COMPLEX GEOMETRICAL OPTICS SOLUTIONS FOR ANISOTROPIC EQUATIONS AND APPLICATIONS

HIDEKI TAKUWA, GUNTHER UHLMANN, AND JENN-NAN WANG

ABSTRACT. In this article, we construct complex geometrical optics solutions with general phase functions for the second order elliptic equation in two dimensions. We then use these special solutions to treat the inverse problem of reconstructing embedded inclusions by boundary measurements.

1. INTRODUCTION

In previous works [21] and [22], special complex geometrical optics (CGO) solutions for certain isotropic systems in the plane were constructed and their applications to the object identification problem were also demonstrated theoretically and numerically. Those systems include the conductivity equation, the elasticity equations, and the Stokes equations, all with isotropic inhomogeneous coefficients. In this paper we will extend the method in [21] and [22] to scalar equations with anisotropic inhomogeneous coefficients in the plane. We shall focus on the conductivity equation:

$$L_\gamma u =: \sum_{i,j=1}^{2} \partial_{x^j}(\gamma^{ij}(x)\partial_{x^i}u) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and the matrix $\gamma(x) = [\gamma^{ij}(x)]$ is symmetric and there exists $C_0 > 0$ such that

$$\gamma(x) \geq C_0 I \quad \text{for all} \quad x \in \Omega.$$

The precise regularity of γ will be specified later on. Our method can treat the equation with L_γ as the leading order without essential modification. The main aim here is to construct solutions to (1.1) with

The first author was partially supported by Grant-in-Aid for Young Scientists (B) 18740073, Japan.

The second author was partially supported by NSF and a Walker Family Endowed Professorship.

The third author was supported in part by the National Science Council of Taiwan (NSC 96-2628-M-002-009).

general complex phases. These types of solutions are very useful in the reconstruction of objects embedded in Ω by boundary measurements.

The key step in treating (1.1) is to transform it to an isotropic equation using isothermal coordinates, which are closely related to quasiconformal mappings [1]. This idea is crucial in studying the Calderón problem in two dimensions (see, for example, [2] and [20]). After reducing the anisotropic equation (1.1) to an isotropic one, we can apply the method in [21] to construct CGO solutions for the isotropic equation and then transform the solutions back to the original coordinates.

A useful application of these CGO solutions is to study the object identification problem by boundary measurements when the known background medium is anisotropic. Using the complex geometrical optics solutions in the reconstruction of embedded objects from boundary measurements was first introduced by Ikehata [11], [12]. He called this method the *enclosure method*. Our method shares the same spirit as the enclosure method. For brevity, we will not give a detailed Ikehata's enclosure method here. We refer the reader to a nice survey article by Ikehata [13] for more details and related applications.

In the implementation of the enclosure type method, how much information of the embedded object one can reconstruct depends on the phase functions of the CGO solutions used in the method. For example, one can reconstruct the convex hull of the object with Calderón type solutions [11], [12]. With complex spherical waves, one may be able to reconstruct some nonconvex parts of the object [8], [19]. In the two-dimensional case, we are able to get much more information of the object by using Mittag-Leffler functions [14], [15] or the special solutions constructed in [21]. Note that Mittag-Leffler functions can be only applied to the case where the background equation is the Laplacian. Like the results in [21], the method of this paper works for any general second order elliptic equations with coefficients having appropriate finite smoothness.

In this paper, we provide theoretical grounds for our method. In order to find the coordinates transformation, we need to solve the Beltrami equation. In Section 4, we will describe a numerical scheme to solve the Beltrami equation. This algorithm is a combination of the fast algorithm developed by Daripa *et al.* [3], [4], [5] (also see [6]) and the NUFFT (nonuniform fast Fourier transform) [7]. The rest of the paper is organized as follows. In Section 2, we will show how to construct complex geometrical optics solutions with general phases for (1.1). The key step is to reduce the equation to the isotropic equation with the help of a quasiconformal mapping. In Section 3, we provide theoretical backgrounds of reconstructing the embedded object by boundary measurements with these special solutions.

2. Construction of CGO solutions

In this section we will give a framework of constructing complex geometrical optics solutions for L_γ. We first recall a fundamental fact: let $F:\Omega\to\widetilde{\Omega}$, $y=F(x)$, be a C^1 bijective map (orientation-preserving), then $u(x)$ solves (1.1) if and only if $v(y)=u\circ F^{-1}(y)$ solves

$$L_{\widetilde{\gamma}}v=0 \quad \text{in} \quad \widetilde{\Omega}, \tag{2.1}$$

where $\widetilde{\gamma}=F_*\gamma$ defined by

$$(F_*\gamma)^{k\ell}(y)=\frac{1}{\det(DF)}\sum_{i,j=1}^{2}\partial_{x^i}F^k\partial_{x^j}F^\ell\gamma^{ij}\big|_{x=F^{-1}(y)}.$$

Our aim here is to find a suitable F so that $F_*\gamma$ is isotropic. In the plane, the existence of such map F is well-known and it is a quasiconformal map. In the following, we will review some essential properties we need from the theory of quasiconformal mapping. For more details, we refer to [1], [16], and [18].

Let us denote $\mathbb{R}^2$ as the complex plane $\mathbb{C}$ and $z=x^1+\imath x^2$ from now on. Assume that $\gamma\in C^1(\overline{\Omega})$. We now extend γ to $\mathbb{C}$, still denoted by γ, such that $\gamma\in C^1(\mathbb{C})$ and $\gamma=I$ for $|z|>R$ for some $R>0$.

Theorem 2.1. *There exists a C^1 bijective map $F:\mathbb{C}\to\mathbb{C}$ such that*

$$F_*\gamma=(\det\gamma\circ F^{-1})^{1/2}I. \tag{2.2}$$

Proof. The proof of this theorem can be found in [20] for C^3 or $C^{1,1}$ conductivities and in [2] for L^∞ conductivities. For the purpose of numerical computation in the following sections, we sketch the proof here. The relation (2.2) holds if and only if F is a quasiconformal map with complex dilatation

$$\mu_\gamma=\frac{\gamma^{22}-\gamma^{11}-2i\gamma^{12}}{\gamma^{11}+\gamma^{22}+2\sqrt{\det\gamma}},$$

i.e.,

$$\bar{\partial}F=\mu_\gamma\partial F, \tag{2.3}$$

where

$$\bar{\partial}=\frac{1}{2}(\partial_{x^1}+i\partial_{x^2}) \quad \text{and} \quad \partial=\frac{1}{2}(\partial_{x^1}-i\partial_{x^2}).$$

Equation (2.3) is the well-known Beltrami equation. Note that here μ_γ is supported in a disc of radius R and

$$\|\mu_\gamma\|_{L^\infty} \le K < 1.$$

Following Ahlfors' approach [1, Chapter V], there are two integral operators which play key roles in solving (2.3):

$$Pf(\xi) = \frac{\imath}{2\pi}\int_{\mathbb{C}} \Big(\frac{1}{z-\xi} - \frac{1}{z}\Big) f(z) d\bar{z}\wedge dz$$

and

$$Tf(\xi) = \lim_{\varepsilon\to 0} \frac{\imath}{2\pi}\int_{|z-\xi|>\varepsilon} \frac{f(z)}{(z-\xi)^2} d\bar{z}\wedge dz.$$

The existence of F to (2.3) can be achieved in two steps [1, page 92]. Firstly, we solve the integral equation

$$h = T(\mu_\gamma h) + T\mu_\gamma. \tag{2.4}$$

Let h be a solution of (2.4) then

$$F = P(\mu_\gamma(h+1)) + z \tag{2.5}$$

solves (2.3). Since $\partial\mu_\gamma \in L^p$ for any $p > 2$, it follows from [1, Lemma 3, page 94] that $F \in C^1$ and is bijective.

Before ending the proof, we say a few words on the existence of (2.4). The celebrated Calderón-Zygmund inequality indicates that for $p > 1$

$$\|Tf\|_{L^p} \le C_p \|f\|_{L^p}$$

with the constant C_p only depending on p. Also, one has $C_p \to 1$ as $p \to 2$. Now by taking $KC_p < 1$, the existence of (2.4) can be obtained using the Neumann series. On the other hand, the operator is well-defined on L^p functions with $p > 2$. □

Now we discuss how to construct complex geometrical optics solutions for L_γ. Suppose that the original conductivity function γ, defined on Ω, was suitably extended and a quasiconformal map F described in Theorem 2.1 was found. Let us define $\tilde{F} = F|_\Omega$. Then $\tilde{F} : \Omega \to \tilde{\Omega}$ with $\tilde{\Omega} = \tilde{F}(\Omega)$ is a C^1 diffeomorphism and

$$\tilde{\gamma}(y) = \tilde{F}_*\gamma = (\det\gamma \circ \tilde{F}^{-1})^{1/2}(y) \in C^1(\tilde{\Omega}).$$

As before, we know that $v(y)$ solves

$$\frac{1}{\tilde{\gamma}} L_{\tilde{\gamma}} v = \Delta v + \frac{\nabla\tilde{\gamma}}{\tilde{\gamma}}\cdot\nabla v = 0 \quad \text{in} \quad \tilde{\Omega} \tag{2.6}$$

if and only if $u(x) = v\circ F(x)$ solves

$$L_\gamma u = 0 \quad \text{in} \quad \Omega.$$

Therefore, it suffices to construct CGO solutions for (2.6). We recall that some CGO solutions with general phase functions for (2.6) have been established in [21]. We now demonstrate how to adapt the arguments in [21] to our setting here. Pick a $y_0 \in \mathbb{C}$. Without loss of generality, we take $y_0 = 0$. For $N \in \mathbb{N}$, let

$$\tilde{\phi}_N(y) = c_N(y-y_0)^N = c_N y^N = \tilde{\varphi}_N(y) + \imath\tilde{\psi}_N(y),$$

where $y = y_1 + \imath y_2 \in \mathbb{C}$, $\tilde{\varphi}_N = \mathrm{Re}(c_N y^N)$, $\tilde{\psi}_N = \mathrm{Im}(c_N y^N)$, $c_N \in \mathbb{C}$ with $|c_N| = 1$. Some level curves of $\tilde{\varphi}_N$ were shown below (also see [21]).

As in [21], let Γ_N be the open cone with vertex at 0 and opening angle π/N in which $\tilde{\varphi}_N > 0$ in Γ_N and $\tilde{\varphi}_N = 0$ on the boundary of Γ_N. We assume that $\Gamma_N \cap \tilde{\Omega} \neq \emptyset$. According to [21], we can construct solutions of the form

$$\tilde{v}_{N,h} = e^{\tilde{\phi}_N/h}(\tilde{a}_N(y) + \tilde{r}_N(y,h))$$

to (2.6), where $\tilde{a}_N(y)$ never vanishes in $\Gamma_N \cap \tilde{\Omega}$ and

$$\|\partial_y^\beta \tilde{r}_N\|_{L^2(\Gamma_N\cap\tilde{\Omega})} \le Ch^{1-|\beta|} \quad \forall\, |\beta| \le 1.$$

We now extend $\tilde{v}_{N,h}$ defined in $\Gamma_N \cap \tilde{\Omega}$ to the whole $\tilde{\Omega}$ by introducing a suitable cut-off function.

For $s > 0$, let $\tilde{\ell}_s = \{y \in \Gamma_N : \tilde{\varphi}_N = s^{-1}\}$. This is the level curve of $\tilde{\varphi}_N$ in Γ_N. Let $0 < t < t_0$ such that

$$(\cup_{s\in(0,t)}\tilde{\ell}_s) \cap \tilde{\Omega} \neq \emptyset$$

and choose a small $\varepsilon > 0$. Define a cut-off function $\chi_{N,t}(y) \in C^\infty(\mathbb{R}^2)$ so that $\chi_{N,t}(y) = 1$ for $y \in \overline{(\cup_{s\in(0,t+\varepsilon/2)}\tilde{\ell}_s) \cap \tilde{\Omega}}$ and is zero for $y \in \bar{\tilde{\Omega}} \setminus (\cup_{s\in(0,t+\varepsilon)}\tilde{\ell}_s)$. We want to remark that $\cup_{s\in(0,t+\varepsilon/2)}\tilde{\ell}_s \subset \cup_{s\in(0,t+\varepsilon)}\tilde{\ell}_s$. Now we define

$$\tilde{v}_{N,t,h}(y) = \chi_{N,t} e^{-t^{-1}/h} \tilde{v}_{N,h} = \chi_{N,t} e^{(\tilde{\varphi}_N - t^{-1} + \imath\tilde{\psi}_N)/h}(\tilde{a}_N + \tilde{r}_N)$$

for $y \in (\cup_{s\in(0,t+\varepsilon)}\tilde{\ell}_s) \cap \tilde{\Omega}$. So $\tilde{v}_{N,t,h}$ can be regarded as a function in $\tilde{\Omega}$ which is zero outside of $\Gamma_N \cap \tilde{\Omega}$. Note that $\tilde{v}_{N,t,h}$ does not solve (2.6) in $\tilde{\Omega}$. We thus take $\tilde{f}_{N,t,h} = \tilde{v}_{N,t,h}|_{\partial\tilde{\Omega}}$ and consider the boundary value problem:

$$\begin{cases} \Delta\tilde{u}_{N,t,h} + \frac{\nabla\tilde{\gamma}}{\tilde{\gamma}} \cdot \nabla\tilde{u}_{N,t,h} = 0 & \text{in} \quad \tilde{\Omega} \\ \tilde{u}_{N,t,h} = \tilde{f}_{N,t,h} & \text{on} \quad \partial\tilde{\Omega}. \end{cases} \tag{2.7}$$

Clearly (2.7) is uniquely solvable. Also it was shown in [21] that there exist $\tilde{C} > 0$ and $\tilde{\varepsilon} > 0$ such that

$$\|\tilde{u}_{N,t,h} - \tilde{v}_{N,t,h}\|_{H^1(\tilde{\Omega})} \le \tilde{C} e^{-\tilde{\varepsilon}/h}$$

for $h \ll 1$.

Next we define

$$u_{N,t,h}(x) = (\tilde{u}_{N,t,h} \circ F)(x),$$

$$f_{N,t,h} = \tilde{f}_{N,t,h} \circ F|_{\partial\Omega},$$

and

$$v_{N,t,h}(x) = (\tilde{v}_{N,t,h} \circ F)(x) = (\chi_{N,t} \circ F)e^{(\varphi_N - t^{-1} + i\psi_N)/h}(a_N + r_N),$$

where $\varphi_N = \tilde{\varphi}_N \circ F$, $\psi_N = \tilde{\psi}_N \circ F$, $a_N = \tilde{a}_N \circ F$, and $r_N = \tilde{r}_N \circ F$. Then, a_N is never zero in $F^{-1}(\Gamma_N \cap \tilde{\Omega})$ and

$$\|\partial_x^\beta r_N\|_{L^2(F^{-1}(\Gamma_N \cap \tilde{\Omega}))} \le Ch^{1-|\beta|} \quad \forall\ |\beta| \le 1.$$

Moreover, we have that $u_{N,t,h}$ satisfies

$$\begin{cases} L_\gamma u_{N,t,h} = 0 \quad \text{in} \quad \Omega \\ u_{N,t,h} = f_{N,t,h} \quad \text{on} \quad \partial\Omega \end{cases}$$

and

$$\|u_{N,t,h} - v_{N,t,h}\|_{H^1(\Omega)} \le Ce^{-\varepsilon'/h} \tag{2.8}$$

for some positive constants C and ε'. The solution $u_{N,t,h}(x)$ is a CGO solution of (1.1) and is *approximated* by $v_{N,t,h}$ with exponentially small errors. We shall pay attention to the level curves of $\varphi_N = \tilde{\varphi}_N \circ F$ defined in $F^{-1}(\Gamma_N)$, which are key to our reconstruction method for the object identification problem.

3. Applications to inverse problems

In this section we demonstrate how to use CGO solutions constructed previously in the object identification problem. To simplify our presentation, we will only discuss the case of identifying inclusions inside of the domain Ω filled with known conductivity. This inverse problem has been extensively studied both theoretically and numerically. We refer to [8] for related references. Let D be an open bounded domain with C^1 boundary such that $\bar{D} \subset \Omega$ and $\Omega \setminus \bar{D}$ is connected. Assume that $\gamma_0(x) \in C^1(\bar{\Omega})$ is a symmetric matrix satisfying

$$\gamma_0(x) \ge \delta I$$

with some $\delta > 0$ for all $x \in \bar{\Omega}$. The conductivity $\gamma(x)$ is a perturbation of γ_0 described by $\gamma(x) = \gamma_0 + \chi_D\gamma_1$, where χ_D is the characteristic function of D and $\gamma_1 \in C(\bar{D})$ is a positive semi-definite matrix, i.e,

$$\gamma_1 \ge 0 \quad \text{on} \quad \bar{D}. \tag{3.1}$$

Then we have $\gamma(x) \geq \delta I$ almost everywhere in Ω. Let v be the solution of

$$\begin{cases} L_\gamma v = 0 & \text{in} \quad \Omega, \\ v = f & \text{on} \quad \partial\Omega. \end{cases} \tag{3.2}$$

The meaning of the solution to (3.2) is understood in the following way. Define

$$[w]_{\partial D} = \mathrm{tr}^+ w - \mathrm{tr}^- w$$

the jump of the function across ∂D, where tr^+ and tr^- denote respectively the trace of w on ∂D from inside and outside of D. For $f \in H^{3/2}(\partial\Omega)$, we define

$$\mathcal{V}_f = \{w \in H^2(D) \oplus H^2(\Omega \setminus \bar{D}) : w|_{\partial\Omega} = f, [w]_{\partial D} = 0, [\gamma\partial_\nu w]_{\partial D} = 0\},$$

where ν is unit normal of ∂D and

$$\gamma\partial_\nu w = \sum_{j,k=1}^{2} \gamma^{jk}\nu_k \partial_{x^j} w.$$

We say that v is the solution of (3.2) if $v \in \mathcal{V}_f$ and $L_\gamma v = 0$ in D and $\Omega \setminus \bar{D}$. The Dirichlet-to-Neumann map is given as

$$\Lambda_D : f \to \gamma\partial_n v|_{\partial\Omega} = \sum_{j,k=1}^{2} \gamma_0^{jk} n_k \partial_{x^j} v|_{\partial\Omega},$$

where n is the unit outer normal of $\partial\Omega$. The inverse problem is to determine the inclusion D from Λ_D. Here we are interested in the reconstruction question.

To begin, we would like to derive two important integral inequalities. Let $u(x) \in H^2(\Omega)$ be the solution of

$$\begin{cases} L_{\gamma_0} u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega \end{cases} \tag{3.3}$$

for $f \in H^{3/2}(\partial\Omega)$. The Dirichlet-to-Neumann map corresponding to (3.3) is denoted by

$$\Lambda_0 : f \to \gamma_0 \partial_n u|_{\partial\Omega}.$$

Then we can prove

Lemma 3.1.

$$\langle (\Lambda_D - \Lambda_0)\bar{f}, f\rangle_{\partial\Omega} \leq \langle \gamma_1 \nabla u, \nabla\bar{u}\rangle_D \tag{3.4}$$

and

$$\langle (\gamma_0^{-1} - \gamma^{-1})\gamma_0 \nabla u, \gamma_0 \nabla\bar{u}\rangle_D \leq \langle (\Lambda_D - \Lambda_0)\bar{f}, f\rangle_{\partial\Omega}, \tag{3.5}$$

where

$$\langle A\nabla u, \nabla \bar{u}\rangle_D = \int_D (\nabla \bar{u})^T A \nabla u dx$$

for any matrix A.

Proof. When γ_0 and γ_1 are isotropic, derivations of (3.4) and (3.5) can be found in [8] (or in [9]). Here we adapt their arguments to treat anisotropic conductivities. Let v and u be solutions of (3.2) and (3.3) with same Dirichlet condition f, respectively. From Green's formula, we have that

$$\langle \gamma\nabla v, \nabla \bar{v}\rangle_\Omega = \langle \gamma\nabla v, \nabla \bar{u}\rangle_\Omega \tag{3.6}$$

and

$$\langle \gamma_0\nabla u, \nabla \bar{u}\rangle_\Omega = \langle \gamma_0\nabla u, \nabla \bar{v}\rangle_\Omega. \tag{3.7}$$

Next we observe that

$$\langle \Lambda_0 \bar{f}, f\rangle_{\partial\Omega} = \langle \gamma_0\nabla \bar{u}, \nabla u\rangle_\Omega = \langle \gamma_0\nabla u, \nabla \bar{u}\rangle_\Omega \tag{3.8}$$

and

$$\langle \Lambda_D \bar{f}, f\rangle_{\partial\Omega} = \langle \gamma\nabla \bar{v}, \nabla u\rangle_\Omega = \langle \gamma\nabla u, \nabla \bar{v}\rangle_\Omega. \tag{3.9}$$

Using (3.6), (3.8), (3.9), and some simple computations, we get that

$$\begin{aligned}
&\langle (\Lambda_D - \Lambda_0)\bar{f}, f\rangle_{\partial\Omega}\\
\le&\langle (\Lambda_D - \Lambda_0)\bar{f}, f\rangle_{\partial\Omega} + \langle \gamma\nabla(v-u), \nabla(\bar{v}-\bar{u})\rangle_\Omega\\
=&\langle \gamma\nabla u, \nabla \bar{v}\rangle_\Omega - \langle \gamma_0\nabla u, \nabla \bar{u}\rangle_\Omega + \langle \gamma\nabla(v-u), \nabla(\bar{v}-\bar{u})\rangle_\Omega\\
=&\langle (\gamma-\gamma_0)\nabla u, \nabla \bar{u}\rangle_\Omega\\
=&\langle \gamma_1\nabla u, \nabla \bar{u}\rangle_D,
\end{aligned}$$

which is exactly (3.4).

By similar computations, we can show that

$$\langle (\Lambda_D - \Lambda_0)\bar{f}, f\rangle_{\partial\Omega} = \langle \gamma_0\nabla(v-u), \nabla(\bar{v}-\bar{u})\rangle_\Omega + \langle (\gamma-\gamma_0)\nabla v, \nabla \bar{v}\rangle_\Omega. \tag{3.10}$$

On the other hand, it is not hard to check that

$$\begin{aligned}
&\langle \gamma_0\nabla(v-u), \nabla(\bar{v}-\bar{u})\rangle_\Omega + \langle (\gamma-\gamma_0)\nabla v, \nabla \bar{v}\rangle_\Omega\\
=&\langle \gamma(\nabla v - \gamma^{-1}\gamma_0\nabla u), (\nabla \bar{v} - \gamma^{-1}\gamma_0\nabla \bar{u})\rangle_\Omega + \langle \gamma_0(\gamma_0^{-1} - \gamma^{-1})\gamma_0\nabla u, \nabla \bar{u}\rangle_\Omega.
\end{aligned} \tag{3.11}$$

Using (3.10), (3.11), and the positive-definiteness of γ immediately leads to (3.5). □

Now we are ready to consider the reconstruction of D from the measurements Λ_D. To make sure that the medium has jumps across ∂D, we assume that for any $q \in \partial D$, there exist $\delta_q > 0$ and $\varepsilon_q > 0$ such that

$$\gamma_1(x) \ge \varepsilon_q I \quad \text{for all} \quad x \in D \cap B(q, \delta_q). \tag{3.12}$$

To set up the measurements, we choose the following Dirichlet condition

$$f_{N,t,h} = u_{N,t,h}|_{\partial\Omega} = v_{N,t,h}|_{\partial\Omega} = (\chi_{N,t} \circ F)e^{(\varphi_N - t^{-1} + \imath\psi_N)/h}(a_N + r_N)|_{\partial\Omega}.$$

Note that the Dirichlet condition $f_{N,t,h}$ is localized on $\partial\Omega$. With $f_{N,t,h}$, we now solve the boundary value problem

$$\begin{cases} L_\gamma u_{N,t,h} = 0 \quad \text{in} \quad \Omega, \\ u_{N,t,h} = f_{N,t,h} \quad \text{on} \quad \partial\Omega \end{cases}$$

and then evaluate the quadratic term

$$E(N,t,h) := \langle (\Lambda_D - \Lambda_0)\bar{f}_{N,t,h}, f_{N,t,h}\rangle_{\partial\Omega}.$$

It is clear that $E(N,t,h)$ is determined entirely by boundary measurements. Before proving the main reconstruction result, we would like to remark that

$$\{x \in F^{-1}(\Gamma_N) : \varphi_N > t_1^{-1}\} \subset \{x \in F^{-1}(\Gamma_N) : \varphi_N > t_2^{-1}\} \quad \text{for} \quad t_1 < t_2.$$

Theorem 3.2. *Let $t > 0$ be given. Then we have:*
(i) *if $\bar{D} \cap \{x \in F^{-1}(\Gamma_N) : \varphi_N > t^{-1}\} = \emptyset$ then there exist $C_1 > 0$, $\varepsilon_1 > 0$, and $h_1 > 0$ such that $E(N,t,h) \le C_1 e^{-\varepsilon_1/h}$ for all $h \ge h_1$;*
(ii) *if $D \cap \{x \in F^{-1}(\Gamma_N) : \varphi_N > t^{-1}\} \ne \emptyset$ then there exist $C_2 > 0$, $\varepsilon_2 > 0$, and $h_2 > 0$ such that $E(N,t,h) \ge C_2 e^{\varepsilon_2/h}$ for all $h \ge h_2$.*

Proof. For statement (i), we use integral inequality (3.4) and (2.8) to obtain that

$$\begin{aligned} E(N,t,h) &\le \langle \gamma_1 \nabla u_{N,t,h}, \nabla \bar{u}_{N,t,h}\rangle_D \le C\int_D |\nabla u_{N,t,h}|^2 dx \\ &\le C\int_D |\nabla v_{N,t,h}|^2 dx + Ce^{-\varepsilon'/h}. \end{aligned} \tag{3.13}$$

Using the structure of $v_{N,t,h}$ and (3.13) immediately leads to (i).

To prove (ii), we need to investigate the matrix $\gamma_0^{-1} - \gamma^{-1}$ a little bit more. For $x \in D$, we see that

$$\begin{aligned} &\gamma_0^{-1} - \gamma^{-1} = \gamma_0^{-1} - (\gamma_0 + \gamma_1)^{-1} \\ &= (\gamma_0 + \gamma_1)^{-1}\{(\gamma_0 + \gamma_1)\gamma_0^{-1}(\gamma_0 + \gamma_1) - (\gamma_0 + \gamma_1)\}(\gamma_0 + \gamma_1)^{-1} \\ &= (\gamma_0 + \gamma_1)^{-1}(\gamma_1 + \gamma_1\gamma_0^{-1}\gamma_1)(\gamma_0 + \gamma_1)^{-1} \end{aligned}$$

(see similar computations in [10] for anisotropic elastic body). Therefore, $\gamma_0^{-1} - \gamma^{-1}$ is positive semi-definite and from (3.12) we have

$$(\gamma_0^{-1} - \gamma_1^{-1})(x) \ge \tilde{\varepsilon}_q I \quad \forall\, x \in D \cap B(q, \delta_q) \tag{3.14}$$

for some $\tilde{\varepsilon}_q > 0$. Now assume that

$$D \cap B(q, \delta_q) \subset D \cap \{x \in F^{-1}(\Gamma_N) : \varphi_N > t^{-1}\}$$

for some $q \in \partial D$ and $\delta_q > 0$. Using (3.14), (3.5), and (2.8), we can compute

$$\begin{aligned} E(N,t,h) &\geq \langle (\gamma_0^{-1} - \gamma^{-1})\gamma_0 \nabla u_{N,t,h}, \gamma_0 \nabla \bar{u}_{N,t,h} \rangle_D \\ &\geq C\tilde{\varepsilon}_q \int_{D\cap B(q,\delta_q)} |\nabla v_{N,t,h}|^2 dx - Ce^{-\varepsilon'/h}. \end{aligned} \tag{3.15}$$

Combining the exponentially growing behavior of $v_{N,t,h}$ and (3.15) yields (ii). □

With the help of Theorem 3.2, we can determine whether the level curve $\{x \in F^{-1}(\Gamma_N) : \varphi_N = t^{-1}\}$ intersects D or not by looking into the behavior of $E(N,t,h)$ as $h \ll 1$. In practice, we can start with a smaller t and move the level curve deeper into the domain Ω by choosing a larger t. In addition, we are able to determine more parts of ∂D by increasing N. We refer to [21] for precise arguments.

4. Algorithm of constructing the quasiconformal map

In the study of our reconstruction method mentioned above, the forward problem is solved with CGO solutions as the Dirichlet boundary condition. Thus, comparing with the reconstruction algorithm in [21] for the isotropic medium case, the only difference is the boundary condition used in the forward problem (3.2). In other words, the numerical scheme used in [21] can be adapted to the problem studied here by merely changing the boundary condition. Therefore, we do not intend to repeat computational results in the paper. Nonetheless, since the CGO solutions constructed here involve the quasiconformal map F, we want to discuss how to find this map F numerically. We will leave the implementation of this algorithm to interested readers.

A fast algorithm using FFT for solving the Beltrami equation has been proposed by Daripa *et al.* [3], [4], [5]. Based on Daripa's algorithm, a slightly modified version was given in [6]. Here we shall follow the description of Daripa's algorithm given in [6]. Since it is often desirable to refine the grid in the part of boundary where the measurements are taken, we shall replace FFT in Darpia's algorithm by NUFFT described in [7].

As outlined in the proof of Theorem 2.1, a solution F to the Beltrami equation (2.3) can be constructed by solving integral equations:

$$h = T(\mu_\gamma h) + T(\mu_\gamma) \tag{4.1}$$

and

$$F = P(\mu_\gamma(h+1)) + z. \tag{4.2}$$

Here the anisotropic conductivity function $\gamma(x)$ has been extended to $\mathbb{C}$ with $\gamma \in C^1$ and $\gamma = I$ for $x \in B(0,R)^c$. Viewing the definition of the complex dilatation μ_γ, we get $\mu_\gamma \in C^1$ and $\mu_\gamma(x) = 0$ for all $x \in B(0,R)^c$.

As shown in the proof of Theorem 2.1, a solution to (4.1) can be constructed by a fix-point type iteration:

$$h^{n+1} = T(\mu_\gamma h^n) + T(\mu_\gamma) \tag{4.3}$$

and $h^n \to h^*$ in some L^p norm with $p > 2$. Let g be Hölder continuous and supported in $B(0,R)$. For such g, it is known that $T(g)$ exists in the sense of the Cauchy principal value [1]. Let us express g and $T(g)$ in Fourier series

$$g(re^{\imath\theta}) = \sum_{-\infty}^{\infty} g_k(r) e^{\imath k\theta}$$

and

$$T(g)(re^{\imath\theta}) = \sum_{-\infty}^{\infty} t_k(r) e^{\imath k\theta}.$$

It was proved in [5] that if $T(g)$ exists as the Cauchy principal value, then the Fourier coefficients of $T(g)$ are given by

$$t_0(0) = -2\lim_{\varepsilon\to 0}\int_\varepsilon^R \frac{g_2(\rho)}{\rho}d\rho, \text{ and } t_k(0) = 0, \text{ for } k \neq 0, \tag{4.4}$$

$$t_k(r) = A_k\int_0^r \frac{r^k}{\rho^{k+1}} g_{k+2}(\rho)d\rho + B_k\int_r^R \frac{r^k}{\rho^{k+1}} g_{k+2}(\rho)d\rho + g_{k+2}(r), \tag{4.5}$$

where

$$A_k = \begin{cases} 0, \ k \geq 0, \\ 2(k+1), \ k < 0, \end{cases} \quad \text{and} \quad B_k = \begin{cases} -2(k+1), \ k \geq 0, \\ 0, \ k < 0 \end{cases}$$

(see also [4]). Note that $g_k(0) = g(0)$ if $k = 0$ and $g_k(0) = 0$ if $k \neq 0$.

Having found a solution to (4.1), we can find a solution of the Beltrami equation by solving (4.2). Let us define

$$\tilde{P}(\tilde{g})(z) = \frac{\imath}{2\pi}\int_{\mathbb{C}} \frac{\tilde{g}(z')}{z'-z} d\bar{z}' \wedge dz',$$

then

$$P(\tilde{g})(z) = \tilde{P}(\tilde{g})(z) - \tilde{P}(\tilde{g})(0).$$

An effective algorithm for evaluating the integral transform $\tilde{P}$ was proposed in [3]. Precisely, let $\tilde{g}$ and $\tilde{P}(\tilde{g})$ be expressed as

$$\tilde{g}(re^{\imath\theta}) = \sum_{-\infty}^{\infty} \tilde{g}_k(r)e^{\imath k\theta} \quad \text{and} \quad \tilde{P}(\tilde{g})(re^{\imath\theta}) = \sum_{-\infty}^{\infty} p_k(r)e^{\imath k\theta},$$

then

$$p_k(r) = \begin{cases} 2\int_0^r (\frac{r}{\rho})^k \tilde{g}_{k+1}(\rho)d\rho, \ k < 0, \\ -2\int_r^R (\frac{r}{\rho})^k \tilde{g}_{k+1}(\rho)d\rho, \ k \geq 0 \end{cases} \tag{4.6}$$

(see [3]). In view of the formulae (4.5) and (4.6), one can easily derive useful relations for $t_k(r)$ and $p_k(r)$ at any pair of nonnegative r and r'. Define

$$d_k(r, r') = 2(k+1)\int_r^{r'} \frac{1}{\rho}(\frac{r'}{\rho})^k g_{k+2}(\rho)d\rho$$

and

$$e_k(r, r') = 2\int_r^{r'} (\frac{r}{\rho})^k \tilde{g}_{k+1}(\rho)d\rho,$$

then we can derive

$$t_k(r) = (\frac{r}{r'})^k(t_k(r') - g_{k+2}(r') - d_k(r, r')) + g_{k+2}(r) \tag{4.7}$$

and

$$p_k(r) = (\frac{r}{r'})^k p_k(r') - e_k(r, r') \tag{4.8}$$

(see [3], [6]). Armed with recurrence formulae (4.7), (4.8), we can evaluate the Fourier coefficients $t_k(r)$ and $p_k(r)$ more efficiently.

We now describe how to implement the above algorithm in actual computation. In view of (4.2), since supp $(\mu_\gamma) \subset \overline{B(0,R)}$, we only need to construct h in $B(0,R)$. We discretize $\overline{B(0,R)}$ with a circular $M_1 \times M_2$ grid. In other words, each grid point is represented by (r_i, θ_j), where $0 < r_i \leq R$ and $\theta_j \in [0, 2\pi]$, for $1 \leq i \leq M_1$, $0 \leq j \leq M_2 - 1$. Note that $\{\theta_j\}$ are not necessarily equi-distributed on $[0, 2\pi]$. We may take $r_0 = 0$ and $r_{M_1} = R$. To initiate the iteration (4.3), we can take $h^0 = 0$. Then we have $h^1 = T(\mu_\gamma)$. Since in each iteration we need to compute $T(\mu_\gamma)$ and $T(\mu_\gamma h^n)$, it is helpful to see how to calculate $T(g)$ for a given function g. Suppose that the value of g at each grid point (r_i, θ_j) is given, say $g(r_i e^{\imath\theta_j}) = g_{i,j}$. Then implementing the type-1 NUFFT [7], one can compute

$$g_k(r_i) = \frac{1}{M_2}\sum_{j=0}^{M_2-1} g_{i,j}e^{-\imath k\theta_j} \tag{4.9}$$

for $-\frac{N}{2} \le k \le \frac{N}{2} - 1$ and $1 \le i \le M_1$. To find the Fourier coefficients of $T(g)$, we start from $r_{M_1} = R$. Using (4.5), we see that

$$t_k(r_{M_1}) = A_k \int_0^{r_{M_1}} \frac{r_{M_1}^k}{\rho^{k+1}} g_{k+2}(\rho) d\rho + g_{k+2}(r_{M_1}). \tag{4.10}$$

We then approximate the integral in (4.10) with available values of $g_k(r)$ at r_i $(0 \le i \le M_1)$ using a suitable approximation formula. Having determined $t_k(r_{M_1})$, we then determine $t_k(r_i)$ for $i \le M_1 - 1$ by the recurrence formula (4.7) with $r' = r_{M_1}$. Applying (4.3), we can determine the discrete Fourier transform of h^{n+1}. Let h^* be an approximate solution of (4.3) with Fourier coefficients $h^*_{i,k}$ for $0 \le i \le M_1$ and $-\frac{N}{2} \le k \le \frac{N}{2} - 1$.

Our next step is to construct a quasiconformal map F from (4.2). To this end, we need to evaluate $\mu_\gamma(h^* + 1) =: \tilde{g}$. We do so by using the point-wise multiplication over grid points. To get the values of h^* at grid points, we simply perform type-2 NUFFT [7] with coefficients $h^*_{i,k}$, i.e.,

$$h^*(r_i e^{\imath\theta_j}) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} h^*_{i,k} e^{\imath k\theta_j}.$$

Then applying the type-1 NUFFT again, we can compute $\tilde{g}_k(r_i)$. By (4.6), we see that for $r_{M_1} = R$

$$p_k(r_{M_1}) = \begin{cases} 2\int_0^R (\frac{R}{\rho})^k \tilde{g}_{k+1}(\rho) d\rho, & k < 0, \\ 0, & k \ge 0. \end{cases}$$

To find $p_k(r_i)$ for $i \le M_1 - 1$, we simply apply the recurrence formula (4.8). Having found Fourier coefficients $p_k(r_i)$, we obtain the values of $\tilde{P}(\tilde{g})$ at grid points by using the type-2 NUFFT. Finally, we can get

$$P(\tilde{g})(r_i e^{\imath\theta_j}) = \tilde{P}(\tilde{g})(r_i e^{\imath\theta_j}) - \tilde{P}(\tilde{g})(0)$$

and

$$F(r_i e^{\imath\theta_j}) = P(\tilde{g})(r_i e^{\imath\theta_j}) + r_i e^{\imath\theta_j},$$

where

$$\tilde{P}(\tilde{g})(0) = p_0(0) = -2\int_0^R \tilde{g}_1(\rho) d\rho.$$

To get F at other points in $\overline{B(0,R)}$, we can use appropriate interpolation or extrapolation techniques.

References

[1] L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Norstrand, Princeton, NJ, 1966. Reprinted by Wadsworth Inc. Belmont, CA, 1987.

[2] K. Astala, L. Päivärinta, and M. Lassas, *Calderón's inverse problem for anisotropic conductivity in the plane*, Comm. PDE, **30** (2005), 207-224.

[3] P. Daripa, *A fast algorithm to solve non-homogeneous Cauchy-Riemann equations in the complex plane*, SIAM J. Sci. Statist. Comput., **13** (1992), 1418-1832.

[4] P. Daripa, *A fast algorithm to solve the Beltrami equation with applications to quasiconformal mappings*, J. Comp. Phys., **106** (1993), 355-365.

[5] P. Daripa and D. Mashat, *Singular integral transforms and fast numerical algorithms*, Numer. Algor., **18** (1998), 133-157.

[6] D. Gaydashev and D. Khmelev, *On numerical algorithms for the solution of a Beltrami equation*, SIAM J. Numer. Anal., **46** (2008), 2238-2253.

[7] L. Greengard and J.-Y. Lee, *Accelerating the nonuniform fast Fourier transform*, SIAM Review, **46** (2004), 443-454.

[8] T. Ide, H. Isozaki, S. Nakata, S. Siltanen, and G. Uhlmann, *Probing for electrical inclusions with complex spherical waves*, to appear in Comm. Pure Appl. Math.

[9] M. Ikehata, *Identification of the shape of the inclusion having essentially bounded conductivity*, J. Inv. Ill-Posed Problems, **7** (1999), 533-540.

[10] M. Ikehata, G. Nakamura, and K. Tanuma, *Identification of the shape of the inclusion in the anisotropic body*, Applicable Analysis, **72** (1999), 17-26.

[11] M. Ikehata, *Enclosing a polygonal cavity in a two-dimensional bounded domain from Cauchy data*, Inverse Problems, **15** (1999), 1231-1241.

[12] M. Ikehata, *How to draw a picture of an unknown inclusion from boundary measuremets. Two mathematical inversion algorithms*, J. Inv. Ill-Posed Problems, **7** (1999), 255-271.

[13] M. Ikehata, *The enclosure method and its applications*, Analytic extension formulas and their applications (Fukuoka, 1999/Kyoto, 2000), 87-103, Int. Soc. Anal. Appl. Comput., 9, Kluwer Acad. Publ., Dordrecht, 2001.

[14] M. Ikehata, *Mittag-Leffler's function and extracting from Cauchy data*, Inverse problems and spectral theory, 41-52, Contemp. Math., **348**, Amer. Math. Soc., Providence, RI, 2004.

[15] M. Ikehata and S. Siltanen, *Electrical impedance tomography and Mittag-Leffler's function*, Inverse Problems, **20** (2004), 1325-1348.

[16] T. Iwaniec and G. Martin, *Geometric Function Theory and Non-linear Analysis*, Oxford University Press Inc., New York, 2001.

[17] K. Knudsen, J.L. Mueller, and S. Siltanen, *Numerical solution method for the dbar-equation in the plane*, Journal of Computational Physics, **198** (2004), 500-517.

[18] O. Lehto and K.I. Virtanen, *Quasiconformal Mapping in the Plane*, 2nd ed., Springer-Verlag, New York, 1973.

[19] G. Nakamura and K. Yoshida, *Identification of a non-convex obstacle for acoustical scattering*, J. Inv. Ill-Posed Problems, **15** (2007), 1-14.

[20] J. Sylvester, *An anisotropic inverse boundary value problem.* Comm. Pure Appl. Math., **43** (1990), 201-232.

[21] G. Uhlmann and J.N. Wang, *Reconstructing discontinuities using complex geometrical optics solutions*, SIAM J. Appl. Math., **68** (2008), 1026-1044.

[22] G. Uhlmann, J.N. Wang, and C.T. Wu *Reconstruction of inclusions in an elastic body*, preprint.

Faculty of Science and Engineering, Doshisha University, Kyotanabe City, 610-0394 Japan
E-mail address: htakuwa@mail.doshisha.ac.jp

Department of Mathematics, University of Washington, Box 354305, Seattle, WA 98195-4350, USA.
E-mail address: gunther@math.washington.edu

Department of Mathematics, Taida Institute for Mathematical Sciences, and NCTS (Taipei), National Taiwan University, Taipei 106, Taiwan.
E-mail address: jnwang@math.ntu.edu.tw