INVERSE PROBLEMS IN THREE-BODY SCATTERING

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Scattering theory is the analysis of the motion of several interacting particles. Inverse problems in scattering theory seek the answer to the question: can one determine the interactions between particles by a scattering experiment? That is, if one shoots a number of particles at each other in an accelarator and observes the outcome, can one find out how the particles interact? In this note we attempt to explain this problem and the existing results in a somewhat informal fashion. However, we always give references for the reader to turn to if further details are desired.

This inverse question really consists of several different problems corresponding to different initial and final configuration of the particles. For example, one can shoot three electrons at each other, let them interact, and observe the outcome. Since the electrons cannot bind each other, we will have three (asymptotically) free particles as the outcome. This is three-cluster to three-cluster scattering: there were three clusters of particles (each consisting of a single electron) both initially and finally.

Alternatively, one can shoot an electron at an atom or ion. If the energy of the electron is high enough, the ion may break up e.g. into the nucleus and an electron (for the helium ion). This is two-cluster to three-cluster scattering: initially there were two clusters, namely the electron and the ion, and finally there are three, the helium nucleus, and the two electrons (taken separately). Or, the atom/ion may not break up, though it may become excited, and eventually one will have an atom/ion and an electron, just as initially. This is two-cluster to two-cluster scattering. If in addition the ion is in the same final and initial states, the problem may be instead considered as a two-body problem, with the ion and the electron comprising the two (now composite) particles; indeed, this is the usual way one thinks of such an experiment physically, at least at low energies. A natural question is then whether this is an accurate description, i.e. whether the more complete three-particle analysis gives the same predictions as the simplified two-particle approach.

Much is known about high energy problems due to the work of Enss and Weder [2, 3], Novikov [9] and Wang [17]. Namely, at high energies the potential (the interaction) is lower order than the Laplacian (the kinetic energy) in an appropriate sense. Thus, one can use the Born approximation to study inverse problems and recover information about the interactions. In this note we discuss fixed energy problems, and problems where the scattering matrix is known in a bounded interval of energies. Here the S-matrix encodes the outcome of a scattering experiment, and has several components corresponding to the various incoming and outgoing configurations as explained later.

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Before stating the results, we need to introduce some notation due to Agmon and Sigal. We work on $\mathbb{R}^n$, and $\Delta \geq 0$ is the positive Laplacian. We are also given a finite collection $\mathcal{X} = \{X_a : a \in I\}$ of linear subspaces X_a or $\mathbb{R}^n$, called the collision planes. We let $X^a = X_a^\perp$ be the orthocomplement of X_a in $\mathbb{R}^n$, so $\mathbb{R}^n = X_a \oplus X^a$. We write the corresponding coordinates as (x_a, x^a), and denote the projection to X^a by π^a. A many-body Hamiltonian is an operator of the form

$$H = \Delta + \sum_a (\pi^a)^* V_a,$$

where V_a is a real valued function on X^a in a certain class, for example V_a is a symbol on X^a of negative order: $V_a \in S^{-\rho}(X^a)$, $\rho > 0$. We drop the pull-back notation from now on and write $H = \Delta + \sum_a V_a$.

One usually assumes that $\mathcal{X}$ is closed under intersections, and $X_0 = \mathbb{R}^n \in \mathcal{X}$, $X_1 = \{0\} \in \mathcal{X}$ and $V_0 = 0$ (note that $X^0 = \{0\}$, so V_0 would simply play the role of the spectral parameter.) Here, to simplify the notation in the three-body case which is what we are considering mostly, we always assume that $X_0 = \mathbb{R}^n \notin \mathcal{X}$.

The main feature of many-body problems is that even if V_a decays at infinity on X^a, it does *not* decay at infinity in $\mathbb{R}^n$ since it is a constant along X_a as well as along its translates X_a', X_a'', so it does not decay if we go to infinity, say, along X_a.

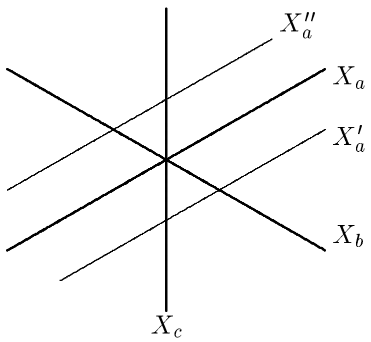

FIGURE 1. Collision planes X_a, X_b and X_c and translates X_a' and X_a'' of X_a. V_a is constant along X_a, is a (typically different) constant along X_a', etc., so it does not decay at infinity unless it is identically zero.

As an example, we recall that the actual N-body problem for d-dimensional particles takes the following form:

$$H = \sum_{i=1}^{N} \frac{\hbar^2}{2m_i} \Delta_{q_i} + \sum_{i<j} V_{ij}(q_i - q_j),$$

where $q_i \in \mathbb{R}^d$ is the position and m_i is the mass of particle i, while V_{ij} is the interaction between particles i and j, and $\hbar$ is Planck's constant. The collison planes are $X_{ij} = \{q_i = q_j\}$, as well as their intersections. Thus, at X_{ij} particles i and j are at the same place, hence the name 'collision plane'. Note that H is an operator on (functions on) $\mathbb{R}^{Nd}$, i.e. $n = Nd$.

In the two-body problem one actually has $H = \Delta_{q_1, q_2} + V_{12}(q_1 - q_2)$, i.e. V_{12} still does not decay at infinity, e.g. if one keeps $q_1 = q_2$ but lets $q_1 \to \infty$. However, one can easily remove the center of mass by performing a Fourier transform along X_{12}, and then one reduces the problem to the study of a Hamiltonian on X^{12} of the form $\Delta + V_{12}$ where V_{12} decays at infinity (we are working on X^{12}!). This reduced Hamiltonian can be considered as a perturbation of Δ, hence its analysis is rather simple.

The center of mass can also be removed in any actual many-body problem, but one still obtains a Hamiltonian with non-decaying potentials as before. For example, in a three-body problem one gets that $X_a \cap X_b = \{0\}$ if $a \neq b$, see Figure 1. (Recall that we do not include $X_0 = \mathbb{R}^n$ among the collision planes.)

We also need to introduce subsystem Hamiltonians to describe bound states such as atoms and ions. For simplicity, I only do this here in the three-body setting. Then a subsystem Hamiltonian H^a is an operator on (functions on) X^a, and it is given by $H^a = \Delta_{X^a} + V_a$. This is now a two-body Hamiltonian, i.e. V_a decays at infinity on X^a. The bound states, i.e. L^2 eigenvalues and eigenfunctions, of H^a are denoted by ϵ_α and ψ_α, so $(H^a - \epsilon_\alpha)\psi_\alpha = 0$. The pair (a, ψ_α) is called a channel. Note that $\epsilon_\alpha \leq 0$ for all α. We write $I_a = \sum_{b \neq a} V_b$ for the intercluster interaction. Thus I_a is a function on $\mathbb{R}^n$.

If we consider the two-cluster to two-cluster problem as a two-body problem by regarding the atom/ion as a single particle, we essentially work on X_a by projecting to a bound state ψ_α of H^a and using the effective interaction

$$V_{\alpha,\mathrm{eff}} = \int_{X^a} I_a |\psi_\alpha|^2 \, dx^a,$$

which is a function on X_a. For various reasons, including the results in high energy settings, it is clear that we can only hope to recover $V_{\alpha,\mathrm{eff}}$ (rather than I_a) by studying the corresponding (i.e. state α to itself) two-cluster to two-cluster scattering matrix.

We can now describe the S-matrices following [16]. We warn the reader that the normalization this gives is geometric as in [6], and is different from the standard wave-operator normalization. The difference is essentially given by pull-back by an antipodal map. Thus, free particles propagate in straight lines, i.e. they leave in the opposite direction from where they came, so in the geometric formulation, in the absence of interactions, the outgoing data are supported in the image of the support of the incoming data under the antipodal map. An incoming wave of energy λ in the channel α is a function of the form

$$u_{\alpha,-} = e^{-i\sqrt{\lambda - \epsilon_\alpha}|x_a|} |x_a|^{-\frac{\dim X_a - 1}{2}} g_{\alpha,-}(\frac{x_a}{|x_a|}) \psi_\alpha(x^a) + \text{lower order terms},$$

and an outgoing wave has the form

$$u_{\alpha,+} = e^{i\sqrt{\lambda - \epsilon_\alpha}|x_a|} |x_a|^{-\frac{\dim X_a - 1}{2}} g_{\alpha,+}(\frac{x_a}{|x_a|}) \psi_\alpha(x^a) + \text{lower order terms},$$

i.e. the sign of the phase has changed. Here $g_{\alpha,\pm}$ may be taken e.g. L^2 functions on S_a, the unit sphere in X_a, or ideally, at least one of them may be taken $\mathcal{C}^\infty$. One is then interested in generalized eigenfunctions of H, i.e. tempered distributions u on $\mathbb{R}^n$ that solves $(H - \lambda)u = 0$. A typical example is of the form

$$u = u_{\alpha,-} - (H - (\lambda + i0))^{-1}((H - \lambda)u_{\alpha,-}); \tag{0.1}$$

here the lower order terms may even be dropped without affecting u and $g_{\alpha,-}$ can be specified to be any smooth function on S_a. In general, even if the incoming data are in a single channel α, as in (0.1), the corresponding generalized eigenfunction u of H will have outgoing waves in all channels. The S-matrix $S_{\alpha\beta}(\lambda)$ picks out the component in channel β by projection in a certain sense, see [16]. Thus, $S_{\alpha\beta}(\lambda)$ maps functions on S_a, the unit sphere in X_a, to functions on S_b, by

$$S_{\alpha\beta}(\lambda)g_{\alpha,-} = g_{\beta,+}$$

for u as in (0.1). For example, the free-to-free (i.e. 3-cluster to 3-cluster in 3-body scattering) S-matrix $S_{00}(\lambda)$ maps functions on S_0, the unit sphere in $\mathbb{R}^n$, to functions on S_0, more precisely $S_{00}(\lambda) : L^2(S_0) \to L^2(S_0)$ is bounded.

Before stating many-body results, we recall that in two-body scattering for compactly supported, or indeed exponentially decaying, potentials the scattering matrix given at a single energy $\lambda > 0$ determines the potential [12, 8, 7, 14].

The first many-body result we state, slightly informally, is an 'old' direct result of the second author [15] and Hassell [5].

Theorem. *Suppose that H is a three-body Hamiltonian and the V_a are Schwartz on X^a for all a. Then $S_{00}(\lambda)$ is a finite sum of Fourier integral operators (FIOs) associated to the broken geodesic relation on S_0 to distance π. Its canonical relation corresponds to the various collision patterns. The principal symbol of the term corresponding to a single collision at X_a is given by, and in turn determines, the 2-body S-matrix of H^a at energies $\lambda' \in (0, \lambda)$.*

Remark. This result presumably extends to short range symbolic potentials, using the same methods, though it is technically more complicated to write down the argument in that case, and it has not been done.

An immediate corollary, when combined with two-body results (e.g. analyticity of the S-matrix in λ' and the Born approximation) is the following inverse result.

Corollary. *If the V_a decay exponentially and* $\dim X_a \geq 2$ *for all a then $S_{00}(\lambda)$ for a single value of λ determines all interactions.*

This result is analogous to the recovery of cracks in a material by directing sound waves at it and observing the singularities of the reflected waves, except the last step which uses two-body results to get the potentials from the two-body S-matrices. Below we sketch some reflecting trajectories. Note that there is only a finite number of possible collision patterns. The dynamics really takes place at infinity: nothing is singular at the collision planes, but it is easier to show it this way.

Remark. The theorem naturally raises the following question. Suppose that $H_a = \Delta_{X^a} + V_a$ is a two-body Hamiltonian and V_a is a Schwartz function on X^a, $\dim X^a \geq 2$. Suppose one knows its scattering matrix on a non-empty open interval of energies λ'. Can one recover V_a? To our knowledge, this question has not been answered.

The new result is in 2-cluster to 2-cluster scattering. To set it up, we first state a theorem of Skibsted [11].

Theorem. [11] *The two-cluster to two-cluster S-matrices have $\mathcal{C}^\infty$ Schwartz kernels, except the conormal singularity of $S_{\alpha\alpha}(\lambda)$ corresponding to free motion, i.e. at the graph of the antipodal map on S_a.*

Thus, principal symbol calculations do not help in this inverse problem. The new result is the following [13].

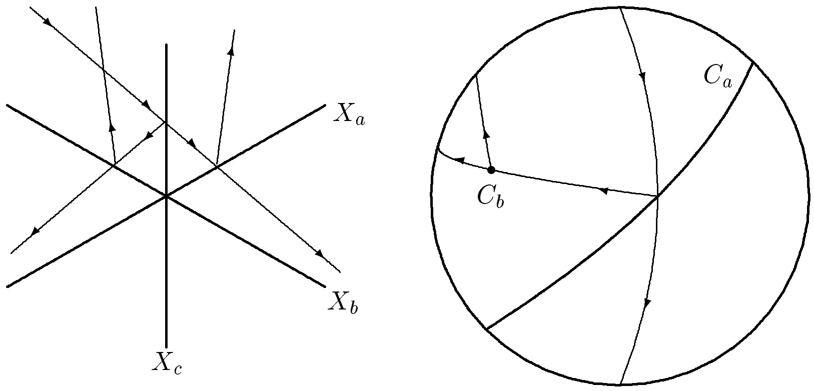

FIGURE 2. On the left, broken geodesics in $\mathbb{R}^n \setminus \{0\}$, $n = 2$, broken at the collision planes X_a, X_b and X_c. On the right, the projection of broken geodesics in $\mathbb{R}^n \setminus \{0\}$, $n = 3$, emanating from the north pole, to the unit sphere S_0, better understood as the sphere at infinity. The C_a, C_b are the intersection of the collision planes X_a, X_b with S_0; $\dim X_a = 2$, $\dim X_b = 1$.

Theorem. *Suppose that α is a channel of energy ϵ_α, V_a is a symbol of negative order (i.e. may be long range). For any $\mu > \dim X_a$ there exists $\delta > 0$ such that the following holds.*

Suppose that $\sup |(1+|x^b|)^\mu V_b(x^b)| < \delta$ *for all* $b \neq a$. *Suppose also that* $I \subset (\epsilon_\alpha, 0)$ *is a non-empty open set, and let*

$$R = 2\sqrt{\sup I - \epsilon_\alpha}.$$

Then $S_{\alpha'\alpha''}(\lambda)$ given for all $\lambda \in I$ and for all bound states α', α'' of H^a with $\epsilon_{\alpha'}, \epsilon_{\alpha''} < \sup I$, determines the Fourier transform of the effective interaction $V_{\alpha,\mathrm{eff}}$ in the ball of radius R centered at 0.

This theorem says that if the unknown interactions are small then the effective interaction can be determined from the knowledge of all S-matrices with incoming and outgoing data in the cluster a in the relevant energy range. In fact, near-forward information suffices as in two-body scattering, where this was observed recently by Novikov [10]. Also, if one is willing to take small R and α is the ground state of H^a, it suffices to know $S_{\alpha\alpha}(\lambda)$ to recover $V_{\alpha,\mathrm{eff}}$.

In case V_b decay exponentially on X^b for all $b \neq a$, then $V_{\alpha,\mathrm{eff}}$ decays exponentially on X_a, hence its Fourier transform is analytic, so $V_{\alpha,\mathrm{eff}}$ itself can be recovered from these S-matrices.

Remark. It is clear from the proof in [13] that there is a natural extension of this theorem to many-body scattering at low energies.

This result should extend to higher energies, i.e. $\sup I \leq 0$ is *not* expected to be essential. But it is hard to make R greater than $2\sqrt{-\epsilon_\alpha}$ even then. The reason is that our method relies on the construction of exponential solutions following Faddeev [4], Calderón [1], Sylvester and the first author [12] and Novikov and Khenkin [7], but in the three-body setting. One thus allows complex momenta $\rho \in \mathbb{C}(X_a)$, the complexification of X_a. Writing $\rho = z\nu + \rho_\perp$ with $|\nu| = 1$, $\rho_\perp \cdot \nu = 0$,

ρ, ν real, $z \in \mathbb{C}$, the atom/ion may break up for $|\rho_\perp| > \sqrt{-\epsilon_\alpha}$ even if $\lambda < 0$, i.e. where the conservation of energy does *not* allow this for *real* frequencies. On the other hand, one needs such large $\rho_\perp$ to recover $V_{\alpha,\mathrm{eff}}$ on larger balls.

We briefly indicate how the break-up can happen. One considers the conjugated Hamiltonian $e^{-i\rho\cdot x_a}(H-\lambda)e^{i\rho\cdot x_a}$ with $\rho \in \mathbb{C}(X_a)$ the complex frequency. The total energy of the system in state α is $\rho\cdot\rho + \epsilon_\alpha$ (kinetic+potential energy), and we are assuming that the total energy is λ, so

$$\rho\cdot\rho + \epsilon_\alpha = \lambda. \tag{0.2}$$

On the other hand, the energy surface of the conjugaed Hamiltonian in the free region is $(\xi+\rho)^2 - \lambda = 0$ where $\xi \in \mathbb{R}^n$. Since ρ is complex, this gives two real equations. Namely, we deduce that $|\xi|^2 + 2\,\mathrm{Re}\,\rho\cdot\xi + \rho^2 - \lambda = 0$ and $\mathrm{Im}\,\rho\cdot\xi = 0$. Here we used that $\rho^2 = \lambda - \epsilon_\alpha$ is real. Writing $\rho = z\nu + \rho_\perp$ as above, $\mathrm{Im}\,\rho\cdot\xi = 0$ gives (for z non-real) that $\nu\cdot\xi = 0$, hence $|\xi|^2 + 2\,\mathrm{Re}\,\rho\cdot\xi + \rho^2 - \lambda = 0$ yields that $(\xi+\rho_\perp)^2 = \epsilon_\alpha + \rho_\perp^2$. Here we used again (0.2). These equations have solutions if $\epsilon_\alpha + \rho_\perp^2 \geq 0$. i.e. if $|\rho_\perp| \geq \sqrt{-\epsilon_\alpha}$ as stated above. This means that the conjugated Hamiltonian allows asymptotically free particles at this energy, i.e. the ion may break up. We refer to [13] for details.

The most unfortunate restriction of the theorem is the smallness assumption on the unknown interactions. While $|\rho_\perp| < \sqrt{-\epsilon_\alpha}$ ensures that we cannot have three asymptotically free particles, it does not rule out the existence of different channels, associated to other 2-clusters, at complex frequencies. The presence of such channels would significantly complicate the analysis. In particular, the crucial analyticity in z would not be clear at all.

Finally, we return to the electron-ion scattering experiment, comparing its treatment as a two-body problem and as a three-body problem. If one only uses a fixed energy λ, and the potentials are exponentially decaying, in the two-body model the S-matrix determines the interaction, which is $V_{\alpha,\mathrm{eff}}$ in this model (here α is the ground state of the ion). No such result exists if we consider the ion as a composite particle, i.e. if the experiment is treated as a three-body problem, although our theorem shows that if the 2-cluster to 2-cluster S-matrix is known in an *interval* of energies and if the interactions are weak, then we can determine $V_{\alpha,\mathrm{eff}}$. Indeed, in the two-body fixed energy inverse result one has to let $\rho_\perp \to \infty$. If we consider the scattering experiment as a three-body problem, $\rho_\perp \to \infty$ causes serious complications, namely the possibility of the break up of the ion. This seems to limit the use of two-body results for composite particles, such as ions.

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