, $exp_x \eta = \gamma(x, \eta)$ is a diffeomorphism onto U. Let D_x, $x \in M$ be the inverse image of M, then $exp_x(D_x) = M$ and $exp_x|_{D_x} : D_x \to M$ is a diffeomorphism.

Now we change variables in (3.1), $y = \gamma(x, \xi t)$. Then $t = d_g(x, y)$ and

$$(I^* I f)(x) = \int\limits_M K(x, y)\, f(y)\, dy,$$

where

$$K(x, y) = 2 \frac{\det (exp_x^{-1})'(x, y) \sqrt{\det g(x)}}{d_g^{n-1}(x, y)}.$$

Notice, that since

$$\gamma(x,\eta) = x + \eta + O(|\eta|^2), \tag{3.2}$$

it follows, that the Jacobian matrix of the exponential map is 1 at 0, and then $\det(exp_x^{-1\prime})(x,x) = 1/\det\left(exp_x\right)'(x,0) = 1$. ¿From (3.2) we also conclude that

$$d^2(x,y) = G_{ij}(x,y)(x-y)^i(x-y)^j\,,\quad G_{ij}(x,x) = g_{ij}(x)\,,\quad G_{ij} \in C^\infty(M\times M)$$

Therefore the kernel of I^*I can be written in the form

$$K(x,y) = \frac{2\det\left(exp_x^{-1}\right)'(x,y)\sqrt{\det g(x)}}{\left(G_{ij}(x,y)(x-y)^i(x-y)^j\right)^{(n-1)/2}}.$$

Thus the kernel K has at the diagonal $x = y$ a singularity of type $|x-y|^{-n+1}$. The kernel

$$K_0(x,y) = \frac{2\sqrt{\det g(x)}}{\left(g_{ij}(x)(x-y)^i(x-y)^j\right)^{(n-1)/2}}$$

has the same singularity. Clearly, the difference $K - K_0$ has a singularity of type $|x-y|^{-n+2}$. Therefore the principal symbols of both operators coincide. The principal symbol of the integral operator, corresponding to the kernel K_0 coincides with its full symbol and is easily calculated. As a result

$$\sigma(I^*I)(x,\xi) = 2\sqrt{\det g(x)}\int \frac{e^{-i(y,\xi)}}{\left(g_{ij}(x)y^iy^j\right)^{(n-1)/2}}dy = c_n|\xi|^{-1}.$$

□

Let r_M denotes the restriction from S onto M.

Theorem 3.1. *Let U be a simple neighborhood of the simple manifold M. Then for any function $h \in H^s(M)$, $s \geq 0$ there exists function $f \in H^{s-1}(U)$, $r_M I^* I f = h$.*

Proof. Let (M,g) be simple and embedded into a compact Riemannian manifold (S,g) without boundary, of the same dimension. Choose a finite atlas of S, which consist of simple open sets U_k with coordinate maps $\kappa_k : U_k \to R^n$. Let $\{\varphi_k\}$ be the subordinated partition of unity: $\varphi_k \geq 0$, $supp\varphi_k \subset U_k$, $\sum\varphi_k = 1$. We assume without loss of generality that $M \subset U_1$ and $\varphi_1|_M = 1$. We consider the operators I_k, I_k^* for the domain U_k, and the pseudodifferential operator on (S,g)

$$Pf = \sum_k \varphi_k\left(I_k I_k^*\right)\left(f|_{U_k}\right),\quad f \in D'(X).$$

Every operator $I_k I_k^* : C_0^\infty(U_k) \to C^\infty(U_k)$ is an elliptic pseudodifferential operator of order -1 with principal symbol $c_n|\xi|^{-1}$, $\xi \in T(U_k)$. Then P is an elliptic pseudodifferential operator with principal symbol $c_n |\xi|^{-1}$, $\xi \in T(S)$ and, therefore, is a Fredholm operator from $H^s(S)$ into $H^{s+1}(S)$. We have that $\operatorname{Ker} P$ has finite dimension, $\operatorname{Ran} P$ is closed and has finite codimension. Notice, that $P^* = P$ (more precisely if $P^s = P : H^s(S) \to H^{s+1}(S)$, then $(P_s)^* = P_{-s-1}$).

For arbitrary $s \geq 0$ the operator $r_M : H^s(S) \to H^s(M)$ is bounded and $r_M(H^s(S)) = H^s(M)$. Then the range of $r_M P : H^s(S) \to H^{s+1}(M)$, $s \geq -1$ is closed.

Since M is only covered by U_1 and $\varphi_1|_M = 1$ we have that $r_M P f = r_M I_1^* I_1 (f|_{U_1})$. Thus, the range of the operator $r_M I_1^* I_1 : H^s(U_1) \to H^{s+1}(M)$, $s \geq -1$ is closed. Now to prove the solvability of the equation

$$r_M I_1^* I_1 f = h \in H^{s+1}(M), \ s \geq -1$$

in $H^s(U_1)$ is sufficient to show that the kernel of the adjoint $(r_M I_1^* I_1)^* : \left(H^{(s+1)}(M)\right)^* \to (H^s(U_1))^*$ is zero.

Let $\langle , \rangle_M$ and $\langle , \rangle$ be dualities between $H^s(M)$ and $(H^s)^*(M)$ or $H^s(S)$ and $H^{-s}(S)$ respectively. The dual space $(H^s(M))^*$, $s \geq 0$ can be identified with the subspace of $H^{-s}(S)$:

$$(H^s(M))^* = H^{-s}(M) = \left\{u \in H^{-s}(S) : \operatorname{supp} u \subset M\right\}.$$

For any $f \in H^s(U_1)$, $u \in H^{-(1+s)}(M)$ we have

$$\langle r_M I_1^* I_1 f, u\rangle_M = \langle P_s \tilde{f}, u\rangle = \langle \tilde{f}, P_{-s-1} u\rangle,$$

where $\tilde{f}$ is an arbitrary continuation of f on the manifold S. On the other hand

$$\langle r_M I_1^* I_1 f, u\rangle_M = \langle f, (r_M I_1^* I_1)^* u\rangle_M .$$

Since $\tilde{f}$ is arbitrary, then equality $\langle \tilde{f}, P_{-s-1} u\rangle = \langle f, (r_M I_1^* I_1)^* u\rangle_M$ implies $(r_M I_1^* I_1)^* = r_{U_1} P_{-s-1} = r_{U_1} I_1^* I_1$.

Because of ellipticity the equality $r_{U_1} P u = 0$ implies smoothness $u|_{U_1}$, and then $u \in H^{-s-1}(M)$ implies of $u \in C_0^\infty(U_1)$. Since $r_{U_1} P u = I_1^* I_1 u$, then

$$I_1^* I_1 u = 0 \Longrightarrow \|I_1 u\|^2_{L^2_\mu(\partial_+ \Omega(\overline{U_1}))} = 0 \Longrightarrow I_1 u = 0 \Longrightarrow u = 0.$$

□

Now we are ready to prove Theorem 1.4.

Proof. Let I, I_1 be the geodesic X-ray transforms on M and $\overline{U_1}$ respectively. From Theorem 3.1 it follows that for any $h \in C^\infty(M)$ there exists $f \in C^\infty(\overline{U_1})$, such that $r_M I_1^* I_1 f = h$. Then $u^f \in C^\infty(\Omega(U_1))$. Let $w = 2u_+^f|_{\partial_+\Omega(M)}$, where u_+^f is the even part with respect to ξ. Then it easy to see, that $w_\psi = 2u_+^f|_{\Omega(M)}$ and $I^* w = h$. The function $w \in C_\alpha^\infty(\partial_+\Omega(M))$ since $w_\psi \in C^\infty(\Omega(M))$. □

4 Scattering relation and folds

In this section we prove Lemma 1.1.

As indicated before we embed (M,g) into a compact manifold (S,g) with no boundary. Let (N,g) be an arbitrary neighborhood in (S,g) of the manifold (M,g), such that any geodesic $\gamma(x,\xi,t), (x,\xi)\in\Omega(N)$ intersects the boundary ∂N transversally. Then the length of the geodesic ray τ is a smooth function on $\Omega(\dot{N})$ and the map $\phi:\partial\Omega(M)\to\partial_-\Omega(N)$, defined by

$$\phi(x,\xi)=\varphi_{\tau(x,\xi)}(x,\xi),\quad (x,\xi)\in\partial\Omega(M), \tag{4.1}$$

is smooth as well. Moreover it turns out ϕ is a fold map with fold $\partial_0\Omega(M)$. This fact will be proved in the next Theorem. Once this is proven Lemma 1.1 follows from [H] Theorem C.4.4. From the assumption $A_+w\in C^\infty(\partial\Omega(M))$ we deduce the existence of a smooth function v on a neighborhood of the range $\phi(\partial\Omega(M))$ such that $w=v\circ\phi$. Consider the function $w_\psi=w\circ\alpha\circ\psi$. Change notation ψ to ψ_M, keeping w_ψ. Denote by ψ_N the map, analogous to ψ_M,

$$\psi_N\left(x,\xi\right)=\varphi_{\tau(x,\xi)}\left(x,\xi\right),\ \ \left(x,\xi\right)\in\Omega\left(N\right).$$

Then $w_\psi=v\circ\phi\circ\alpha\circ\psi_M$. It easy to see, that $\phi\circ\alpha\circ\psi_M=\psi_N|_{\Omega(M)}$. Since the map ψ_N is smooth on $\Omega\left(M\right)$, then $w_\psi\in C^\infty\left(\Omega\left(M\right)\right)$, i.e. $w\in C^\infty_\alpha(\partial_+\Omega(M))$. Thus Lemma 1.1 is proven once we show that ϕ is a fold.

Theorem 4.1. *Let (M,g) be a strictly convex, non-trapping manifold and N an arbitrary neighborhood of M, such, that any geodesic $\gamma(x,\xi,t),(x,\xi)\in\Omega(\dot{N})$ intersects the boundary ∂N transversally. Then the map ϕ, defined by (4.1) is a fold with fold $\partial_0\Omega(M)$.*

First we recall the definition of a Whitney fold.

Definition 4.1. *Let M,N be C^∞ manifolds of the same dimension and let $f:M\longrightarrow N$ be a C^∞ map with $f(m)=n$. The function f is a Whitney fold (with fold L) at m if f drops rank by one simply at m, so that $\{x;df(x)$ is singular $\}$ is a smooth hypersurface near m and Ker $(df(m))$ is transversal to T_mL.*

Now we prove Theorem 4.1.

Proof. Firstly notice, that $\partial_0\Omega(M)$ is a smooth non-singular hypersurface in $\partial\Omega(M)$. It is given by the equation $f(x,\xi)=(\xi,\nu(x))=0,(x,\xi)\in\partial\Omega(M)$. It is easy to see that the map $f'(x,\xi)$ at any point $(x,\xi)\in\partial_0\Omega(M)$ is non-singular.

If a submanifold Σ of the manifold M is locally given near a point m by equations $h_k(x)=0$, then the vector $X\in T_m(M)$ belongs to $T_m(\Sigma)$ iff $h_k'(m)(X)=0$.

Let us find $T_{(x,\xi)}(\partial_0\Omega(M))$, as a subspace in $T_{(x,\xi)}(T(M))$. Denote by $\rho(x)=$ dist $(x,\partial M)$ the distance to ∂M in M and smoothly continue it into $N\setminus M$. The submanifold $\partial_0\Omega(M))\in$

$T(M)$ is given by the three equations: $\rho = 0$, $|\xi| = 1$ and $(\xi, \nabla\rho) = 0$. Then, using (2.1) and $\nabla\rho|_{\partial M} = -\nu$ we have

$$T_{(x,\xi)}(\partial_0\Omega(M)) = \{X \in T_{(x,\xi)}(T(M)) : (\nu(x), X_h) = 0, (\xi, X_v) = 0, (\nabla(\xi, \nu(x)), X_h) + (\nu, X_v) = 0\}$$

Consider Ker $\phi'(x,\xi)$ also as a subspace of $T_{(x,\xi)}(T(M))$. It easy to show that Ker $\phi'(x,\xi)$ is 1-dimensional and generated by the vector $(\xi, 0)$ (i.e. $X_h = \xi, X_v = 0$). Then this vector is transversal to $T_{(x,\xi)}(\partial_0\Omega(M))$, since $(\nabla(\xi,\nu(x)),\xi) \neq 0$ if $(\xi,\nu(x)) = 0$ given that ∂M is strictly convex. □

5 The Hilbert transform and geodesic flow

In this section we prove Theorem 1.5 in the introduction.

Let H be the Hilbert transform as defined in (1.3). We have that H is a unitary operator in the space $L^2_0(\Omega_x) = \{u \in L^2(\Omega_x) : u_0 = 0\}$,

$$(u, v) = (Hu, Hv), \quad \forall u, v \in L^2_0(\Omega_x),$$

$$H^2(u) = -u, \quad \forall u \in L^2_0(\Omega_x).$$

Clearly, all these properties remain the same if we change Ω_x to $\Omega(M)$.

In order to prove Theorem 1.4 we need the following commutator formula which is valid for Riemannian manifolds of any dimension

Lemma 5.1. *Let u be a smooth function on the manifold $\Omega^2(M) = \bigcup_{x\in M} \Omega^2_x$, $\Omega^2_x = \{(x,\xi,\eta) : \xi,\eta \in \Omega_x\}$. Then*

$$\nabla \int\limits_{\Omega_x} u(x,\xi,\eta)d\Omega_x(\eta) = \int\limits_{\Omega_x} \nabla^{(2)} u(x,\xi,\eta)d\Omega_x(\eta)\,, \tag{5.1}$$

where $\nabla^{(2)}$ under the integral sign in (5.1) denotes the horizontal derivative on $\Omega^2(M)$,

$$\nabla^{(2)}_j u(x,\xi,\eta) = (\frac{\partial}{\partial x^j} - \Gamma^i_{jk}\xi^k\partial_{i(\xi)} - \Gamma^i_{jk}\eta^k\partial_{i(\eta)})u(x,\xi,\eta).$$

Notice that the horizontal tangential derivative can be defined on $T(M) \times T(M)$ in a similar fashion to the case of $T(M)$ in section 2.

Proof. Let $\varphi \in C_0^\infty\left(\mathbf{R}^+\right)$ be arbitrary function. We define the function v on $T^2\left(M\right)$ by

$$v(x,\xi,\eta) = \varphi\left(|\eta|\right) u(x,\xi/\left|\xi\right|,\eta/\left|\eta\right|)$$

Let us consider the integral

$$S(x,\xi) = \int\limits_{T_x(M)} v(x,\xi,\eta) dT_x\left(\eta\right).$$

Identifying $T_x(M)$ with R^n we have

$$S(x,\xi) = \int\limits_{R^n} v(x,\xi,\eta)\sqrt{\det g\left(x\right)}d\eta.$$

Then

$$\nabla_j S = \frac{\partial S}{\partial x^j} - \Gamma^i_{jk}\xi^k\frac{\partial S}{\partial \xi^i} = \int\limits_{R^n}(\frac{\partial v}{\partial x^j} - \Gamma^i_{jk}\xi^k\frac{\partial v}{\partial \xi^i})\sqrt{\det g}d\eta + \int\limits_{R^n} v\frac{\partial \ln\sqrt{\det g\left(x\right)}}{\partial x^j}\sqrt{\det g}d\eta.$$

Since $\partial \ln\sqrt{\det g}/dx^j = \Gamma^k_{jk}$ we rewrite the last integral in the form

$$\int\limits_{R^n} v\frac{\partial}{\partial \eta^k}\left(\Gamma^k_{jl}\eta^l\right)\sqrt{\det g}d\eta.$$

Then

$$\nabla_j S = \int\limits_{R^n}(\frac{\partial v}{\partial x^j} - \Gamma^i_{jk}\xi^k\frac{\partial v}{\partial \xi^i} - \Gamma^k_{jl}\eta^l\frac{\partial v}{\partial \eta^k})\sqrt{\det g}d\eta.$$

Since

$$(\frac{\partial}{\partial x^j} - \Gamma^i_{jk}\xi^k\frac{\partial}{\partial \xi^i} - \Gamma^k_{jl}\eta^l\frac{\partial}{\partial \eta^k})\left|\eta\right| = 0,$$

then after changing to spherical coordinates we obtain

$$\nabla S(x,\xi) = \int\limits_0^\infty \varphi\left(t\right)t^{n-1}dt\int\limits_{\Omega_x}\nabla u(x,\xi,\eta)d\Omega_x\left(\eta\right). \tag{5.2}$$

Now S in spherical coordinates is given by

$$S(x,\xi) = \int\limits_0^\infty \varphi\left(t\right)t^{n-1}dt\int\limits_{\Omega_x} u(x,\xi,\eta)d\Omega_x\left(\eta\right) \tag{5.3}$$

We conclude (5.1) using (5.2),(5.3).

□

Now we prove Theorem 1.4.

Proof. A straightforward calculation gives that

$$\nabla\frac{1+(\xi,\eta)}{(\xi_\perp,\eta)} = 0$$

and therefore we have

$$\nabla Hu(x,\xi) = \frac{1}{2\pi}\int\limits_{\Omega_x}\frac{1+(\xi,\eta)}{(\xi_\perp,\eta)}\nabla u(x,\eta)d\Omega_x\left(\eta\right).$$

For any pair vectors ξ, $\eta\in\Omega_x$ we have

$$\eta = (\xi,\eta)\xi + (\xi_\perp,\eta)\xi_\perp, \eta_\perp = -(\xi_\perp,\eta)\xi + (\xi,\eta)\xi_\perp, \quad (\xi,\eta)^2 + (\xi_\perp,\eta)^2 = 1.$$

Then

$$\eta\frac{1+(\xi,\eta)}{(\xi_\perp,\eta)} = \xi\frac{(\xi,\eta)+(\xi,\eta)^2}{(\xi_\perp,\eta)} + \xi_\perp(1+(\xi,\eta)) =$$

$$= \xi\frac{(\xi,\eta)+1}{(\xi_\perp,\eta)} - \xi\left(\xi_\perp,\eta\right) + \xi_\perp(\xi,\eta) + \xi_\perp = \xi\frac{1+(\xi,\eta)}{(\xi_\perp,\eta)} + \xi_\perp + \eta_\perp.$$

Thus

$$H\mathcal{H}u = \mathcal{H}Hu + \mathcal{H}_\perp u_0 + (\mathcal{H}_\perp u)_0$$

and Theorem 1.4 is proved.

□

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