Inverse problems, invisibility, and artificial wormholes

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Abstract

We will describe recent theoretical and experimental progress on making objects invisible to electromagnetic waves. Maxwell's equations have transformation laws that allow for design of electromagnetic parameters that would steer light around a hidden region, returning it to its original path on the far side. Not only would observers be unaware of the contents of the hidden region, they would not even be aware that something was hidden. The object would have no shadow. New advances in metamaterials have given some experimental evidence that this indeed can be made possible at certain frequencies.

1 Introduction

There have recently been many studies [AE, GKLU1, MN, Le, PSS1, MBW, W] on the possibility, both theoretical and practical, of a region or object

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being shielded, or cloaked from detection via electromagnetic waves. The interest in cloaking was raised in particular in 2006 when it was realized that practical cloaking constructions are possible using so-called metamaterials which allow fairly arbitrary specification of electromagnetic material parameters. At the present moment such materials have been implemented at microwave frequencies [Sc]. On the practical limitations of cloaking, we note that, with current technology, above microwave frequencies the required metamaterials are difficult to fabricate and assemble, although research is presently progressing on metamaterial engineering at optical frequencies [Sh]. Furthermore, metamaterials are inherently prone to dispersion, so that realistic cloaking must currently be considered as occurring at a single wavelength, or very narrow range of wavelengths.

The theoretical considerations related to cloaking were introduced already in 2003, before the appearance of practical possibilities for cloaking. Indeed, the cloaking constructions in the zero frequency case, i.e., for electrostatics, were introduced as counterexamples in the study of inverse problems. In [GLU2, GLU3] it was shown that passive objects can be coated with a layer of material with a degenerate conductivity which makes the object undetectable by electrical impedance tomography (EIT), that is, in the electrostatic measurements. This gave counterexamples for uniqueness in the Calderón inverse problem for the conductivity equation. The counterexamples were motivated by consideration of certain degenerating families of Riemannian metrics, which in the limit correspond to singular conductivities, i.e., that are not bounded below or above. A related example of a complete but noncompact two-dimensional Riemannian manifold with boundary having the same Dirichlet-to-Neumann map as a compact one was given in [LTU].

Before discussing the recent results on cloaking and counterexamples in inverse problems, let us briefly discuss the positive results for inverse problems. The paradigm problem is Calderón's inverse problem, which is the question of whether an unknown conductivity distribution inside a domain in $\mathbb{R}^n$, modelling for example the human thorax, can be determined from voltage and current measurements made on the boundary. For isotropic conductivities this problem can be mathematically formulated as follows: Let Ω be the measurement domain, and denote by σ a bounded and strictly positive function describing the conductivity in Ω. In Ω the voltage potential u satisfies the equation

$$\nabla \cdot \sigma \nabla u = 0. \tag{1}$$

To uniquely fix the solution u it is enough to give its value, f, on the boundary. In the idealized case, one measures for all voltage distributions $u|_{\partial\Omega} = f$ on the boundary the corresponding current flux, $\nu \cdot \sigma \nabla u$, through the boundary, where ν is the exterior unit normal to $\partial\Omega$. Mathematically this amounts to the knowledge of the Dirichlet–Neumann map Λ corresponding to σ, i.e., the map taking the Dirichlet boundary values of the solution to (1) to the corresponding Neumann boundary values,

$$\Lambda : \quad u|_{\partial\Omega} \mapsto \nu \cdot \sigma \nabla u|_{\partial\Omega}.$$

Calderón's inverse problem is then to reconstruct σ from Λ. The problem was originally proposed by Calderón [C] in 1980. Sylvester and Uhlmann [SyU] proved unique identifiability of the conductivity in dimensions three and higher for isotropic conductivities which are C^∞–smooth, and Nachman gave a reconstruction method [N]. In three dimensions or higher unique identifiability of the conductivity is known for conductivities with 3/2 derivatives [BT], [PPU]. In two dimensions the first global result for C^2 conductivities is due to Nachman [N1]. This was improved in [BU] to Lipschitz conductivities. Astala and Päivärinta showed in [AP] that uniqueness holds also for general isotropic conductivities merely in L^∞.

The Calderón problem with an anisotropic, i.e., matrix-valued, conductivity that is uniformly bounded from above and below has been studied in two dimensions [S, N1, SuU, ALP] and in three dimensions or higher [LaU, LeU, LTU]. For example, for the anisotropic inverse conductivity problem in the two dimensional case, it is known that Cauchy data determines the conductivity tensor up to a diffeomorphism $F : \overline{\Omega} \longrightarrow \overline{\Omega}$. Thus, the inverse problem is not uniquely solvable, but the non-uniqueness of the problem can be characterized. This makes it possible, e.g., to find the unique conductivity that is closest to isotropic ones [KLO]. Another related inverse problem is the Gel'fand problem, which uses boundary measurements at all frequencies, rather than at a fixed one. For this problem, uniqueness results are available; see, e.g., [BeK, KK], with a detailed exposition in [KKL].

We emphasize that for the above positive results for inverse problems it is assumed that the eigenvalues of the conductivity are bounded below and above by positive constants. Thus, a key point in the current works on invisibility that allows one to avoid the known uniqueness theorems is the lack of positive lower and/or upper bounds on the eigenvalues of these symmetric tensor fields.

For Maxwell's equations the inverse problem with the isotropic permittivity ε and permeability μ and the data given at one frequency was solved in [OPS]. The inverse problem with the anisotropic permittivity and permeability has been studied with data given at all frequencies (or in the time domain) when the permittivity and permeability tensors ε and μ are conformal to each other, i.e., multiples of each other by a positive scalar function; this condition has been studied in detail in [KLS]. For Maxwell's equations in the time domain, this condition corresponds to polarization-independent wave velocity. This seemingly special condition arises quite naturally also in the invisibility constructions, since the pushforward $(\widetilde{\varepsilon}, \widetilde{\mu})$ of an isotropic pair (ε, μ) by a diffeomorphism need not be isotropic but does satisfy this conformality.

Let us now return to the recent results on cloaking and the counterexamples for inverse problems. In 2006, several cloaking constructions were proposed. The constructions in [Le] are based on conformal mapping in two dimensions and are justified via change of variables on the exterior of the cloaked region. At the same time, [PSS1] proposed a cloaking construction for Maxwell's equations based on a singular transformation of the original space, again observing that, outside the cloaked region, the solutions of the homogeneous Maxwell equations in the original space become solutions of the transformed equations. The transformations used there are the same as used in [GLU2, GLU3] in the context of Calderón's inverse conductivity problem. The paper [PSS2] contained analysis of cloaking on the level of ray-tracing, full wave numerical simulations were discussed in [CPSSP], and the cloaking experiment at 8.5Ghz is in [Sc].

The electromagnetic material parameters used in cloaking constructions are degenerate and, due to the degeneracy of the equations at the surface of the cloaked region, it is important to consider rigorously (weak) solutions to Maxwell's equations on *all* of the domain, not just the exterior of the cloaked region. This analysis was carried out in [GKLU1]. There, various constructions for cloaking from observation are analyzed on the level of physically meaningful electromagnetic waves, i.e., finite energy distributional solutions of the equations. In the analysis of the problem, it turns out that the cloaking structure imposes hidden boundary conditions on such waves at the surface of the cloak. When these conditions are overdetermined, finite energy solutions typically do not exist. The time-domain physical interpretation of this was at first not entirely clear, but it now seems to be intimately related with blow-up of the fields, which would may compromise the desired cloaking effect [GKLU3]. We review the results here and give the possible remedies to restore invisibility.

We note that [GLU2, GLU3] gave, in dimensions $n \geq 3$, counterexamples to uniqueness for the inverse conductivity problem. Such counterexamples have now also been given and studied further in two dimensional case [KSVW, ALP2].

2 Basic constructions

The material parameters of electromagnetism, the electrical permittivity, $\varepsilon(x)$; magnetic permeability, $\mu(x)$; and the conductivity $\sigma(x)$ can be considered as coordinate invariant objects. If $F : \Omega_1 \longrightarrow \Omega_2, \quad y = F(x)$, is a diffeomorphism between domains in $\mathbb{R}^n$, then $\sigma(x) = [\sigma^{jk}(x)]_{j,k=1}^n$ on Ω_1 pushes forward to $(F_*\sigma)(y)$ on Ω_2, given by

$$(F_*\sigma)^{jk}(y) = \frac{1}{\det\left[\frac{\partial F}{\partial x}(x)\right]} \sum_{p,q=1}^{n} \frac{\partial F^j}{\partial x^p}(x) \frac{\partial F^k}{\partial x^q}(x)\sigma^{pq}(x)\Bigg|_{x=F^{-1}(y)}. \tag{2}$$

The same transformation rule is valid for permittivity ε and permeability μ. It was observed by Luc Tartar (see [KV]) that it follows that if F is a diffeomorphism of a domain Ω fixing $\partial\Omega$, then the conductivity equations with the conductivities σ and $\widetilde{\sigma} := F_*\sigma$ have the same Dirichlet-to-Neumann map, producing infinite-dimensional families of indistinguishable conductivities. This can already be considered as a weak form of invisibility, with distinct conductivities being indistinguishable by external observations; however, nothing has been hidden yet.

On the other hand, a Riemannian metric $g = [g_{jk}(x)]_{j,k=1}^n$ is a covariant symmetric two-tensor. Remarkably, in dimension three or higher, a material parameter tensor and a Riemannian metric can be associated with each other by

$$\sigma^{jk} = |g|^{1/2} g^{jk}, \quad \text{or} \quad g^{jk} = |\sigma|^{2/(n-2)} \sigma^{jk}, \tag{3}$$

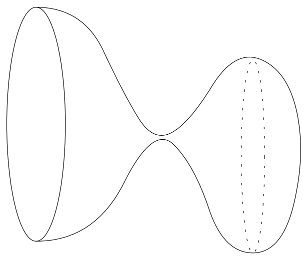


Figure 1: A family of manifolds that develops a singularity when the width of the neck connecting two parts goes to zero.

where $[g^{jk}] = [g_{jk}]^{-1}$ and $|g| = \det(g)$. Using this correspondence, examples of singular anisotropic conductivities in $\mathbb{R}^n, n \geq 3$, that are indistinguishable from a constant isotropic conductivity, in that they have the same Dirichlet-to-Neumann map, are given in [GLU3]. This construction is based on degenerations of Riemannian metrics, whose singular limits can be considered as coming from singular changes of variables. If one considers Figure 1, where the "neck" of the surface (or a manifold in the higher dimensional cases) is pinched, the manifold contains in the limit a pocket about which the boundary measurements do not give any information. If the collapsing of the manifold is done in an appropriate way, in the limit we have a Riemannian manifold which is indistinguishable from flat surface. This can be considered as a singular conductivity that appears the same as a constant conductivity to all boundary measurements.

To consider the above precisely, let $B(0,R) \subset \mathbb{R}^3$ be an open ball with center 0 and radius R. We use in the sequel the set $N = B(0,2)$, decomposed to two parts, $N_1 = B(0,2) \setminus \overline{B}(0,1)$ and $N_2 = B(0,1)$. Let $\Sigma = \partial N_2$ the the interface (or "cloaking surface") between N_1 and N_2.

We use also a "copy" of the ball $B(0,2)$, with the notation $M_1 = B(0,2)$. Let $g_{jk} = \delta_{jk}$ be the Euclidian metric in M_1 and let $\gamma = 1$ be the corresponding homogeneous conductivity. Define a singular transformation

$$F : M_1 \setminus \{0\} \to N_1, \qquad F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}, \quad 0 < |x| \leq 2. \tag{4}$$

The pushforward $\widetilde{g} = F_* g$ of the metric g in F is the metric in N_1 given by

$$(F_* g)_{jk}(y) = \sum_{p,q=1}^{n} \frac{\partial F^p}{\partial x^j}(x) \frac{\partial F^q}{\partial x^k}(x) g_{pq}(x) \bigg|_{x=F^{-1}(y)}. \tag{5}$$

We use it to define a singular conductivity

$$\widetilde{\sigma} = \begin{cases} |\widetilde{g}|^{1/2} \widetilde{g}^{jk} & \text{for } x \in N_1, \\ \delta^{jk} & \text{for } x \in N_2 \end{cases} \tag{6}$$

in N. Then, denoting by $(r, \phi, \theta) \mapsto (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta)$ the spherical coordinates, we have

$$\widetilde{\sigma} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}, \quad 1 < |x| \leq 2.$$

This means that in the Cartesian coordinates the conductivity $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma}(x) = 2(I - P(x)) + 2|x|^{-2}(|x|-1)^2 P(x), \quad 1 < |x| < 2,$$

where I is the identity matrix and $P(x) = |x|^{-2} x x^t$ is the projection to the radial direction. We note that the anisotropic conductivity $\widetilde{\sigma}$ is singular on Σ in the sense that it is not bounded from below by any positive multiple of I. (See [KSVW] for a similar calculation for $n = 2$.)

Consider now the *Cauchy data* of all $H^1(N)$-solutions of the conductivity equation corresponding to $\widetilde{\sigma}$, that is,

$$C_1(\widetilde{\sigma}) = \{(u|_{\partial N}, \nu\cdot \widetilde{\sigma}\nabla u|_{\partial N}) \ : \ u \in H^1(N),\ \nabla\cdot \widetilde{\sigma}\nabla u = 0\},$$

where ν is the Euclidian unit normal vector of ∂N.

Theorem 2.1 *([GLU3]) The Cauchy data of H^1-solutions for all conductivities $\widetilde{\sigma}$ and γ on N coincide, that is, $C_1(\widetilde{\sigma}) = C_1(\gamma)$.*

This means that all boundary measurements for the homogeneous conductivity $\gamma = 1$ and the degenerated conductivity $\widetilde{\sigma}$ are the same. In the figure below there are analytically obtained solutions on a disc with metric $\widetilde{\sigma}$

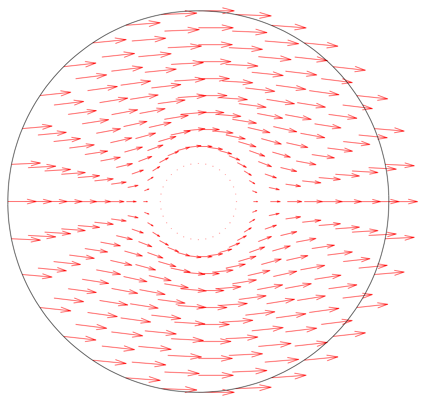


Figure 2: Analytic solutions for the currents

As seen in the figure, no currents appear near the center of the disc, so that if the conductivity is changed near the center, the measurements on the boundary ∂N do not change.

We note that a similar type of theorem is valid also for a more general class of solutions. Consider an unbounded quadratic form, A in $L^2(N)$,

$$A_{\widetilde{\sigma}}[u,v] = \int_N \widetilde{\sigma}\nabla u \cdot \nabla v \, dx$$

defined for $u, v \in \mathcal{D}(A_{\widetilde{\sigma}}) = C_0^\infty(N)$. Let $\overline{A}_{\widetilde{\sigma}}$ be the closure of this quadratic form and say that

$$\nabla \cdot \widetilde{\sigma}\nabla u = 0 \quad \text{in } N$$

is satisfied in the finite energy sense if there is $u_0 \in H^1(N)$ supported in N_1 such that $u - u_0 \in \mathcal{D}(A_{\widetilde{\sigma}})$ and

$$\overline{A}_{\widetilde{\sigma}}[u - u_0, v] = -\int_N \widetilde{\sigma}\nabla u_0 \cdot \nabla v \, dx, \quad \text{for all } v \in \mathcal{D}(\overline{A}_{\widetilde{\sigma}}).$$

Then Cauchy data set of the finite energy solutions, denoted

$$C_f(\widetilde{\sigma}) = \{(u|_{\partial N}, \nu \cdot \widetilde{\sigma}\nabla u|_{\partial N}) \ : \ u \text{ is finite energy solution of } \nabla \cdot \widetilde{\sigma}\nabla u = 0\}$$

coincides with $C_f(\gamma)$. Using the above more general class of solutions, one can consider the non-zero frequency case,

$$\nabla \cdot \widetilde{\sigma} \nabla u = \lambda u,$$

and show that the Cauchy data set of the finite energy solutions to the above equation coincides with the corresponding Cauchy data set for γ, cf. [GKLU1].

3 Maxwell's equations

In what follows, we treat Maxwell's equations in non-conducting media, that is, for which $\sigma = 0$. We consider the electric and magnetic fields, E and H, as differential 1-forms, given in some local coordinates by

$$E = E_j(x) dx^j, \quad H = H_j(x) dx^j.$$

For 1-form $E(x) = E_1(x)dx^1 + E_2(x)dx^2 + E_3(x)dx^3$ we define the push-forward of E in F, denoted $\widetilde{E} = F_* E$, by

$$\begin{aligned} \widetilde{E}(\widetilde{x}) &= \widetilde{E}_1(\widetilde{x}) d\widetilde{x}^1 + \widetilde{E}_2(\widetilde{x}) d\widetilde{x}^2 + \widetilde{E}_3(\widetilde{x}) d\widetilde{x}^3 \\ &= \sum_{j=1}^{3} \Big(\sum_{k=1}^{3} (DF^{-1})_j^k(\widetilde{x})\, E_k(F^{-1}(\widetilde{x})) \Big) d\widetilde{x}^j, \quad \widetilde{x} = F(x). \end{aligned}$$

A similar kind of transformation law is valid for 2-forms. We interpret the curl operator for 1-forms in $\mathbb{R}^3$ as being the exterior derivative, d. Maxwell's equations then have the form

$$\operatorname{curl} H = -ikD + J, \quad \operatorname{curl} E = ikB$$

where we consider the D and B fields and the external current J (if present) as 2-forms. The constitutive relations are

$$D = \varepsilon E, \quad B = \mu H,$$

where the material parameters ε and μ are linear maps mapping 1-forms to 2-forms.

Let g be a Riemannian metric in $\Omega \subset \mathbb{R}^3$. Using the metric g, we define a specific permittivity and permeability by setting

$$\varepsilon^{jk} = \mu^{jk} = |g|^{1/2} g^{jk}.$$

To introduce the material parameters $\widetilde{\varepsilon}(x)$ and $\widetilde{\mu}(x)$ that make cloaking possible, we consider the map F given by (4), the Euclidean metric g in M_1 and $\widetilde{g} = F_* g$ in N_1 as before, and define the singular permittivity and permeability by the formula analogous to (6),

$$\varepsilon^{jk} = \mu^{jk} = \begin{cases} |\widetilde{g}|^{1/2} \widetilde{g}^{jk} & \text{for } x \in N_1, \\ \delta^{jk} & \text{for } x \in N_2. \end{cases} \tag{7}$$

These material parameters are singular on Σ, requiring that what it means for fields $(\widetilde{E}, \widetilde{H})$ to form a solution to Maxwell's equations must be defined carefully.

3.1 Definition of solutions of Maxwell equations

Since the material parameters $\widetilde{\varepsilon}$ and $\widetilde{\mu}$ are again singular, we need to define solutions carefully.

Definition 3.1 *We say that $(\widetilde{E}, \widetilde{H})$ is a finite energy solution to Maxwell's equations on N,*

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{on } N, \tag{8}$$

if $\widetilde{E}$, $\widetilde{H}$ are one-forms and $\widetilde{D} := \widetilde{\varepsilon}\,\widetilde{E}$ and $\widetilde{B} := \widetilde{\mu}\,\widetilde{H}$ two-forms in N with $L^1(N, dx)$-coefficients satisfying

$$\|\widetilde{E}\|^2_{L^2(N,|\widetilde{g}|^{1/2} dV_0(x))} = \int_N \widetilde{\varepsilon}^{jk}\, \widetilde{E}_j\, \overline{\widetilde{E}_k}\, dV_0(x) < \infty, \tag{9}$$

$$\|\widetilde{H}\|^2_{L^2(N,|\widetilde{g}|^{1/2} dV_0(x))} = \int_N \widetilde{\mu}^{jk}\, \widetilde{H}_j\, \overline{\widetilde{H}_k}\, dV_0(x) < \infty; \tag{10}$$

where dV_0 is the standard Euclidean volume, $(\widetilde{E}, \widetilde{H})$ is a classical solution of Maxwell's equations on a neighborhood $U \subset \overline{N}$ of ∂N:

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\varepsilon(x)\widetilde{E} + \widetilde{J} \quad \text{in } U,$$

and finally,

$$\int_N ((\nabla \times \widetilde{h}) \cdot \widetilde{E} - ik\widetilde{h} \cdot \widetilde{\mu}(x)\widetilde{H})\, dV_0(x) = 0,$$
$$\int_N ((\nabla \times \widetilde{e}) \cdot \widetilde{H} + \widetilde{e} \cdot (ik\widetilde{\varepsilon}(x)\widetilde{E} - \widetilde{J}))\, dV_0(x) = 0$$

for all $\widetilde{e}, \widetilde{h} \in C_0^\infty(\Omega^1 N)$.

Here, $C_0^\infty(\Omega^1 N)$ denotes smooth 1-forms on N whose supports do not intersect ∂N, and the inner product "$\cdot$" denotes the Euclidean inner product.

Surprisingly, the finite energy solutions do not exists for generic currents. To consider this, let M be the disjoint union of a ball $M_1 = B(0,2)$ and a ball $M_2 = B(0,1)$. These will correspond to sets N, N_1, N_2 after an appropriate changes of coordinates. We thus consider a map $F : M \setminus \{0\} = (M_1 \setminus \{0\}) \cup M_2 \to N \setminus \Sigma$, where F mapping $M_1 \setminus \{0\}$ to N_1 is the the map defined by formula (4) and F mapping M_2 to N_2 as the identity map.

Theorem 3.2 *([GKLU1]) Let E and H be 1-forms with measurable coefficients on $M \setminus \{0\}$ and $\widetilde{E}$ and $\widetilde{H}$ be 1-forms with measurable coefficients on $N \setminus \Sigma$ such that $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$. Let J and $\widetilde{J}$ be 2-forms with smooth coefficients on $M \setminus \{0\}$ and $N \setminus \Sigma$, that are supported away from $\{0\}$ and Σ. Then the following are equivalent:*

1. *The 1-forms $\widetilde{E}$ and $\widetilde{H}$ on N satisfy Maxwell's equations*

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{on } N, \tag{11}$$
$$\nu \times \widetilde{E}|_{\partial N} = f$$

   in the sense of Definition 3.1.

2. *The forms E and H satisfy Maxwell's equations on M,*

$$\nabla \times E = ik\mu(x)H, \quad \nabla \times H = -ik\varepsilon(x)E + J \quad \text{on } M_1, \tag{12}$$
$$\nu \times E|_{\partial M_1} = f$$

   and

$$\nabla \times E = ik\mu(x)H, \quad \nabla \times H = -ik\varepsilon(x)E + J \quad \text{on } M_2 \tag{13}$$

with Cauchy data

$$\nu \times E|_{\partial M_2} = b^e, \quad \nu \times H|_{\partial M_2} = b^h \tag{14}$$

that satisfies $b^e = b^h = 0$.

Moreover, if E *and* H *solve (12), (13), and (14) with non-zero* b^e *or* b^h*, then the fields* $\widetilde{E}$ *and* $\widetilde{H}$ *are not solutions of Maxwell equations on* N *in the sense of Definition 3.1.*

The above theorem can be interpreted by saying that the cloaking of active objects is difficult, as the idealized model with non-zero currents present within the region to be cloaked, leads to non-existence of finite energy distributional solutions. We find two ways of dealing with this difficulty. One is to simply augment the above coating construction around a ball by adding a perfect electrical conductor (PEC) lining at Σ, so that $\nu \times \widetilde{E} = 0$ at the inner surface of Σ, i.e., when approaching Σ from N_2. Physically, this corresponds to a surface current J along Σ which shields the interior of N_2 of N and make the object inside the coating material to appear like a passive object. Other boundary conditions making the problem solvable in some sense, using a different definition based on self-adjoint extensions of the operators, have been recently characterized in [W, W2]. Alternatively to considering a boundary condition on Σ, one can introduce a more elaborate construction, which we refer to as the *double coating.* Mathematically, this corresponds to a singular Riemannian metric which degenerates in the same way as one approaches Σ from both sides; physically it would correspond to surrounding both the inner and outer surfaces of Σ with appropriately matched metamaterials.

4 Cloaking an infinite cylindrical domain

In the following we change the geometrical situation where we do our considerations, and redefine the meaning of the used notations.

We consider next an infinite cylindrical domain. Below, $B_2(0,r) \subset \mathbb{R}^2$ is Euclidian disc with center 0 and radius r. Let us use in the following the notations $N = B_2(0,2) \times \mathbb{R}$, $N_1 = (B_2(0,2) \setminus B_2(0,1)) \times \mathbb{R}$, and $N_2 = B_2(0,1) \times \mathbb{R}$. Moreover, let M be the disjoint union of $M_1 = B_2(0,2) \times \mathbb{R}$ and $M_2 = B_2(0,1) \times \mathbb{R}$. Finally, let us denote in this section $\Sigma = \partial B_2(0,1) \times \mathbb{R}$,

$L = \{(0,0)\} \times \mathbb{R} \subset M_1$. We define the map $F : M \setminus L \to N \setminus \Sigma$ in cylindrical coordinates by

$$F(r, \theta, z) = (1 + \frac{r}{2}, \theta, z), \quad \text{on } M_1 \setminus L,$$
$$F(r, \theta, z) = (r, \theta, z), \quad \text{on } M_2.$$

Again, let g be the Euclidian metric on M, and $\varepsilon = 1$ and $\mu = 1$ be homogeneous material parameters in M. Using map F we define $\widetilde{g} = F_* g$ in $N \setminus \Sigma$ and define $\widetilde{\varepsilon}$ and $\widetilde{\mu}$ as in formula (7). By finite energy solutions of Maxwell's equations on N we will mean one-forms $\widetilde{E}$ and $\widetilde{H}$ satisfying the the conditions analogous to Definition 3.1. Next, let us denote by $u|_{\Sigma-}$ the trace of u defined on M_2 at the boundary ∂M_2. The proof of [GKLU1, Thm. 7.1] yields the following result:

Theorem 4.1 *Let E and H be 1-forms with measurable coefficients on $M \setminus L$ and $\widetilde{E}$ and $\widetilde{H}$ be 1-forms with measurable coefficients on $N \setminus \Sigma$ such that $E = F^* \widetilde{E}$, $H = F^* \widetilde{H}$. Let J and $\widetilde{J}$ be 2-forms with smooth coefficients on $M \setminus L$ and $N \setminus \Sigma$, that are supported away from L and Σ, respectively.*

Then the following are equivalent:

1. *On N, the 1-forms $\widetilde{E}$ and $\widetilde{H}$ satisfy Maxwell's equations*

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{in } N, \tag{15}$$
$$\nu \times \widetilde{E}|_{\partial N} = f$$

   and $\widetilde{E}$ and $\widetilde{H}$ are finite energy solutions.

2. *On M, the forms E and H are classical solutions to Maxwell's equations on M, with data*

$$\begin{aligned} b_1^e &= \zeta \cdot E|_L, & b_2^e &= \zeta \cdot E|_{\Sigma-}, \\ b_1^h &= \zeta \cdot H|_L, & b_2^h &= \zeta \cdot H|_{\Sigma-}, \\ t_2^e &= \eta \cdot E|_{\Sigma-}, & t_2^h &= \eta \cdot H|_{\Sigma-} \end{aligned} \tag{16}$$

   that satisfy

$$b_2^e(r, \theta, z)|_{r=1} = b_1^e(z), \quad b_2^h(r, \theta, z)|_{r=1} = b_1^h(z), \text{ and } t_2^e = t_2^h = 0 \tag{17}$$

   for all $z \in \mathbb{R}$ and $\theta \in S^1$. Here, $\zeta = \partial_z$ and $\eta = \partial_\theta$ are the vertical and the angular vector fields tangential to Σ, correspondingly.

Moreover, if E and H solve Maxwell's equations on M with the boundary values (16) that do not satisfy (17), then then the fields $\widetilde{E}$ and $\widetilde{H}$ are not finite energy solutions of Maxwell equations on N.

Further analysis and numerical simulations, exploring the consequences of this non-existence result for cloaking, can be found in [GKLU3].

5 Cloaking a cylinder with the Soft-and-Hard boundary condition

Next, we consider N_2 as an obstacle, while the domain N_1 is equipped with a metric corresponding to the above coating in the cylindrical geometry. Motivated by the conditions at Σ in the previous section, we impose the soft-and-hard surface (SHS) boundary condition on the boundary of the obstacle. In classical terms, the SHS condition on a surface Σ [HLS, Ki1] is

$$\zeta \cdot E|_\Sigma = 0 \quad \text{and} \quad \zeta \cdot H|_\Sigma = 0,$$

where $\zeta = \zeta(x)$ is a tangential vector field on Σ, that is, $\zeta \times \nu = 0$. In other words, the part of the tangential component of the electric field E that is parallel to ζ vanishes, and the same is true for the magnetic field H. This was originally introduced in antenna design and can be physically realized by having a surface with thin parallel gratings filled with dielectric material [Ki1, Ki2, Li, HLS]. Here, we consider this boundary condition when ζ is the vector field $\eta = \partial_\theta$, that is, the angular vector field that is tangential to Σ.

To this end, let us give still one more definition of weak solutions, appropriate for this construction. We consider only solutions on the set N_1; nevertheless, we continue to denote $\partial N = \partial N_1 \setminus \Sigma$.

Definition 5.1 *We say that the 1-forms $\widetilde{E}$ and $\widetilde{H}$ are* finite energy solutions *of Maxwell's equations on N_1 with the soft-and-hard (SH) boundary conditions on Σ,*

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{on } N_1, \tag{18}$$

$$\eta \cdot \widetilde{E}|_\Sigma = 0, \quad \eta \cdot \widetilde{H}|_\Sigma = 0, \tag{19}$$

$$\nu \times \widetilde{E}|_{\partial N} = f,$$

if $\widetilde{E}$ and $\widetilde{H}$ are 1-forms on N_1 and $\widetilde{\varepsilon}\widetilde{E}$ and $\widetilde{\mu}\widetilde{H}$ are 2-forms with measurable coefficients satisfying

$$\|\widetilde{E}\|^2_{L^2(N_1,|\widetilde{g}|^{1/2}dV_0)} = \int_{N_1} \widetilde{\varepsilon}^{jk}\, \widetilde{E}_j\, \overline{\widetilde{E}_k}\, dV_0(x) < \infty, \tag{20}$$

$$\|\widetilde{H}\|^2_{L^2(N_1,|\widetilde{g}|^{1/2}dV_0)} = \int_{N_1} \widetilde{\mu}^{jk}\, \widetilde{H}_j\, \overline{\widetilde{H}_k}\, dV_0(x) < \infty; \tag{21}$$

Maxwell's equation are valid in the classical sense in a neighborhood U of ∂N:

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\varepsilon(x)\widetilde{E} + \widetilde{J} \quad \textit{in } U,$$
$$\nu \times \widetilde{E}|_{\partial N} = f;$$

and finally,

$$\int_{N_1} ((\nabla \times \widetilde{h}) \cdot \widetilde{E} - ik\widetilde{h} \cdot \widetilde{\mu}(x)\widetilde{H})\, dV_0(x) = 0,$$
$$\int_{N} ((\nabla \times \widetilde{e}) \cdot \widetilde{H} + \widetilde{e} \cdot (ik\widetilde{\varepsilon}(x)\widetilde{E} - \widetilde{J}))\, dV_0(x) = 0,$$

for all $\widetilde{e}, \widetilde{h} \in C_0^\infty(\Omega^1 N_1)$ satisfying

$$\eta \cdot \widetilde{e}|_\Sigma = 0, \quad \eta \cdot \widetilde{h}|_\Sigma = 0. \tag{22}$$

We then have the following invisibility result.

Theorem 5.2 *([GKLU1]) Let E and H be 1-forms with measurable coefficients on $M_1 \setminus L$ and $\widetilde{E}$ and $\widetilde{H}$ be 1-forms with measurable coefficients on N_1 such that $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$. Let J and $\widetilde{J}$ be 2-forms with smooth coefficients on $M_1 \setminus L$ and $N_1 \setminus \Sigma$, that are supported away from L and Σ.*

Then the following are equivalent:

1. *On N_1, the 1-forms $\widetilde{E}$ and $\widetilde{H}$ satisfy Maxwell's equations with SH boundary conditions in the sense of Definition 5.1.*

2. *On M_1, the forms E and H are classical solutions of Maxwell's equations,*

$$\nabla \times E = ik\mu(x)H, \quad \textit{in } M_1 \tag{23}$$
$$\nabla \times H = -ik\varepsilon(x)E + J, \quad \textit{in } M_1,$$
$$\nu \times E|_{\partial M_1} = f.$$

This result implies that when the surface Σ is lined with a material implementing the SHS boundary condition, the finite energy distributional solutions exist for all incoming waves.

6 Artificial wormholes

Cloaking a ball or cylinder are particularly extreme examples of what has come to be known as *transformation optics* in the physics literature, and other interesting effects are possible. We sketch the construction of artificial electromagnetic wormholes, introduced in [GKLU2, GKLU4]. Consider first as in Fig. 3 a 3-dimensional wormhole manifold (or handlebody) $M = M_1 \# M_2$ where the components

$$\begin{aligned} M_1 &= \mathbb{R}^3 \setminus (B(O,1) \cup B(P,1)), \\ M_2 &= \mathbb{S}^2 \times [0,1] \end{aligned}$$

are glued together smoothly.

An optical device that acts as a wormhole for electromagnetic waves at a given frequency k can be constructed by starting with a two-dimensional finite cylinder

$$T = \mathbb{S}^1 \times [0,L] \subset \mathbb{R}^3$$

and taking its neighborhood $K = \{x \in \mathbb{R}^3 : \ \mathrm{dist}(x,T) < \rho\}$, where $\rho > 0$ is small enough and $N = \mathbb{R}^3 \setminus K$. Let us put on ∂K the SHS boundary condition and cover K with "invisibility cloaking material", that in the boundary normal coordinates around K has the same representation as $\widetilde{\varepsilon}$ and $\widetilde{\mu}$ when cloaking an infinite cylinder. Finally, let

$$U = \{x : \ \mathrm{dist}(x,K) > 1\} \subset \mathbb{R}^3.$$

The set U can be considered both as a subset of $N \subset \mathbb{R}^3$ and the wormhole manifold M, $U \subset M_1$. Then all measurements of fields E and H in $U \subset M$ and $U \subset N$ coincide with currents that are supported in U, that is, thus $(N, \widetilde{\varepsilon}, \widetilde{\mu})$ behaves as the wormhole M in all external measurements.

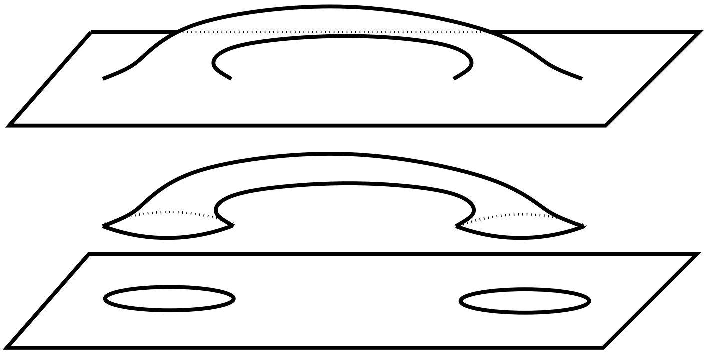
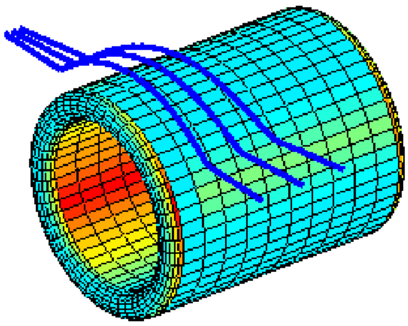


Figure 3: A schematic figure of two dimensional wormhole construction by gluing surfaces. Note that in the artificial wormhole construction components are three dimensional.



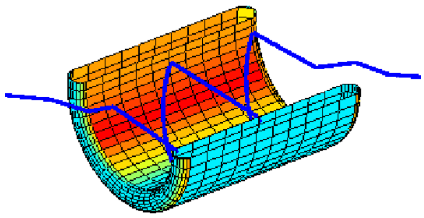


Fig. 4 (a) Rays travelling outside. (b) A ray travelling inside.

In Fig. 4, we give ray-tracing simulations in and near the wormhole. The obstacle in the figures is K, and the metamaterial corresponding to $\widetilde{\varepsilon}$ and $\widetilde{\mu}$ is not shown.

7 A general framework for singular transformation optics

We now formulate an informal general principle, rigorously established for and governing all of the above cloaking and wormhole constructions. By appropriate choice of the components and transformations, this will allow one to design describe new optical devices having interesting and useful effects on electromagnetic waves.

For the general construction, we start with an "abstract" 3-manifold M (possibly with a boundary), which is the disjoint union of components $M_j, 1 \leq j \leq p$. One also specifies (possibly empty) 0- or 1-dimensional submanifolds $\gamma_j \subset M_j$, which are the points or curves to be blown up. Then, one decomposes $\mathbb{R}^3 = N$, the "device", into a disjoint union of $N_j, 1 \leq j \leq p$, with (possibly empty) cloaking surfaces $\Sigma_j \subset \partial N_j$. Finally, one specifies singular transformations $F_j : M_j \setminus \gamma_j \longrightarrow N_j \setminus \Sigma_j$. On each M_j one specifies a Riemannian metric G_j, corresponding to permittivity and permeabilty ϵ_j, μ_j, which get pushed forward to material parameters $\widetilde{\epsilon}_j, \widetilde{\mu}_j$, singular at Σ_j, which we denote by $(\widetilde{\epsilon}, \widetilde{\mu})$ on N. If one has designed the device correctly, then the following should hold:

"Metatheorem about metamaterials"

There is a one-to-one correspondence F_* between the solutions (E, H) to Maxwell's equations for (ϵ, μ) on M satisfying certain bounadry conditions and the finite energy distributional solutions $(\widetilde{E}, \widetilde{H})$ for Maxwell's equations for $(\widetilde{\epsilon}, \widetilde{\mu})$ on N satisfying certain boundary conditions at the cloaking surfaces Σ_j.

The resulting pairs (E, H) and $(\widetilde{E}, \widetilde{H})$ are indistinguishable by external measurements, and thus EM waves passing through the concrete optical device N in $\mathbb{R}^3$, which can be (at least approximately) realized at some frequency k by a metamterial construction with appropriate linings at the cloaking surfaces, behave as though they are propagating on the model space M. The ability to vastly extend the range of conventional optical design by this "singular transformation optics" has the potential, if accompanied by sufficient development of metamaterial technology, to revolutionize the manipulation of electromagnetic waves.

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