# Approximate quantum cloaking and almost trapped states

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We describe potentials which act as approximate cloaks for matter waves, i.e., for solutions of the time-independent Schrödinger equation at energy $E$, with applications to the design of tunable ion traps. These potentials are derived from ideal cloaks for the conductivity and Helmholtz equations, by a procedure we call isotropic transformation optics. For most $E$, if $V_0$ is a potential which is surrounded by a sequence of approximate cloaks, then asymptotically (i) $V_0$ is both undetectable and unaltered by matter waves originating externally to the cloak; and (ii) the combined potential of $V_0$ surrounded by the cloak does not perturb waves outside the cloak. On the other hand, for $E$ near certain resonant energies, cloaking *per se* fails and the approximate cloaks support wave functions concentrated, or *almost trapped*, inside the cloaked region and negligible outside. Applications include ion traps, dc or magnetically tunable, or customizable to support almost trapped states of arbitrary multiplicity. Possible uses include simulation of abstract quantum systems, magnetically tunable quantum beam switches, and illusions of singular magnetic fields.

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*Introduction.* A fundamental problem is to describe the scattering of waves at energy $E$ by a potential $V(\mathbf{r})$, as governed by the time-independent Schrödinger equation; the related inverse problem consists of determination of $V$ from scattering data or from boundary measurements. In this Letter, for a generic energy $E$, we construct sequences $V_n^E, n = 1, 2, 3, \ldots$, of bounded potentials which act as an *approximate quantum cloaks*: for any potential $V_0$ whose support is surrounded by the support of $V_n^E$, the scattering amplitude of $V_0 + V_n^E$ goes to zero asymptotically in $n$, so that $V_0$ is undetectable by matter waves at energy $E$. The central potentials $V_n^E$ are layered, supported in a spherical annulus $\{r_1 \le r \le r_2\}$ in $\mathbb{R}^3$, with spatial oscillations of increasing amplitude as $r \searrow r_1$ and decreasing layer thickness as $n \to \infty$, so that their local (resp., long range) effect on wave propagation becomes stronger (resp., weaker) as $n$ increases. For generic $E$, the potential $V_0 + V_n^E$ has a negligible effect on matter waves originating outside of its support. Alternatively, for $E$ close to special values, $V_n^E$ allows the core $\{r \le r_1\}$ to support *almost trapped states* and be used to form traps for ions (here denoting any charged particles), almost invisible to external matter waves. An approximate version of the dichotomy regarding ideal cloaks for the Helmholtz equation [4, Thm. 1] holds: If $E$ is sufficiently separated from all interior eigenvalues, then with high probability the approximate cloak keeps particles of energy $E$ from entering the cloaked region, see Figs. 1(r,red) and 2(l); on the other hand, for $E$ close to an eigenvalue, the cloaked region supports almost trapped states, accepting and binding such particles, leading to a new type of ion trap, cf. Figs. 1(r,blue) and 2(r) and formula (4).

Recently, Zhang, et al., [1], using ideas from transformation optics [2, 3], described an ideal quantum mechanical cloak at any fixed energy $E$ and proposed a physical implementation. The construction starts with a homogeneous, isotropic mass tensor $\widehat{m}_0$ and zero potential, and subjects this pair to the singular transformation (1) below. The resulting $\widehat{m}, V$ yield a Schrödinger equation that is in fact the Helmholtz equation for the corresponding singular Riemannian metric and thus covered by the analysis of cloaking in [4]. The treatment there shows that potentials within the cloaked region are undetectable by exterior measurements, whether far- or near-field, and that finite energy waves must satisfy the perfectly reflective Neumann boundary condition on the inside $\Sigma^-$ of the cloaking surface. The cloaking mass tensor $\widehat{m}$ is highly anisotropic, and infinitely so at $\Sigma^+$, making such a quantum cloak challenging to construct, with ultracold atoms in an optical lattice proposed in [1] as a possible realization.

The approach in this Letter is to forgo the perfect functioning of the ideal quantum cloak, and to describe sequences of bounded potentials $V_n^E, n = 1, 2, 3, \ldots$, which act as approximate cloaks with respect to $\widehat{m}_0$, thus not requiring extreme conditions to realize. The failure to cloak perfectly is in fact a controllable feature that can be taken advantage of for applications described below. The $V_n^E$ are found by means of *isotropic transformation optics*, a technique we introduce for avoiding the singular and anisotropic behavior, difficult to physically realize, of material parameters that commonly occur in transformation optics-based designs [3, 5, 6]; more details and proofs can be found in [7].

*Inverse scattering and conductivity cloaks.* There is an enormous literature on unique determination of a potential $V$ from scattering of plane waves at energy $E$ by the Schrödinger equation $\left(-\nabla^2 + V\right)\psi = E\psi$, as encoded in the scattering amplitude. Equivalently, for compactly supported $V$ one may consider the near-field measurements of wave functions at the boundary $\partial\Omega$ of a larger region $\Omega$, as encoded in the Dirichlet-to-Neumann (DN) operator, $\Lambda_V^E(\psi|_{\partial\Omega}) = \partial_\nu\psi|_{\partial\Omega}$ where $\nu$ is the normal of $\partial\Omega$ [8]. Ideal cloaking gives highly singular examples of nonuniqueness, but in order to construct approximately cloaking potentials, we first need to recall the ideal electrostatic cloaking of [9]. For simplicity, we describe the cloak on $B_3 - B_1$, with $B_R = \{r \le R\}$ denoting the central ball of radius $R$ in $\mathbb{R}^3$, so that the *cloaking surface*, the interface between the cloaked and uncloaked regions, is $\Sigma = \{r = 1\}$. Subjecting a homogeneous, isotropic conductivity $\sigma_0$ to a singular transformation, we constructed certain singular, anisotropic conductivity tensor fields on $B_3 - B_1$ which, when augmented by any conductivity bounded above and below on $B_1$, results in a total conductivity on $B_3$ giving the same electrostatic boundary measurements as $\sigma_0$. (For related results in dimension two, see [10, 11].) The same construction, applied to the electric permittivity and magnetic permeability rather than the conductivity, was used to propose an electromagnetic (EM) cloak [3], and a microwave realization of a variant of that design reported [13]. Ray-based cloaking for 2D was proposed in [5], while potentials transparent for rays are in [12].

Let $F = (F^1, F^2, F^3) : B_3 - \{0\} \to B_3 - B_1$ be the singular transformation, for $\mathbf{r} = (x^1, x^2, x^3) \in \mathbb{R}^3$,

$$\widetilde{\mathbf{r}} = F(\mathbf{r}) = \mathbf{r},\ 2 < r \le 3; F(\mathbf{r}) = \left(1 + \frac{r}{2}\right)\frac{\mathbf{r}}{r},\ r \le 2, \quad (1)$$

which results in the transformed version of $\sigma_0$ on $B_3 - B_1$, augmented for simplicity by $2\sigma_0$ on $B_1$,

$$\sigma_1 = F_*\sigma_0,\ \mathbf{r} \in B_3 - B_1; \quad \sigma_1 = 2\,\sigma_0,\ \mathbf{r} \in B_1. \qquad (2)$$

$F_*$ denotes the change-of-variables action of $F$ on tensors,

$$(F_*\sigma)^{jk}(\widetilde{\mathbf{r}}) = \left.\frac{1}{\det[\frac{\partial F^j}{\partial x^k}]} \sum_{p,q=1}^{3} \frac{\partial F^j}{\partial x^p}\,\frac{\partial F^k}{\partial x^q}\sigma^{pq}\right|_{\mathbf{r}=F^{-1}(\widetilde{\mathbf{r}})}.$$

The ideal cloak $\sigma_1$ has a singularity at $\Sigma$, both in that one of the eigenvalues (corresponding to the radial direction) tends to 0 as $r \searrow 1$, and that there is a jump across $\Sigma$, within which $\sigma_1$ is non-singular. Aside from the radius of the outer ball and the factor 2 in the second part of (2), $\sigma_1$ is the conductivity introduced in [9] and shown to be indistinguishable from $\sigma_0$, *vis-a-vis* boundary measurements at $\partial B_3$ of electrostatic fields.

Consider also the corresponding acoustic equation,

$$\partial_i\left(\sigma_1^{ij}\partial_j u\right) + E a_1 u = 0, \quad a_1 = \left(\det[\sigma_1^{ij}]\right)^{-1}, \qquad (3)$$

where we consider $\sigma_1$ as a mass density and $a_1$ as a bulk modulus. Then, using measurements at $\partial B_3$ of acoustic waves of frequency $\sqrt{E}$, the pair $\sigma_1$, $a_1$ is indistinguishable from $\sigma_0$, $a_0 = 1$ [4, 14–16]. The waves $u$ within the cloaked region have a simple description [4, Thm.1]. Either (I) if $E$ is *not* a Neumann eigenvalue of the cloaked region, then $u$ must vanish there; or (II) if $E$ *is* an eigenvalue, then $u$ can be an associated eigenfunction there, while possibly vanishing on $B_3 - B_1$. The Dirichlet eigenvalues and eigenfunctions of (3) on $B_3$ can be separated into $(E_j^{+,1}, u_j^+)$ and $(E_j^{-,1}, u_j^-)$, $j = 1, 2, 3, \ldots$, with $u_j^+$ and $u_j^-$ supported in $B_3 - B_1$ and $B_1$, resp.

*Approximate cloaks and failure of cloaking.* For $1 < R \le 2$, let $\sigma_R$ be given by the same formulae as in (2), but applied on $B_3 - B_R$ and $B_R$, resp., and similarly for $a_R$ in (3). Observe that for each $R > 1$, $\sigma_R$ and $a_R$ are *nonsingular*; however, their lower (resp., upper) bounds go to 0 (resp., $\infty$) as $R \searrow 1$. (Similar truncations of EM cloaks have been studied in [11, 17, 18].)

When $\sigma_1$, $a_1$ are replaced by $\sigma_R$, $a_R$ there is a decomposition similar to the one above for (3), with eigenvalues and eigenfunctions $(E_j^{+,R}, v_j^+)$ and $(E_j^{-,R}, v_j^-)$ concentrating in $B_3 - B_1$ and $B_1$, resp., and $E_j^{\pm,R}$ converging to $E_j^{\pm,1}$ as $R \to 1$. The solution of the boundary value problem $\partial_i\left(\sigma_R^{ij}\partial_j v\right) + E a_R v = 0$ on $B_3$, $v|_{\partial B_3} = f$, has an eigenfunction expansion

$$v(x) = \sum_{\pm}\sum_{j=1}^{\infty}\left(\int_{\partial B_3} f\frac{\partial v_j^{\pm}}{\partial\nu}\,dS\right)\frac{v_j^{\pm}(x)}{E - E_j^{\pm,R}}. \qquad (4)$$

An approximate version of dichotomy (I)-(II) holds for approximate acoustic cloaks: When $E$ is not equal to any $E_j^{\pm,1}$, one can show the DN operators for the $\sigma_R, a_R$ converge to that for $\sigma_1, a_1$ as $R \to 1$; physically, this means that the boundary measurements of pressure and the normal component of the particle velocity for the approximate cloaks tend to those for the ideal cloak, which are themselves the same as for $\sigma_0, a_0$. However, if $E$ is close to some $E_j^{-,R}$, the corresponding term in (4) may dominate the others, in which case the solution $v$, having a large coefficient of $v_j^-$, concentrates in $B_1$. Since $v_j^-$ cannot vanish identically in $B_3 - B_1$, both the near-field measurements on the boundary $\partial B_3$ and the far-field patterns differ noticeably from those corresponding to $\sigma_0$, $a_0$. This interior resonance corresponds to an acoustic wave almost trapped within the cloak.

*Isotropic transformation optics.* A well known phenomenon in effective medium theory is that homogenization of isotropic material parameters may lead, in the small-scale limit, to anisotropic ones [19]. We exploit this, using ideas from [20, 21], to approximate the anisotropic, almost cloaking $\sigma_R$ by *isotropic* conductivities $\sigma_{R,\epsilon}$ so that for $\varepsilon > 0$ the pairs $\sigma_{R,\epsilon}$, $a_R$ also function as approximate acoustic cloaks [7]. The $\sigma_{R,\epsilon}(\mathbf{r})$ are layered and spatially highly oscillating. (In the context of EM cloaking, thin concentric layers of homogeneous, isotropic media were considered in [22, 23].)

*Approximate Schrödinger cloaks.* The gauge transformation $\psi = \sqrt{\sigma}u$ reduces the acoustic equation (3), with nonsingular isotropic conductivity $\sigma = \sigma_{R,\epsilon}$ in place of the anisotropic $\sigma_1$, and $a_R$ in place of $a_1$, to the Schrödinger equation at the same energy $E$, $\left(-\nabla^2 + V_{R,\epsilon}^E\right)\psi = E\psi$, where $V_{R,\epsilon}^E =$

$\nabla^2(\sqrt{\sigma}_{R,\epsilon})/\sqrt{\sigma}_{R,\epsilon} + E\left(1 - a_R^{1/2}\sigma_{R,\epsilon}^{-1}\right)$. As $\sigma_{R,\epsilon}$ is highly oscillatory, $V_{R,\epsilon}^E$ consists of a layered pattern of concentric central potential barriers and wells of increasing amplitudes and decreasing widths as $\varepsilon \searrow 0$. The radial profile of the potential over one spherical layer is in Fig. 1(l). The boundary measurements of solutions of these Schrödinger equations at $\partial B_3$ coincide with those for the corresponding acoustic equations. By the convergence of the acoustic equations, we can choose $R \searrow 1$, $\epsilon \searrow 0$, so that the boundary measurements for these Schrödinger equations converge to those for the acoustic equation (3) at energy $E$, which in turn are the same as for the Schrödinger equation in free space. The nonresonant case is summarized by:

**Approximate Quantum Cloaking.** *Let $V_0$ be a bounded potential on $B_1$, and $E$ be neither a Dirichlet eigenvalue of the free Hamiltonian $-\nabla^2$ on $B_3$ nor a Neumann eigenvalue of $-\nabla^2 + V_0$ on $B_1$. Then there exists a sequence of cloaking potentials $V_n^E$ on $B_3$ such that the DN operators $\Lambda^E_{V_0+V_n^E} \to \Lambda_0^E$ as $n \to \infty$. I.e., at energy $E$ the potential $V_0 + V_n^E$ is indistinguishable by near-field measurements, asymptotically in $n$, from the zero potential; a similar result holds for far-field patterns. $V_0$ is thus approximately cloaked when surrounded by $V_n^E$.*

As any specific measurement device has a limited precision, this means that it is possible to design a potential to cloak an object within from any single-particle measurements made using that device at energy $E$.

**Numerics:** We use analytic expressions to compute the wave function $\psi$ for an incident plane wave with $\psi_{inc}(x) = ae^{ik\mathbf{r}\cdot\vec{d}}$. The computations are made without reference to physical units, using $a = 1, E = 0.5, k = \sqrt{E}$. The cloak is based on $R = 1.005$, corresponding to an anisotropy ratio of $\sigma_R$ at $\Sigma_R = \{r = R\}$ of $4 \times 10^4$. In the simulations we use a cloak consisting of 20 homogenized layers inside and 30 homogenized layers outside $\Sigma_R$. Inside the cloak we have located a centrally symmetric step potential, $W(x) = c_{inn}\chi_{[0,0.9]}(r)$. The cloaking potential $V_n^E$ and the energy $E$ are the same in all figures, but we vary the constant $c_{inn}$. In Fig. 1(r, red) and Fig. 2(l) we have $c_{inn} = -98.5$, and $\psi$ is the wave produced by an incoming plane wave. In Fig. 1(r, blue), with $c_{inn} = +1.858$, and in Fig. 2(r), with $c_{inn} = -71.45$, there is no incoming wave, but rather an excited almost trapped state in the cloaked region.

**Applications:** *Almost trapped states and ion traps.* A version of the dichotomy for approximate acoustic cloaks described above also holds for approximate quantum cloaks, since the $u$ and $\psi$ waves are equivalent by the gauge transformation. As a consequence, given an energy $E$, the approximate quantum cloak may be such that $E$ either is or is not an eigenvalue for the ideal cloak. This results in $B_1$ becoming either (I') an almost cloaked region that with a high probability does not accept energy $E$ particles from outside $\Sigma$; or (II') a trap that supports *almost trapped* states, which correspond to a particle at energy $E$ trapped in $B_1$ with high probability. A design of either type could possibly be implemented by an array of ac and dc electrodes with total effective potential approximating $V_n^E$ for large $n$ [24]. This leads to a new type of trap for ions, differing from, e.g., the Paul [25], Penning [26] or Zajfman [27] traps, and justified on the level of quantum mechanics. Furthermore, the trap may be made tunable by including a dc electrode in the trapped region, corresponding to a Coulomb $V_0$; varying the charge changes whether or not $E$ is as eigenvalue of $-\nabla^2 + V_0$ and thus which of (I') or (II') holds. Alternatively, one can switch between (I') and (II') by application of homogeneous magnetic fields; see below.

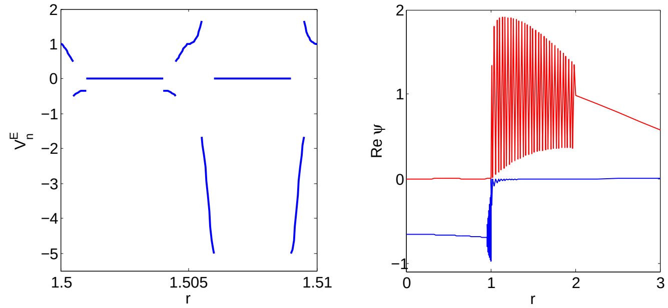


FIG. 1: **Left:** The radial profile of the potential $V_n^E$ over a typical layer $1.5 < r < 1.51$. The potential $V_n^E$ in $\{R = 1.005 \le r \le 2\}$ is obtained by repeating similar profiles, with increasing amplitudes as $r \searrow R$. **Right:** $\mathrm{Re}\,\psi$ on a segment $\{(x,0,0):\ 0 \le x \le 3\}$ for the same cloaking potential $V_n^E$ and two different cloaked $V_0$'s. For the red curve, $E$ is *not* close to an interior eigenvalue and $\psi$ is produced by an incoming plane wave. For the blue curve, $E$ *is* a Dirichlet eigenvalue on $B_3$ and $\psi$ is an almost trapped state.

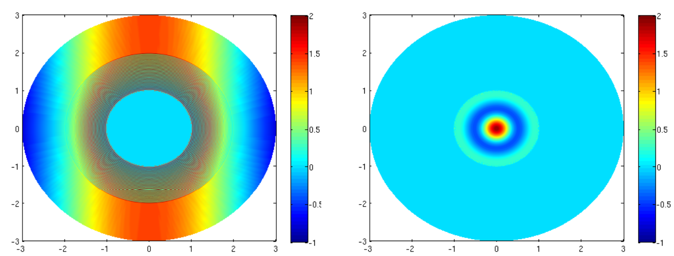

FIG. 2: **Left:** $\mathrm{Re}\,\psi$ in $B_3$ for $\psi$ resulting from an incident plane wave. $E$ is not near an interior Neumann eigenvalue; the matter wave passes unaltered. Moiré pattern is an artifact. **Right:** An excited almost trapped state. $E = E_j^{-,R}$ is an energy close to a Neumann eigenvalue of $B_1$, for which the ideal cloak supports an trapped state.

*Topological ion traps.* The basic construction outlined above can be modified to make the wave function on $B_1$ behave as though it were confined to a compact, boundaryless three-dimensional manifold, topologically but not metrically the three-dimensional sphere, $\mathbb{S}^3$. By suitable choice of metric, the energy level $E$ can have arbitrary multiplicity for the interior of the resulting trap, allowing one to implement physical systems mimicking matter waves on abstract spaces. As the starting point one uses not the original cloaking conductivity $\sigma_1$ (the *single coating* construction), but rather a *double* coating [4, Sec.2], which we denote here by $\sigma^{(2)}$. This is singular from both sides of $\Sigma$, and in the EM cloak setting corresponds to coating both sides of $\Sigma$ with metamaterials. See [7, Fig. 7]. By [4, Sec. 3.3], the finite energy solutions of the resulting Helmholtz equation on $B_3$ split into direct sums of waves on $B_3 - B_1$, as for $\sigma_1$, and waves on $B_1$ which are identifiable with eigenfunctions of the Laplace-Beltrami operator $-\nabla_g^2$ on $(\mathbb{S}^3, g)$ with eigenvalue $\sqrt{E}$. If one takes $g$ to be the standard metric on $\mathbb{S}^3$, then nonground states are degenerate and of high multiplicity, while a generic choice of $g$ yields nondegenerate energy levels [28]. On the other hand, by suitable choice of $g$ any finite number of energy levels and multiplicities can be specified [29], allowing traps supporting almost trapped states at energy $E$ of arbitrary degeneracy.

*Magnetically tunable quantum beam switch.* Consider a beam of ions of energy $E$, leaving an oven and traversing a tube $T = \{0 \le \rho \le \rho_0, 0 \le \theta \le 2\pi, 0 \le z \le L\}$ (in cylindrical coordinates). Treating the ions as matter waves, place in $T$ several almost trapping traps of the type described above, centered at points $z_j, j = 1, 2, \ldots, N$ on the $z$-axis, forming a potential $V(\mathbf{r}) = \sum_{j=1}^N V_n^E\big(\mathbf{r} - (0, 0, z_j)\big)$. The techniques above may be applied to the Schrödinger equation with magnetic potential $A(\mathbf{r})$ on a region $\Omega$,

$$\Big( -(\nabla + iA)^2 + V\Big)\psi = E\psi \quad \text{on } \Omega,$$
$$\Lambda^E_{V,A}(f) = \partial_\nu \psi|_{\partial\Omega} + i(A\cdot\nu)f \quad \text{on } \partial\Omega.$$

We design the traps so that, in the absence of a magnetic field, or for small field strengths, the traps act as cloaks and thus the ions pass through $T$ unhindered. However, if a homogeneous magnetic field is then applied to the tube, chosen so that the magnetic Schrödinger operator has $E$ as a Neumann eigenvalue inside each trap, then there is a large probability that an ion passing the $j$th trap will bind to that trap. If $N$ is large enough, then the probability that any ion traveling the length of $T$ will become bound is $\sim 1$, and $T$ thus functions as a magnetically controlled switch for the beam of ions.

*Magnified magnetic fields.* For a homogeneous magnetic field with linear magnetic potential $A$, one can obtain a sequence of electrostatic potentials $W_n$ for which $\lim_{n\to\infty} \Lambda^E_{W_n,A} = \Lambda^E_{0,\widetilde{A}}$, with $\widetilde{A}$ singular at a point. I.e., in the presence of a homogeneous magnetic field, the $W_n$ produce far- or near-field measurements that tend, as $n \to \infty$, to those of the zero electrostatic potential in the presence of a magnetic field blowing up at a point, giving the illusion of a locally singular magnetic field [7].

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