HYPERBOLIC GEOMETRY AND LOCAL DIRICHLET-NEUMANN MAP

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ABSTRACT. We show in three dimensions that if we know the Dirichlet-to-Neumann map associated to the Schrödinger equation with a potential around the intersection of a sphere centered outside the convex hull of a bounded domain with smooth boundary, we can determine uniquely the corresponding spherical mean of the potential with an appropriate measure. This allows one to determine the potential near the boundary where the Dirichlet-to-Neumann map is measured. Similar results are valid for Electrical Impedance Tomography.

1. INTRODUCTION

In this paper we investigate what information can be obtained on the internal parameters of a bounded body by making measurements only on part of the boundary of the medium. In particular our results apply to the problem of Electrical Impedance Tomography (EIT) where one makes electrical measurements on part of the boundary and attempts to determine the conductivity inside. We start by describing a more general problem for the Schrödinger equation associated to a potential.

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary. Let $q \in L^\infty(\Omega)$. Assume that 0 is not a Dirichlet eigenvalue for the Schrödinger operator $-\Delta + q$ in Ω. The Dirichlet-to-Neumann (DN) map for $-\Delta + q$ is defined to be

$$\Lambda : f \to \left(\frac{\partial u}{\partial n}\right)\Big|_{\partial\Omega}, \tag{1.1}$$

where n is the outer unit normal to $\partial\Omega$, and u is the solution to the equation

$$\begin{cases} (-\Delta + q)u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

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Let Σ be a surface in $\mathbf{R}^3$ and $\Sigma_\epsilon = \{x \in \mathbf{R}^3; \mathrm{dist}(x, \Sigma) < \epsilon\}$. Two DN maps Λ_1 and Λ_2 are said to coincide around $\partial\Omega \cap \Sigma$ if there exists $\epsilon > 0$ such that

$$\Lambda_1 f = \Lambda_2 f \quad \text{on} \quad \partial\Omega \cap \Sigma_\epsilon \quad \forall f \in H_0^{3/2}(\partial\Omega \cap \Sigma_\epsilon). \tag{1.3}$$

The main result of this paper is the following local result. For $x_0 \in \mathbf{R}^3$ and $\omega \in S^2$, we define

$$L(x_0, \omega) = \{x_0 + t\omega; t \in \mathbf{R}^1\}. \tag{1.4}$$

Let $S(x_0, R)$ and $B(x_0, R)$ be the sphere and the ball of radius $R > 0$ centered at x_0, respectively.

Theorem 1.1. *Let $q_1, q_2 \in C^7(\overline{\Omega})$ be real-valued. Assume that 0 is not a Dirichlet eigenvalue for $-\Delta + q_i, i = 1, 2$. Let Λ_1, Λ_2 be the associated DN maps. Let x_0 be outside the closure of the convex hull of Ω. Suppose for some $R > 0$ Λ_1 and Λ_2 coincide around $\partial\Omega \cap S(x_0, R)$. Let $q(x) = q_1(x) - q_2(x)$ and extend it to be 0 outside Ω. Then we have*

$$\int_{S^2} q(x_0 + R\theta) \frac{d\theta}{(1 + \theta \cdot \omega)^2} = 0, \tag{1.5}$$

for any $\omega \in S^2$ satisfying

$$\mathrm{dist}(L(x_0, \omega), \Omega) > 0 \tag{1.6}$$

where $d\theta$ denotes the standard measure on S^2.

We also have

$$\int_\Pi q(x) d\Pi = 0, \tag{1.7}$$

for any plane Π where $d\Pi$ is the surface measure induced from the Lebesgue measure on $\mathbf{R}^3$, provided $\Lambda_1 = \Lambda_2$ around $\partial\Omega \cap \Pi$.

This theorem is inspired by the work [GU] where it is proven that one can determine the Radon transform of the potential on a plane by measuring the Cauchy data of certain approximate solutions of the Schrödinger equation on a neighborhood of the intersection of the plane with the boundary. We derive the same conclusion (with an appropriate measure) from the local knowledge of the DN map around the intersection of the sphere and the surface of the body by embedding the problem into the hyperbolic space. This idea of embedding is introduced in [Is]. We go further than [GU] by constructing solutions with a large parameter of the Schrödinger equation which decay rapidly except on the sphere. On the other hand we assume more regularity of the potential than [GU].

Using Carleman estimates it is shown in [BU] that if one knows the set of Cauchy data for the Schrödinger equation in special subsets of the boundary then the potentials are the same. From the integral geometric information (1.5) we can find the potential near the boundary under appropriate conditions. We prove

Theorem 1.2. *Assume that 0 is not a Dirichlet eigenvalue for $-\Delta + q_i, i = 1, 2$. Let Λ_1, Λ_2 be the associated DN maps. Let x_0 be on the boundary of the convex hull of Ω. Suppose for some $R > 0$ Λ_1 and Λ_2 coincide around $\partial\Omega \cap B(x_0, R)$, i.e. (1.3) holds with Σ_ϵ replaced by $B(x_0, R)$. Then we have $q_1 = q_2$ on $B(x_0, R) \cap \Omega$.*

In other words, knowledge of the DN map near a point in the boundary implies that we can determine uniquely the potential in the interior of the domain near the boundary. This allows for localization of the problem. The larger the set where we know the DN map the larger the set in the domain where we can determine uniquely the potential. As a consequence we obtain the global result in [SU] that if the DN maps for two potentials are the same in the whole boundary then the potentials are the same in the interior. (This result holds also for $L^\infty(\Omega)$ potentials.)

This problem is closely related to Electrical Impedance Tomography (EIT). We recall briefly this inverse method. Let $\gamma \in C^2(\overline{\Omega})$ be a strictly positive function on $\overline{\Omega}$. The equation for the potential in the interior, under the assumption of no sinks or sources of current in Ω, is

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = f. \tag{1.8}$$

The Dirichlet-to-Neumann map is defined in this case as follows:

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial n}\right)\Big|_{\partial\Omega}. \tag{1.9}$$

If $q = \left(\Delta\sqrt{\gamma}\right)/\sqrt{\gamma}$ then

$$\Lambda_q(f) = \gamma^{-\frac{1}{2}} \Lambda_\gamma(\gamma^{-\frac{1}{2}} f) + \left(\frac{1}{2\gamma}\frac{\partial \gamma}{\partial n}\right) f, \tag{1.10}$$

where Λ_q denotes the associated DN map for the Schrödinger equation (1.2). As a direct consequence of this formula and Theorem 1.1 we have

Corollary 1.3. *Let $\gamma_i \in C^9(\overline{\Omega})$, $i = 1, 2$, be strictly positive. Let x_0 be outside the closure of the convex hull of Ω. Suppose for some $R > 0$ $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ around $\partial\Omega \cap S(x_0, R)$. Then we have the same conclusion as in Theorem 1.1 with $q_i = (\Delta\sqrt{\gamma_i})/\sqrt{\gamma_i}, i = 1, 2$.*

Here let us note that under the assumptions of Corollary 1.3, by using the boundary determination result of [KV], we have that $\gamma_1|_{\partial\Omega\cap S_\epsilon} = \gamma_2|_{\partial\Omega\cap S_\epsilon}$ and $(\partial\gamma_1/\partial n)|_{\partial\Omega\cap S_\epsilon} = (\partial\gamma_2/\partial n)|_{\partial\Omega\cap S_\epsilon}$.

Theorem 1.1 also implies a rigorous justification of the backprojection algorithm developed in [C] for the linearized EIT problem at a constant conductivity which involves integration of the conductivity over spheres, which are equipotential surfaces of the electric field created by injecting current at an array of electrodes at the boundary. We note that in our case we are considering the non-linear problem. In turn [C] is a generalization to three dimensions of the Barber and Brown algorithm for the linearized EIT problem at constant conductivities in two dimensions. This algorithm has been extensively studied in [BB], [SV], [BT]. A result similar to Theorem 1.2 is valid for EIT.

Theorem 1.4. *Take x_0 from the boundary of the convex hull of Ω. Suppose $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ on $B(x_0, R) \cap \partial\Omega$. Then we have $\gamma_1 = \gamma_2$ on $B(x_0, R) \cap \Omega$.*

To prove this theorem, let $\psi_i = \sqrt{\gamma_i}$. Then by virtue of Theorem 1.2, ψ_1 and ψ_2 satisfy the same equation $\Delta\psi_i + a(x)\psi_i = 0$ on $B(x_0, R) \cap \Omega$. Since ψ_1 and ψ_2 have the same Cauchy data on $B(x_0, R) \cap \partial\Omega$, they coincide by Calderón's uniqueness theorem for the Cauchy problem (see e.g. [Mi]. p.252).

We sketch below the proof of Theorem 1.1. The first step of the proof of this theorem is to embed the above boundary value problem (1.2) in the hyperbolic space $\mathbf{H}^3$. Without loss of generality, we assume that $x_0 = 0$ and that

$$\Omega \subset \mathbf{R}^3_+ = \{(x, x_3) \in \mathbf{R}^3; x_3 > 0\}. \tag{1.11}$$

Let u be a solution to the equation

$$(-\Delta + q)u = 0 \quad \text{in} \quad \Omega. \tag{1.12}$$

Then $v = x_3^{1/2}u$ satisfies

$$(-\Delta_g + x_3^2 q - \frac{3}{4})v = 0 \quad \text{in} \quad \Omega, \tag{1.13}$$

where $\Delta_g = x_3^2\partial_3^2 - x_3\partial_3 + x_3^2\Delta_x$ is the Laplace-Beltrami operator on $\mathbf{H}^3$, $\partial_3 = \partial/\partial x_3$ and Δ_x is the Laplacian in $\mathbf{R}^2$.

Let us recall the well-known hyperbolic geometry on $\mathbf{H}^3$. We represent $(x_1, x_2, x_3) \in \mathbf{R}^3_+ = \mathbf{H}^3$ by quaternions : $\mathbf{z} = x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{C})$, the action $\mathbf{z} \to \gamma \cdot \mathbf{z} = (a\mathbf{z} + b)(c\mathbf{z} + d)^{-1}$ is an isometry on $\mathbf{H}^3$. When $x_3 = 0$, this is a linear fractional transformation on $\mathbf{C} \simeq \{x_3 = 0\}$. We choose γ in such a way that this induced transformation maps the circle $S(x_0, R) \cap \{x_3 = 0\}$ to the line $\{x_1 = x_3 = 0\}$. Then the hemisphere $S(x_0, R) \cap \{x_3 > 0\}$ is mapped to the plane $\{x_1 = 0, x_3 > 0\}$. (This is proved in the same way as in the arguments in §5.) Therefore in order to prove Theorem 1.1 we have only to consider the case that $S(x_0, R)$ is replaced by the plane $\{x_1 = 0\}$.

We next introduce the change of variable $y = \log x_3$ and the unitary operator

$$L^2(\mathbf{H}^3; x_3^{-3}dxdx_3) \ni v \to w = e^{-y}v \in L^2(\mathbf{R}^3; dxdy).$$

Then the equation (1.13) is transformed into

$$(-\partial_y^2 - e^{2y}\Delta_x + e^{2y}q + \frac{1}{4})w = 0 \quad \text{in} \quad \Omega, \tag{1.14}$$

where $\partial_y = \partial/\partial y$. Let $n = (n_x, n_y)$ and $n' = (n'_x, n'_3)$ be outer unit normals to $\partial\Omega$ with respect to the metrics $(dx)^2 + (dy)^2$ and $(dx)^2 + (dx_3)^2$, respectively. One can then show that

$$(e^{2y}n_x \cdot \partial_x + n_y\partial_y)\Big|_{\partial\Omega} = \lambda(x, x_3)x_3^2(n'_x \cdot \partial_x + n'_3\partial_3)\Big|_{\partial\Omega} \tag{1.15}$$

for a smooth function $\lambda(x, x_3) > 0$ on $\partial\Omega$ (see [Is], (6.9)). Therefore the boundary value problem (1.2) is equivalent to

$$\begin{cases} (-\partial_y^2 - e^{2y}\Delta_x + e^{2y}q + \frac{1}{4})u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega, \end{cases} \tag{1.16}$$

with the associated DN map

$$\Lambda f = e^{2y}n_x \cdot \partial_x u + n_y\partial_y u. \tag{1.17}$$

We are now in a position to reformulate our problem. Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary. Fix a constant $\lambda > 0$ and consider the following boundary value problem

$$\begin{cases} (-\partial_y^2 - e^{2y}\Delta_x + V - \lambda)u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega, \end{cases} \tag{1.18}$$

The associated DN map is defined by (1.17). The proof of Theorem 1.1 is reduced to show the following result.

Theorem 1.5. *Let $V_1, V_2 \in C^7(\overline{\Omega})$ be real-valued and Λ_1, Λ_2 the associated D-N maps. Suppose $\Lambda_1 = \Lambda_2$ around $\partial\Omega \cap \{x_1 = 0\}$. Let $M = \Omega \cap \{x_1 = 0\}$. Then we have*

$$\int_M V_1(0, x_2, y)e^{-y}dx_2dy = \int_M V_2(0, x_2, y)e^{-y}dx_2dy.$$

The proof of this last theorem is done by combining the ideas in [GU] and [Is]. In §2, we derive a decay estimate for the Green operator of $-\partial_y^2 + e^{2y}(-i\partial_x + \theta)^2 - \lambda$. In §3, we construct a family of solutions to the equation $(-\partial_y^2 + e^{2y}(-i\partial_x + \theta)^2 - \lambda)u = 0$ whose supports concentrate on the plane $\{x_1 = 0\}$. Theorem 1.5 is proved in §4. Theorems 1.1 and 1.2 are proved in §5 and §6, respectively.

2. Green operator

In this section, we study properties of the Green operator of

$$P(\theta) = -\partial_y^2 + e^{2y}(-i\partial_x + \theta)^2 - \lambda, \quad \theta \in \mathbf{C}^{n-1}, \quad \lambda > 0, \tag{2.1}$$

on $\mathbf{R}^n, n \geq 2$. Let $G(y, y'; \sigma, \zeta)$ be the Green function of $-\partial_y^2 + e^{2y}\zeta^2 - \sigma^2$ defined by [Is], (2.16). Namely

$$G(y, y'; \sigma, \zeta) = \begin{cases} K_{i\sigma}(\zeta e^y)I_{i\sigma}(\zeta e^{y'}), & y > y', \\ I_{i\sigma}(\zeta e^y)K_{i\sigma}(\zeta e^{y'}), & y' > y, \end{cases}$$

where $K_{i\sigma}, I_{i\sigma}$ are modified Bessel functions. Let

$$G(\sigma, \zeta)f = \int_{-\infty}^{\infty} G(y, y'; \sigma, \zeta)f(y')dy', \tag{2.2}$$

$$\mathbf{G}_0(\sigma, \alpha, z)f = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^{n-1}} e^{ix\cdot\xi}G(\sigma, \zeta(\xi, \alpha, z))\hat{f}(\xi, \cdot)d\xi, \tag{2.3}$$

$$\zeta(\xi, \alpha, z) = \sqrt{(\xi + z\alpha)^2}, \quad \alpha \in S^{n-2}. \tag{2.4}$$

Here $\hat{f}(\xi, y)$ is the partial Fourier transform of $f(x, y)$ with respect to x and for $p = (p_1, \cdots, p_d) \in \mathbf{C}^d$, $p^2 = p\cdot p = \sum_{j=1}^d p_j^2$. We take the branch of $\sqrt{\cdot}$ in such a way that Re $\sqrt{\cdot} \geq 0$ with cut along the negative real axis. We have, by definition, for any $f \in C_0^\infty(\mathbf{R}^3)$

$$P(z\alpha)\mathbf{G}_0(\pm\sqrt{\lambda}, \alpha, z)f = f. \tag{2.5}$$

For $s \in \mathbf{R}$, the space $L^{2,s}$ is defined by

$$f \in L^{2,s} \Longleftrightarrow \int_{\mathbf{R}^n} (1+|y|)^{2s}|f(x,y)|^2 dxdy < \infty \tag{2.6}$$

with the obvious norm. In the following we assume that $\sigma \in \mathbf{R} \setminus \{0\}$.

Theorem 2.1. *Let $s > 1/2$. Then there exists a constant $C > 0$ such that*

$$\|\mathbf{G}_0(\sigma, \alpha, z)f\|_{-s} \leq C\|f\|_s, \quad \forall z \in \mathbf{C}. \tag{2.7}$$

Proof. This follows from the inequality $|G(y, y'; \sigma, \zeta)| \leq C$ by [Is], (2.18). $\diamondsuit$

In the following lemma, $\hat{\varphi}(\xi)$ denotes the (full) Fourier transform :

$$\hat{\varphi}(\xi) = (2\pi)^{-\nu/2} \int_{\mathbf{R}^\nu} e^{-ix\cdot\xi}\varphi(x)dx.$$

Let us recall the following inequality

$$\|\hat{\varphi}(\cdot)\|_{L^2(S^{\nu-1})} \leq C\|\varphi\|_{B(\mathbf{R}^\nu)}, \tag{2.8}$$

where $B(\mathbf{R}^\nu)$ is the space introduced by Agmon-Hörmander [AgHo].

Lemma 2.2. *(1) If $s > 1/2$,*

$$\|\hat{\varphi}(t\cdot)\|_{L^2(S^{\nu-1})} \leq Ct^{(1-\nu)/2}\|\varphi\|_B \leq Ct^{(1-\nu)/2}\|\langle x\rangle^s\varphi(x)\|_{L^2(\mathbf{R}^\nu)}, \quad \forall t \geq 1,$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$.
(2) If $s > \nu/2$,

$$\|\hat{\varphi}(t\cdot)\|_{L^2(S^{\nu-1})} \leq C(1+t)^{(1-\nu)/2}\|\langle x\rangle^s\varphi(x)\|_{L^2(\mathbf{R}^\nu)}, \quad \forall t \geq 0.$$

Proof. The assertion (1) follows from the following two inequalities :

$$\|\hat{\varphi}(t\cdot)\|_{L^2(S^{\nu-1})} \leq Ct^{-\nu}\|\varphi(\cdot/t)\|_B,$$

$$\|\varphi(\cdot/t)\|_B \leq Ct^{(\nu+1)/2}\|\varphi\|_B.$$

The first inequality follows easily by the scaling argument and (2.8). To prove the second, we introduce the notation

$$\Omega(a, b; \varphi) = \left(\int_{a<|x|<b} |\varphi(x)|^2 dx\right)^{1/2}.$$

Then we have

$$t^{-\nu/2}\|\varphi(\cdot/t)\|_B = \Omega(0,t^{-1};\varphi) + \sum_{j\geq 1} 2^{j/2}\Omega(2^{j-1}t^{-1}, 2^j t^{-1};\varphi).$$

If $2^k \leq t \leq 2^{k+1}$, one can estimate the right-hand side as follows :

$$\begin{aligned} t^{-\nu/2}\|\varphi(\cdot/t)\|_B &\leq C\left\{\Omega(0,2^{-k};\varphi) + 2^{k/2}\sum_{j\geq 1} 2^{(j-k)/2}\Omega(2^{j-k-1},2^{j-k};\varphi)\right\} \\ &\leq C2^{k/2}\left\{\Omega(0,1;\varphi) + \sum_{n\geq 1} 2^{n/2}\Omega(2^{n-1},2^n;\varphi)\right\}. \end{aligned}$$

This proves the second inequality.

The assertion (2) is proved by (1) and the following inequality

$$\|\hat{\varphi}(t\cdot)\|_{L^\infty(S^{\nu-1})} \leq C\|\varphi\|_{L^1(\mathbf{R}^\nu)} \leq C\|\langle x\rangle^s\varphi(x)\|_{L^2(\mathbf{R}^\nu)},$$

where $s > \nu/2$. $\diamondsuit$

Theorem 2.3. *Let* $u = \mathbf{G}_0(\sigma,\alpha,z)f$.
(1) If $n = 3$ *and* $s > 1$, *we have for large* $|\mathrm{Im}\ z|$

$$\int_{\mathbf{R}^3}\langle y\rangle^{-2s}e^y|u(x,y)|^2dxdy \leq C\frac{\log|\mathrm{Im}\ z|}{|\mathrm{Im}\ z|}\int_{\mathbf{R}^3}\langle x\rangle^{2s}\langle y\rangle^{2s}e^{-y}|f(x,y)|^2dxdy.$$

(2) If $n = 2$ *and* $s > 1/2$, *we have for large* $|\mathrm{Im}\ z|$

$$\int_{\mathbf{R}^2}\langle y\rangle^{-2s}e^y|u(x,y)|^2dxdy \leq \frac{C}{|\mathrm{Im}\ z|^2}\int_{\mathbf{R}^2}\langle y\rangle^{2s}e^{-y}|f(x,y)|^2dxdy.$$

Proof. We first consider the case $n = 3$. Without loss of generality, we assume that $\mathrm{Re}\ z = 0$, and $\alpha = (1,0)$. We put $\mathrm{Im}\ z = \tau$ and let

$$\Omega_1 = \left\{|\xi|^2 < \frac{\tau^2}{2}\right\} \cup \left\{|\xi|^2 > \frac{3\tau^2}{2}\right\},$$

$$\Omega_2 = \left\{\frac{\tau^2}{2} < |\xi|^2 < \frac{3\tau^2}{2}\right\} \cap \left\{|\xi_1| > \frac{\tau}{10}\right\},$$

$$\Omega_3 = \left\{\frac{\tau^2}{2} < |\xi|^2 < \frac{3\tau^2}{2}\right\} \cap \left\{|\xi_1| < \frac{\tau}{10}\right\}.$$

We define

$$u_j(x,y) = (2\pi)^{-1}\int_{\Omega_j} e^{ix\cdot\xi}\big(G(\sigma,\zeta)\hat{f}(\xi)\big)(y)d\xi.$$

On $\Omega_1 \cup \Omega_2$, $|\zeta| \geq C\tau$ for $C > 0$. Hence using the inequality

$$|G(y,y';\sigma,\zeta)| \leq C|\zeta|^{-1}e^{-(y+y')/2}$$

([Is], (2.19)), we have for $s > 1/2$, $j = 1, 2$,

$$\begin{aligned}\int_{\mathbf{R}^2} |u_j(x,y)|^2 dx &= \int_{\Omega_j} \big|\big(G(\sigma,\zeta)\hat{f}(\xi)\big)(y)\big|^2 d\xi \\ &\le Ce^{-y}\int_{\Omega_j\times\mathbf{R}} \langle y'\rangle^{2s} e^{-y'}\frac{|\hat{f}(\xi,y')|^2}{|\zeta|^2}d\xi dy' \\ &\le \frac{C}{\tau^2}e^{-y}\int_{\mathbf{R}^3}\langle y'\rangle^{2s}e^{-y'}|f(x',y')|^2dx'dy'.\end{aligned}$$

Therefore u_1, u_2 satisfy the desired inequality.

To estimate u_3, we let

$$\Omega_3' = \left\{\frac{1}{2} < |\eta|^2 < \frac{3}{2}\right\} \cap \left\{|\eta_1| < \frac{1}{10}\right\}.$$

Then we have

$$\int_{\mathbf{R}^2} |u_3(x,y)|^2 dx \le Ce^{-y}\int_{\Omega_3'\times\mathbf{R}}\langle y'\rangle^{2s}e^{-y'}\frac{|\hat{f}(\tau\eta,y')|^2}{|\eta^2-1|+|\eta_1|}d\eta dy'.$$

We split Ω_3' into two parts :

$$\Omega_3' = \big(\Omega_3' \cap \{\eta_2 > 0\}\big) \cup \big(\Omega_3' \cap \{\eta_2 < 0\}\big) =: \Omega_{3,+}' \cup \Omega_{3,-}',$$

and estimate the integral over $\Omega_{3,+}' \times \mathbf{R}$.

Let $\eta_0 = (0,1)$ and put

$$h_\tau(x,y) = e^{-i\tau\eta_0\cdot x}f(x,y).$$

Then by the change of variables $\eta = \eta_0 + k$

$$\int_{\Omega_{3,+}'\cap\{|\eta-\eta_0|<\epsilon\}}\frac{|\hat{f}(\tau\eta,y)|^2}{|\eta^2-1|+|\eta_1|}d\eta = \int_{\Omega_{3,+}''\cap\{|k|<\epsilon\}}\frac{|\hat{h}_\tau(\tau k,y)|^2}{|k^2+2k_2|+|k_1|}dk,$$

where $\Omega_{3,+}'' = \{1/2 < |k+\eta_0|^2 < 3/2\}\cap\{|k_1| < 1/10, k_2 > -1\}$. By choosing ϵ small enough,

$$|k^2 + 2k_2| + |k_1| \ge (|k_1| + |k_2|)/2$$

for $|k| < \epsilon$. We have, therefore,

$$\int_{\Omega_{3,+}''\cap\{|k|<\epsilon\}}\frac{|\hat{h}_\tau(\tau k,y)|^2}{|k^2+2k_2|+|k_1|}dk \le C\int_0^\epsilon\Big(\int_{S^1}|\hat{h}_\tau(\tau r\omega,y)|^2d\omega\Big)dr.$$

Lemma 2.2 (2) implies that this is dominated from above by

$$\frac{\log\tau}{\tau}\int_{\mathbf{R}^2}\langle x\rangle^{2s}|h_\tau(x,y)|^2dx = \frac{\log\tau}{\tau}\int_{\mathbf{R}^2}\langle x\rangle^{2s}|f(x,y)|^2dx$$

for $s > 1$.

On the other hand there is a constant $C > 0$ such that $|k^2+2k_2|+|k_1| \geq C$ on $\Omega''_{3,+} \cap \{|k| > \epsilon\}$. Therefore we have for some $\epsilon_0 > 0$

$$\begin{aligned}\int_{\Omega''_{3,+}\cap\{|k|>\epsilon\}} \frac{|\hat{h}_\tau(\tau k, y)|^2}{|k^2+2k_2|+|k_1|}dk &\leq C\int_\epsilon^{\epsilon_0}\Big(\int_{S^1}|\hat{h}_\tau(\tau r\omega, y)|^2 d\omega\Big)dr \\ &\leq \frac{C}{\tau}\int_{\mathbf{R}^2}\langle x\rangle^{2s}|f(x,y)|^2 dx.\end{aligned}$$

The integral over $\Omega'_{3,-} \times \mathbf{R}$ is estimated in a similar manner. This completes the proof of (1).

If $n = 2$, we have only to consider Ω_1 and Ω_2 in the above computation. $\diamondsuit$

For $n = 3$ and $s > 1$, we define the space $\mathcal{H}_s^{(\pm)}$ by

$$u \in \mathcal{H}_s^{(\pm)} \Longleftrightarrow \|u\|^2_{\mathcal{H}_s^{(\pm)}} = \int_{\mathbf{R}^3}\langle x\rangle^{\pm 2s}\langle y\rangle^{\pm 2s}e^{\pm|y|}|u(x,y)|^2 dxdy < \infty, \quad (2.9)$$

and for $n = 2$ and $s > 1/2$

$$u \in \mathcal{H}_s^{(\pm)} \Longleftrightarrow \|u\|^2_{\mathcal{H}_s^{(\pm)}} = \int_{\mathbf{R}^2}\langle y\rangle^{\pm 2s}e^{\pm|y|}|u(x,y)|^2 dxdy < \infty. \quad (2.10)$$

Theorem 2.4. *Let $n = 2, 3$. Then for a constant $C > 1$*

$$\|\mathbf{G}_0(\sigma,\alpha,z)f\|_{\mathcal{H}_s^{(-)}} \leq Cd(|\mathrm{Im}\, z|)\|f\|_{\mathcal{H}_s^{(+)}}, \quad |\mathrm{Im}\, z| > C,$$

where

$$d(t) = (t^{-1}\log t)^{1/2} \quad \text{if} \quad n = 3, \quad d(t) = t^{-1} \quad \text{if} \quad n = 2. \quad (2.11)$$

By Theorem 2.4, one can prove the following theorem.

Theorem 2.5. *Let $n = 2, 3$. Let $V(x,y)$ be a bounded function of compact support on $\mathbf{R}^n$. Then there exists a constant $C_0 > 0$ such that for $\mathrm{Im}\, z > C_0$, $\|\mathbf{G}_0(\sigma,\alpha,z)V\|_{\mathbf{B}(\mathcal{H}_s^{(-)};\mathcal{H}_s^{(-)})} < 1/2$. Define $\mathbf{G}_V(\sigma,\alpha,z) \in \mathbf{B}(\mathcal{H}_s^{(+)};\mathcal{H}_s^{(-)})$ by*

$$\mathbf{G}_V(\sigma,\alpha,z) = (1+\mathbf{G}_0(\sigma,\alpha,z)V)^{-1}\mathbf{G}_0(\sigma,\alpha,z)$$

for $\mathrm{Im}\, z > C_0$. Then there exists a constant $C > 0$ such that

$$\|\mathbf{G}_V(\sigma,\alpha,z)f\|_{\mathcal{H}_s^{(-)}} \leq Cd(|\mathrm{Im}\, z|)\|f\|_{\mathcal{H}_s^{(+)}}, \quad |\mathrm{Im}\, z| > C_0,$$

where $d(t)$ is defined by (2.11). Moreover

$$(P(z\alpha)+V)\mathbf{G}_V(\pm\sqrt{\lambda},\alpha,z)f = f.$$

One can also prove the following theorem using standard elliptic estimates and interpolation.

Theorem 2.6. *Let $n = 2, 3$. Let K be a compact set in $\mathbf{R}^n$. Fix $C_1 > 0$ arbitrarily. Then there exists a constant $C > 0$ such that*

$$\|\partial_{x,y}^{\gamma}\mathbf{G}_V(\sigma,\alpha,z)f\|_{L^2(K)} \leq Cd(|\mathrm{Im}\, z|)|\mathrm{Im}\, z|^{|\gamma|}\|f\|_{\mathcal{H}_s^{(+)}}$$

for $|\gamma| \leq 2$, $|\mathrm{Re}\, z| < C_1$, $|\mathrm{Im}\, z| > C_0$, where $d(t)$ is defined by (2.11).

3. Approximate $\delta^{1/2}$-solutions

In this section we construct solutions $u(\theta)$ of the Schrödinger equation whose supports essentially concentrate near the plane $\{x_1 = 0\}$. Let us call them approximate $\delta^{1/2}$-solutions since as will be seen in Lemma 3.4 below, the product $u^{(1)}(\theta)\overline{u^{(2)}(\overline{\theta})}$ converges to the Dirac measure supported on the pl plane $\{x_1 = 0\}$.

Let $P(\theta)$ be as in (2.1) with $n = 3$. We construct an approximate solution of $P(\theta)v = 0$ for $\theta = (0, i\tau), \tau > 0$, of the form

$$u_0(\theta) = \psi_0(y,\tau)\psi_1(x_1,\tau)\psi_2(x_2,\tau). \tag{3.1}$$

Noting that

$$\begin{aligned} P(\theta)\psi_0\psi_1\psi_2 &= ((-\partial_y^2 - e^{2y}\tau^2 - \lambda)\psi_0)\psi_1\psi_2 \\ &+ (e^{2y}\psi_0)(-\partial_{x_1}^2\psi_1)\psi_2 \\ &+ (e^{2y}\psi_0)\psi_1(-\partial_{x_2}^2 + 2\tau\partial_{x_2})\psi_2, \end{aligned}$$

we put for small $\delta > 0$

$$\psi_0(y,\tau) = \tau^{1/2}H^{(1)}_{i\sqrt{\lambda}}(\tau e^y), \tag{3.2}$$

$$\psi_1(x_1,\tau) = \tau^{\delta/2}\chi(\tau^{\delta}x_1), \tag{3.3}$$

$$\psi_2(x_2,\tau) = \chi(\frac{x_2}{\tau}), \tag{3.4}$$

where $H^{(1)}_{\nu}(z)$ is the Hankel function of the first kind and $\chi(t) \in C_0^{\infty}(\mathbf{R}^1)$, $\chi(t) = 0$ $(|t| > 1)$, $\|\chi\|_{L^2(\mathbf{R}^1)} = 1$. Then we have

$$\begin{aligned} P(\theta)u_0(\theta) &= -\tau^{2\delta}(e^{2y}\psi_0)\tau^{\delta/2}\chi''(\tau^{\delta}x_1)\psi_2 \\ &\quad -\tau^{-2}(e^{2y}\psi_0)\psi_1\chi''(x_2/\tau) \\ &\quad +2(e^{2y}\psi_0)\psi_1\chi'(x_2/\tau). \end{aligned} \tag{3.5}$$

Lemma 3.1. *We have*

$$u_0(\theta) = 0 \quad \text{if} \quad |x_1| > \tau^{-\delta},$$

and for a compact set K in $\mathbf{R}^3$

$$\sup_{\tau > 1} \|u_0(\theta)\|_{L^2(K)} < \infty.$$

Proof. This follows from (3.1) $\sim$ (3.4) and the following asymptotic expansion of the Hankel function (see e.g. [Le] p.122)

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp\Big[i\big(z - \frac{(2\nu+1)\pi}{4}\big)\Big] \tag{3.6}$$

as $z \to \infty$. $\diamondsuit$

We take a bounded domain Ω_0 such that $\overline{\Omega} \subset \Omega_0$ and pick $\chi_{\Omega_0} \in C_0^\infty(\mathbf{R}^3)$ such that $\chi_{\Omega_0} = 1$ on Ω_0. Let V be a bounded function of compact support. We put for large $\tau > 0$

$$u_1(\theta) = -\mathbf{G}_V(\sigma, \alpha, i\tau)\chi_{\Omega_0}(P(\theta) + V)u_0(\theta), \tag{3.7}$$

where $\sigma = \sqrt{\lambda}, \alpha = (0,1)$. By Theorem 2.6, we have for $s > 1$

$$\|u_1(\theta)\|_{\mathcal{H}_s^{(-)}} \leq C\tau^{-1/2+3\delta}. \tag{3.8}$$

An important feature of $u_1(\theta)$ is that its support essentially concentrates near the plane $\{x_1 = 0\}$. To prove this property we prepare the following lemma.

Lemma 3.2. *Let K be a compact set in $\mathbf{R}^3$. Let $f_\tau(x,y) \in C^6(\mathbf{R}^3)$ be such that*

$$\text{supp } f_\tau \subset K \cap \{|x_1| < \tau^{-\delta}\}.$$

We define

$$N(\tau) = \int_{\mathbf{R}^3} |(1 - \partial_{x_1}^2)^2(1 - \partial_{x_2}^2)f_\tau(x,y)|dxdy.$$

Let $w_\tau = \mathbf{G}_0(\sigma, \alpha, i\tau)f_\tau$. Then if $(x,y) \in K$ and $|x_1| > 2\tau^{-\delta}$

$$|w_\tau(x,y)| \leq C\tau^{-2+\delta}N(\tau).$$

Proof. Let $D_\pm = \mathbf{R}^2 \times \{\xi \in \mathbf{R}^2; \pm\xi_2 > 0\} \times \mathbf{R}^1$ and split $w_\tau = w_+ + w_-$, where

$$w_\pm(x,y) = (2\pi)^{-2} \int_{D_\pm} e^{i(x-x')\cdot\xi} G(y,y';\sigma,\zeta)f_\tau(x',y')dx'd\xi dy'.$$

Note that $\zeta = \sqrt{(\xi + i\tau\alpha)^2}$ is smooth if $\xi_2 \neq 0$. We consider w_+. Since $|x_1 - x_1'| \geq \tau^{-\delta}$ on the support of the integrand, we have by integration by parts

$$\begin{aligned} w_+(x,y) &= i\int_{D_+} e^{i(x-x')\cdot\xi}(x_1 - x_1')^{-1} \\ &\quad \times \left(\frac{\partial}{\partial\xi_1}G(y,y';\sigma,\zeta)\right) f_\tau(x',y')dx'd\xi dy'. \end{aligned}$$

Let us estimate $\partial G/\partial\xi_1$. Using

$$I_\nu'(z) = (I_{\nu-1}(z) + I_{\nu+1}(z))/2, \quad K_\nu'(z) = -(K_{\nu-1}(z) + K_{\nu+1}(z))/2$$

(see [Le] p. 110), when $\nu = i\sqrt{\lambda}$ we have as $z \to 0$

$$|I_\nu'(z)| + |K_\nu'(z)| \leq C/|z|. \tag{3.9}$$

On the other hand, the behaviors of $I_\nu'(z)$ and $K_\nu'(z)$ as $z \to \infty$ are the same as those of $I_\nu(z)$ and $K_\nu(z)$ ((2.5) $\sim$ (2.7) of [Is]). Therefore for Re $z \geq 0$

$$|K_{i\sqrt{\lambda}}'(z)| \leq Ce^{-\mathrm{Re}\, z}\left(\frac{1}{|z|} + \frac{1}{\sqrt{|z|}}\right), \tag{3.10}$$

$$|I_{i\sqrt{\lambda}}'(z)| \leq Ce^{\mathrm{Re}\, z}\left(\frac{1}{|z|} + \frac{1}{\sqrt{|z|}}\right). \tag{3.11}$$

Since $\partial\zeta/\partial\xi_1 = \xi_1/\zeta$, we then have

$$\left|\frac{\partial}{\partial\xi_1}G(y,y';\sigma,\zeta)\right| \leq C\left(\frac{|\xi_1|}{|\zeta|^{5/2}} + \frac{|\xi_1|}{|\zeta|^2}\right), \tag{3.12}$$

when y, y' vary over a bounded set in $\mathbf{R}$. Letting

$$g_\tau = (1+\xi_1^2)^{-2}(1+\xi_2^2)^{-1}(1-\partial_{x_1}^2)^2(1-\partial_{x_2}^2)f_\tau,$$

we have by integration by parts

$$\begin{aligned} w_+(x,y) &= i\int_{D_+} e^{i(x-x')\cdot\xi}(x_1 - x_1')^{-1} \\ &\quad \times \left(\frac{\partial}{\partial\xi_1}G(y,y';\sigma,\zeta)\right) g_\tau dx'd\xi dy'. \end{aligned}$$

We split the set $\{\xi \in \mathbf{R}^2; \xi_2 > 0\}$ in the following way :

$$\Omega_1 = \left(\left\{|\xi|^2 < \frac{\tau^2}{2}\right\} \cup \left\{|\xi|^2 > 2\tau^2\right\}\right) \cap \left\{\xi_2 > 0\right\},$$

$$\Omega_2 = \left\{\frac{\tau^2}{2} < |\xi|^2 < 2\tau^2\right\} \cap \left\{\xi_2 > \frac{\tau}{10}\right\},$$

$$\Omega_3 = \left\{\frac{\tau^2}{2} < |\xi|^2 < 2\tau^2\right\} \cap \left\{0 < \xi_2 < \frac{\tau}{10}\right\},$$

and decompose $w_+ = \sum_{i=1}^3 w_{i+}$, accordingly. On $\Omega_1 \cup \Omega_2$ we have that $|\zeta| \geq C\tau$. Therefore by using (3.12), we have

$$\sum_{i=1}^{2} |w_{i+}(x,y)| \leq C\tau^{-2+\delta} N(\tau).$$

On Ω_3, we make the change of variables $\xi = \tau\eta$. Then we have

$$\begin{aligned} |w_{3+}(x,y)| &\leq C\tau^{\delta+2} N(\tau) \int_{\frac{1}{2}<\eta^2<2, |\eta_2|<\frac{1}{10}} (1+\tau|\eta_1|)^{-3} \\ &\quad \times \left\{ \frac{1}{\tau^{5/2}(|\eta^2-1|+|\eta_2|)^{5/4}} + \frac{1}{\tau^2(|\eta^2-1|+|\eta_2|)} \right\} d\eta \\ &\leq C\tau^{\delta-3} N(\tau). \qquad \diamondsuit \end{aligned}$$

We emphasize that in the expression for $N(\tau)$, f_τ is differentiated only with respect to the x-variables. We also remark that since f_τ is compactly supported, we have

$$N(\tau) \leq C \|(1-\partial_{x_1}^2)^2 (1-\partial_{x_2}^2) f_\tau\|_{L^2(\mathbf{R}^3)}.$$

We shall use these facts in §4.

The approximate $\delta^{1/2}$-solution is now defined by

$$u(\theta) = u_0(\theta) + u_1(\theta), \tag{3.13}$$

where $u_0(\theta), u_1(\theta)$ are defined by (3.1) and (3.7).

Lemma 3.3. *(1)* $(P(\theta)+V)u(\theta) = 0$ *on* Ω.
(2) $\|u(\theta)\|_{L^2(\Omega)} \leq C$.
(3) Let K *be a compact set in* $\mathbf{R}^3$ *such that* $K \subset \Omega_0$ *and* $K_\tau = K \cap \{|x_1| > 2\tau^{-\delta}\}$. *Suppose* $V \in C_0^n(\mathbf{R}^3)$ *with* $n \geq 6$. *Then*

$$\sup_{(x,y)\in K_\tau} \left|\partial_x^\gamma \partial_y^m u_1(\theta)\right| \leq C\tau^{m-2+O(\delta)},$$

for $|\gamma| \leq n-6$, $m = 0, 1$.

Proof. The assertion (1) follows from (3.7) and (3.13). The assertion (2) follows from Lemma 3.1 and Theorem 2.6. In order to prove (3), we put $f = \chi_{\Omega_0}(P(\theta)+V)u_0(\theta)$ and $G(\theta) = \mathbf{G}_V(\sigma,\alpha,i\tau)$. Then $u_1(\theta) = -G(\theta)f$. Letting $G_0(\theta) = \mathbf{G}_0(\sigma,\alpha,i\tau)$ and iterating the resolvent equation

$$G(\theta) = G_0(\theta) - G_0(\theta)VG(\theta), \tag{3.14}$$

we see that by using Theorems 2.4 and 2.6, we have only to consider the terms like

$$G_0VG_0V\cdots VG_0f.$$

Let us consider the case that $m=0$. Since the support of f is contained in the region $\{(x,y);|x_1|<\tau^{-\delta}\}$, we can show that $|\partial_x^\gamma u_1(\theta)|\leq C\tau^{-2+O(\delta)}$ by using Lemma 3.2. Here we should note that ∂_x commutes with G_0. To deal with ∂_y, let us note that

$$\Big|\frac{\partial^2}{\partial\xi_1\partial y}G(y,y';\sigma,\zeta)\Big|\leq C\left(\frac{|\xi_1|}{|\zeta|^2}+\frac{|\xi_1|}{|\zeta|}\right),$$

if y,y' vary over a compact set. Then we get the assertion (3) for $m=1$ by the same argument. ♢

Let us note that Lemma 3.3 (3) also holds for $u(\theta)$ by virtue of Lemma 3.1.

In §4, we need solutions of equations $(P(\theta)+V_1)u=0$ and $(P(\overline{\theta})+V_2)v=0$ for $\theta=(0,i\tau)$ such that as $\tau\to\infty$, $u\overline{v}$ tends to a measure supported on the plane $\{x_1=0\}$. A solution for $(P(\theta)+V_1)u=0$ is constructed above, which we denote by $u^{(1)}(\theta)$. To get a solution of $(P(\overline{\theta})+V_2)v=0$, noting that $\overline{\theta}=i\tau(0,-1)$, we put

$$u^{(2)}(\overline{\theta})=u_0(\overline{\theta})+u_1(\overline{\theta}),\tag{3.15}$$

where $u_0(\overline{\theta})$ is defined by

$$u_0(\overline{\theta})=\tau^{1/2}H^{(1)}_{i\sqrt{\lambda}}(\tau e^y)\cdot\tau^{\delta/2}\chi(\tau^\delta x_1)\cdot\chi(-\frac{x_2}{\tau}),\tag{3.16}$$

and $u_1(\overline{\theta})$ is defined by (3.7) with $\alpha,\theta,V,u_0(\theta)$ replaced by $(0,-1),\overline{\theta},V_2$ and $u_0(\overline{\theta})$, respectively. Then $u^{(2)}(\overline{\theta})$ possesses the properties in Lemma 3.3 with θ replaced by $\overline{\theta}$. Moreover we have the following lemma.

Lemma 3.4. *For any bounded continuous function f on Ω, we have as $\tau\to\infty$*

$$\int_\Omega fu^{(1)}(\theta)\overline{u^{(2)}(\overline{\theta})}dxdy\to C\int_M f(0,x_2,y)e^{-y}dx_2dy,$$

where $M=\Omega\cap\{x_1=0\}$ and $C=2\chi(0)^2e^{\pi\sqrt{\lambda}}/\pi$.

Proof. We have only to apply (3.6). ♢

4. Spherical mean transform

We shall prove Theorem 1.5 in this section. Let $P(\theta)$ be defined by (2.1) for $n = 3$ and u a solution to the equation

$$\begin{cases} (P(\theta)+V)u = 0 & \text{in} \quad \Omega, \\ u = \varphi & \text{on} \quad \partial\Omega. \end{cases} \tag{4.1}$$

The DN map $\Lambda(\theta)$ is defined by

$$\Lambda(\theta)\varphi = B(\theta)u\Big|_{\partial\Omega}, \tag{4.2}$$

$$B(\theta) = n_y\partial_y + e^{2y}n_x \cdot (\partial_x + i\theta), \tag{4.3}$$

where $n = (n_x, n_y)$ is the outer unit normal to $\partial\Omega$ with respect to the Euclidean metric $(dx)^2 + (dy)^2$. Let us note that by the gauge transformation

$$e^{-i\theta\cdot x}\Lambda(0)e^{i\theta\cdot x} = \Lambda(\theta), \quad \forall\theta \in \mathbf{C}^2. \qquad (4.4).$$

Let $(\, , \,)$ and $(\, , \,)_{\partial\Omega}$ be the inner products of $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively. The following lemma is a direct consequence of Green's formula.

Lemma 4.1. *(1) For any $u, v \in H^2(\Omega)$,*

$$(P(\theta)u, v) - (u, P(\overline{\theta})v) = -(B(\theta)u, v)_{\partial\Omega} + (u, B(\overline{\theta})v)_{\partial\Omega}.$$

(2) For any $\varphi, \psi \in H^{3/2}(\partial\Omega)$

$$(\Lambda(\theta)\varphi, \psi)_{\partial\Omega} = (\varphi, \Lambda(\overline{\theta})\psi)_{\partial\Omega}.$$

Let $\Lambda_i(\theta)$ be the DN map of $P(\theta) + V_i$. Let $S_\epsilon = \partial\Omega \cap \{|x_1| \leq \epsilon\}$, and assume that $\Lambda_1(0) = \Lambda_2(0)$ around S_0. Then by (4.4), there exists $\epsilon > 0$ such that

$$\Lambda_1(\theta)\varphi = \Lambda_2(\theta)\varphi \quad \text{on} \quad S_\epsilon, \quad \forall\varphi \in H_0^{3/2}(S_\epsilon), \quad \forall\theta \in \mathbf{C}^2. \tag{4.5}$$

We extend V_1 and V_2 as $H^7(\mathbf{R}^3)$-functions of compact support. With $u^{(1)}(\theta)$ and $u^{(2)}(\overline{\theta})$ defined in the last paragraph of §3, we show that

$$\lim_{\tau\to\infty}((V_1 - V_2)u^{(1)}(\theta), u^{(2)}(\overline{\theta})) = 0. \tag{4.6}$$

This together with Lemma 3.4 shows Theorem 1.5.

In order to prove (4.6), we let $u = u^{(1)}(\theta), v = u^{(2)}(\overline{\theta})$ in Lemma 4.1 (1). Then we have

$$\Big((V_1 - V_2)u^{(1)}(\theta), u^{(2)}(\overline{\theta})\Big) = \Big((\Lambda_1(\theta) - \Lambda_2(\theta))u^{(1)}(\theta), u^{(2)}(\overline{\theta})\Big)_{\partial \; Omega}. \tag{4.7}$$

Pick $\epsilon > 0$ small enough, and let $\chi_0(x_1) \in C^\infty(\mathbf{R}^1)$ be such that $\chi_0(x_1) = 1$ for $|x_1| < \epsilon/2$, $\chi_0(x_1) = 0$ for $|x_1| > \epsilon$. Put $\chi_\infty(x_1) = 1 - \chi_0(x_1)$. Then by (4.5), we have

$$\chi_0(x_1)(\Lambda_1(\theta) - \Lambda_2(\theta))\chi_0(x_1) = 0. \tag{4.8}$$

Therefore we have

$$\begin{aligned} & ((\Lambda_1(\theta) - \Lambda_2(\theta))u^{(1)}(\theta), u^{(2)}(\overline{\theta}))_{\partial\Omega} \\ = \ & ((\Lambda_1(\theta) - \Lambda_2(\theta))\chi_\infty(x_1)u^{(1)}(\theta), \chi_0(x_1)u^{(2)}(\overline{\theta}))_{\partial\Omega} \\ + \ & (\chi_0(x_1)u^{(1)}(\theta), (\Lambda_1(\overline{\theta}) - \Lambda_2(\overline{\theta}))\chi_\infty(x_1)u^{(2)}(\overline{\theta}))_{\partial\Omega} \\ + \ & ((\Lambda_1(\theta) - \Lambda_2(\theta))\chi_\infty(x_1)u^{(1)}(\theta), \chi_\infty(x_1)u^{(2)}(\overline{\theta}))_{\partial\Omega}. \end{aligned}$$

By Lemma 3.3 (3) we have for $w = u^{(1)}(\theta)$ and $u^{(2)}(\overline{\theta})$,

$$\|\chi_\infty(x_1)w\|_{H^m(\partial\Omega)} \leq C\tau^{m-2+O(\delta)}, \quad m = 0, 1. \tag{4.9}$$

By virtue of the inequality

$$\|\Lambda(\theta)w\|_{L^2(\partial\Omega)} \leq C(\|w\|_{H^1(\partial\Omega)} + \tau\|w\|_{L^2(\partial\Omega)}), \tag{4.10}$$

we then have

$$((\Lambda_1(\theta) - \Lambda_2(\theta))u^{(1)}(\theta), u^{(2)}(\overline{\theta})) \to 0. \tag{4.11}$$

This together with (4.7) proves (4.6).

5. Proof of Theorem 1.1

Without loss of generality, we assume that Ω is in the upper half space $\mathbf{R}^3_+$ and the sphere S is centered at the origin. The linear fractional transformation

$$w = R\frac{z+1}{-z+1} = \frac{\rho z + \rho}{-z/2\rho + 1/2\rho}, \quad \rho = \sqrt{\frac{R}{2}} \tag{5.1}$$

maps the line $\{\mathrm{Re}\ z = 0\}$ to the circle $\{|w| = R\}$. Recall that the quaternions can be represented by 2×2 matrices :

$$x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j} = \begin{pmatrix} x_1 + ix_3 & x_2 \\ -x_2 & x_1 - ix_3 \end{pmatrix}. \tag{5.2}$$

Putting

$$\mathbf{z} = x_2\mathbf{i} + x_3\mathbf{j},$$

$$\mathbf{w} = \Big(\rho\mathbf{z} + \rho\mathbf{1}\Big)\Big(-\frac{\mathbf{z}}{2\rho} + \frac{\mathbf{1}}{2\rho}\Big)^{-1} = y_1\mathbf{1} + y_2\mathbf{i} + y_3\mathbf{j},$$

we have

$$y_1 = R\frac{1-r^2}{1+r^2}, \quad y_2 = R\frac{2x_2}{1+r^2}, \quad y_3 = R\frac{2x_3}{1+r^2}, \tag{5.3}$$

where $r = (x_2^2 + x_3^2)^{1/2}$. In particular this implies $y_1^2 + y_2^2 + y_3^2 = R^2$. Letting $x_2 = r\cos\theta, x_3 = r\sin\theta$, we also have

$$(dy_1)^2 + (dy_2)^2 + (dy_3)^2 = \frac{4R^2}{(1+r^2)^2}\left((dr)^2 + (rd\theta)^2\right). \tag{5.4}$$

Therefore the measure dS on the sphere $\{y_1^2 + y_2^2 + y_3^2 = R^2\}$ induced from the Lebesgue measure $dy_1dy_2dy_3$ is written as

$$dS = \frac{4R^2}{(1+r^2)^2} r dr d\theta = \frac{4R^2}{(1+r^2)^2} dx_2 dx_3.$$

This implies

$$dx_2 dx_3 = \frac{dS}{(R+y_1)^2}. \tag{5.5}$$

Comparing (1.14) and (1.18) we have $e^{2y}q + 1/4 = V - \lambda$. Therefore by virtue of Theorem 1.5,

$$\int_{\Omega\cap\{x_1=0\}} q dx_2 dx_3 = 0. \tag{5.6}$$

Hence by transforming back to the variable y and using (5.5), we have

$$\int_{\Omega\cap S} q\frac{dS}{(R+y_1)^2} = 0. \tag{5.7}$$

By a rotation in the y_1, y_2-plane, we get Theorem 1.1.

6. Proof of Theorem 1.2

We assume that $x_0 = (0,0,\epsilon)$ and $\Omega \subset \{x_3 > \epsilon\}$. Theorem 1.1 then implies that

$$\int_{S^2} q(x + r\theta)\frac{d\theta}{(1+\theta\cdot\omega)^2} = 0 \tag{6.1}$$

if $|x| < \epsilon, 0 < r < R - \epsilon$, $\mathrm{dist}(L(x,\omega),\Omega) > 0$. On the support of the integrand, $\theta_3 \geq C_0$ for some $C_0 > 0$. Let $\omega = (\sqrt{1-t^2}, 0, t)$. Then for small $\delta > 0$, $(1+\theta\cdot\omega)^{-2}$ is analytic in the domain $|\mathrm{Re}\, t| < \delta$. In fact if $1+\theta\cdot\omega = 0$, we have $\mathrm{Re}\, t = -\theta_3/(\theta_1^2+\theta_3^2)^{1/2}$. Since $\theta_3 > C_0 > 0$, $|\mathrm{Re}\, t| > \delta$ if δ is chosen small enough.

Letting t tend to infinity along the imaginary axis, we obtain

$$\int_{S^2} q(x + r\theta)\frac{d\theta}{(\theta_1 + i\theta_3)^2} = 0, \tag{6.2}$$

for $0 < r < R - \epsilon, |x| < \epsilon$. The proof of Theorem 1.2 is then completed if we show the following lemma.

Lemma 6.1. *Let $f(x) \in L^\infty(\mathbf{R}^n), n \geq 2$, be such that supp $f \subset \{x_n > \delta\}$ for some $\delta > 0$. Suppose there exist $R > 0$ and non-negative integer k such that*

$$\int_{S^{n-1}} f(x + r\theta)(\theta_1 + i\theta_n)^{-k} d\theta = 0$$

for all $0 < r < R, |x| < \delta$. Then we have $f(x) = 0$ for $|x| < R + \delta$.

Proof. By using Friedrich's mollifier, we have only to prove this lemma when $f \in C^\infty(\mathbf{R}^n)$. By the assumption, we have

$$\int_{|y|<r} f(x + y)(y_1 + iy_n)^{-k} dy = 0$$

for all $0 < r < R, |x| < \delta$. Applying $\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_n}$, we get

$$\int_{|y|<r} \left\{ \Big(\frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_n}\Big) f(x + y) \right\} (y_1 + iy_n)^{-k} dy = 0.$$

Green's formula then implies that

$$\int_{S^{n-1}} f(x + r\theta)(\theta_1 + i\theta_n)^{1-k} d\theta = 0.$$

Repeating this procedure, we get

$$\int_{S^{n-1}} f(x + r\theta) d\theta = 0, \quad 0 < r < R, \quad |x| < \delta.$$

Applying $\frac{\partial}{\partial x_i}$, we have

$$\int_{|y|<r} \frac{\partial}{\partial y_i} f(x + y) dy = 0.$$

By Green's formula, we have

$$\int_{|y|=r} y_i f(x + y) dS_y = 0.$$

Repeating this procedure, we finally conclude that

$$\int_{|y|=r} P(y) f(x + y) dS_y = 0$$

for any $0 < r < R, |x| < \delta$ and any polynomial $P(y)$. Then we have $f(x) = 0$ for $|x| < R + \delta$. $\diamondsuit$

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