AN INVERSE BOUNDARY VALUE PROBLEM FOR A SEMILINEAR WAVE EQUATION ON LORENTZIAN MANIFOLDS

PETER HINTZ, GUNTHER UHLMANN, AND JIAN ZHAI

ABSTRACT. We consider an inverse boundary value problem for a semilinear wave equation on a time-dependent Lorentzian manifold with time-like boundary. The time-dependent coefficients of the nonlinear terms can be recovered in the interior from the knowledge of the Neumann-to-Dirichlet map. Either distorted plane waves or Gaussian beams can be used to derive uniqueness.

1. INTRODUCTION

Let (M, g) be a $(1+3)$-dimensional Lorentzian manifold with boundary ∂M, where the metric g is of signature $(-, +, +, +)$. We assume that $M = \mathbb{R} \times N$ where N is a manifold with boundary ∂N, and write the metric g as

$$g = -\beta(t, x') \mathrm{d}t^2 + \kappa(t, x'),$$

where $x = (t, x') = (x^0, x^1, x^2, x^3)$ are local coordinates on M; here, $\beta : \mathbb{R} \times N \to (0, \infty)$ is a smooth function and $\kappa(t, \cdot)$ is a Riemannian metric on N depending smoothly on $t \in \mathbb{R}$. The boundary $\partial M = \mathbb{R} \times \partial N$ of M is then timelike. Let ν denote the unit outer normal vector field to ∂M. Assume that ∂M is null-convex, which means that $\mathrm{II}(V, V) = g(\nabla_V \nu, V) \geq 0$ for all null vectors $V \in T(\partial M)$; see [16] for a discussion of this condition. We consider the semilinear wave equation on M

$$
\begin{aligned}
\Box_g u(x) + H(x, u(x)) &= 0, && \text{on } M, \\
\partial_\nu u(x) &= f(x), && \text{on } \partial M, \\
u(t, x') &= 0, && t < 0,
\end{aligned}
$$

(1)

where $\Box_g = |\det g|^{-1/2} \partial_j (\sqrt{|\det g|} g^{jk} \partial_k)$ is the wave operator (d'Alembertian) on (M, g). We assume that $H(x, z)$ is smooth in z near 0 with Taylor expansion

$$H(x, z) \sim \sum_{k=2}^{\infty} h_k(x) z^k, \quad h_k \in \mathcal{C}^\infty(M).$$

As Neumann data, we take f which are small in $\mathcal{C}^{m+1}$ for fixed large m. The Neumann-to-Dirichlet (ND) map Λ is defined as

$$\Lambda f = u|_{\partial M},$$

where u is the solution of (1). We will investigate the inverse problem of determining $h_j(x)$, $j = 2, 3, \ldots$, from Λ.

We remark that for the *linear* equation $\Box_g u + V u = 0$, the problem of recovering V from the ND map is still open in general. Stefanov and Yang [33] proved that the light ray transform of V can be recovered from boundary measurements; however, the invertibility of the light ray transform is

still unknown on general Lorentzian manifolds. We refer to [28, 12, 37] for an overview and recent results on the light ray transform.

In [24], the nonlinearity was exploited to solve inverse problems for a nonlinear equation where the corresponding inverse problem is still open for linear equations. The starting point of the approach is the higher order linearization, which we shall briefly introduce here. We take boundary Neumann data of the form $f = \sum_{i=1}^{N} \epsilon_i f_i$, where ϵ_i, $i = 1, \ldots, N$ are small parameters. Since Λ is a nonlinear map, $\Lambda(\sum_{i=1}^{N} \epsilon_i f_i)$ contains more information than $\{\Lambda(f_i)\}_{i=1,\ldots,N}$: indeed, useful information can be extracted from

$$\frac{\partial^N}{\partial \epsilon_1 \cdots \partial \epsilon_N}\Big|_{\epsilon_1=\cdots=\epsilon_N=0} \Lambda\left(\sum_{i=1}^{N} \epsilon_i f_i\right).$$

This higher order linearization technique has been extensively used in the literature [34, 18, 24, 30, 22, 29, 8, 38, 7, 35, 5, 1, 4, 26, 27, 14, 20, 21, 25]

The recovery of nonlinear terms from source-to-solution map was considered in [30], where the authors use the nonlinear interactions of distorted plane waves. The approach originated from [24], and has been successfully used to study inverse problems for nonlinear hyperbolic equations [30, 22, 29, 8, 38, 7, 35, 5]. For some similar problems, Gaussian beams are used instead of distorted plane waves [23, 13, 36]. The two approaches are actually closely related; both enable a pointwise recovery of the coefficients in the interior.

In this article, we will study the above inverse boundary value problem using both distorted plane waves and Gaussian beams. The two approaches will be discussed and compared in the last section.

To state our main result, recall that a smooth curve $\mu : (a, b) \to M$ is *causal* if $g(\dot\mu(s), \dot\mu(s)) \leq 0$ and $\dot\mu(s) \neq 0$ for all $s \in (a, b)$. Given $p, q \in M$, we write $p \leq q$ if $p = q$ or p can be joined to q by a future directed causal curve. We say $p < q$ if $p \leq q$ and $p \neq q$. We denote the causal future of $p \in M$ by $J^+(p) = \{q \in M : p \leq q\}$ and the causal past of $q \in M$ by $J^-(q) = \{p \in M : p \leq q\}$. We shall restrict the ND map to $(0, T) \times \partial N$, and correspondingly work in

$$\mathbb{U} = \bigcup_{p,q \in (0,T) \times \partial N} J^+(p) \cap J^-(q).$$

We assume that null geodesics in $\mathbb{U}$ do not have cut points.

Theorem 1. *Consider the semilinear wave equations*

$$\Box_g u(x) + H^{(j)}(x, u(x)) = 0, \qquad j = 1, 2.$$

Assume $H^{(j)}(x, z)$ are smooth in z near 0 and have a Taylor expansion[1]

$$H^{(j)}(x, z) \sim \sum_{k=2}^{\infty} h_k^{(j)}(x) z^k, \qquad h_k^{(j)} \in \mathcal{C}^\infty(\overline{\mathbb{U}}).$$

If the Neumann-to-Dirichlet maps $\Lambda^{(j)}$ acting on $\mathcal{C}^6([0, T] \times \partial N)$ are equal, $\Lambda^{(1)} = \Lambda^{(2)}$, then

$$h_k^{(1)}(x) = h_k^{(2)}(x), \qquad x \in \mathbb{U}, \ k \geq 2.$$

The strategy of the proof is to send in distorted plane waves (or Gaussian beams) from *outside* the manifold M (within a small extension $\widetilde{M}$) and analyze contributions to the ND map from

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[1] The notation means that $h_k^{(j)}(x) = \frac{1}{k!} \frac{\partial^k}{\partial z^k} H^{(j)}(x, 0)$.

nonlinear interactions in the interior of M as well as from subsequent reflections at the boundary ∂M of M.

The rest of this paper is organized as follows. In Section 2, we establish the well-posedness of the initial boundary value problem (1) for small boundary data. In Section 3, we use the nonlinear interaction of distorted plane waves to prove the main theorem. In Section 4, we give another proof of the main theorem using Gaussian beam solutions, assuming h_2 is already known. Finally, the two approaches will be compared and discussed in Section 5.

2. Well-posedness for small boundary data

We establish well-posedness of the initial boundary value problem (1) in this section with small boundary value f. We assume f satisfies the compatibility condition $f = \frac{\partial f}{\partial t} = 0$ at $\{t = 0\}$.

Fix $m \geq 5$. We assume $f \in \mathcal{C}^{m+1}([0,T] \times \partial N)$ and $\|f\|_{\mathcal{C}^{m+1}([0,T] \times \partial N)} \leq \epsilon_0$ for a small number $\epsilon_0 > 0$. We can find a function $h \in \mathcal{C}^{m+1}([0,T] \times N)$ such that $\partial_\nu h|_{[0,T] \times \partial N} = f$ and

$$\|h\|_{\mathcal{C}^{m+1}([0,T] \times N)} \leq C\|f\|_{\mathcal{C}^{m+1}([0,T] \times \partial N)}.$$

Let $\widetilde{u} = u - h$. Then $\widetilde{u}$ satisfies the equation

$$\Box_g \widetilde{u} = F(x, \widetilde{u}, h) := -\Box_g h - H(x, \widetilde{u} + h),$$

supplemented with the boundary condition $\partial_\nu \widetilde{u} = 0$ on $(0,T) \times \partial N$ and initial conditions $\widetilde{u} = \frac{\partial \widetilde{u}}{\partial t} = 0$ at $\{0\} \times N$. The above equation can be written in the form

$$
\text{(2)} \quad
\begin{aligned}
\Box_g \widetilde{u} &= F(x, \widetilde{u}, h), &&\text{in } (0,T) \times \partial N, \\
\partial_\nu \widetilde{u} &= 0, &&\text{on } (0,T) \times \partial N, \\
\widetilde{u} = \frac{\partial \widetilde{u}}{\partial t} &= 0, &&\text{on } t = 0.
\end{aligned}
$$

This equation is of the form [6, equation (5.12)]. For $R > 0$, define $Z(R,T)$ as the set of all functions w satisfying

$$w \in \bigcap_{k=0}^{m} W^{k,\infty}([0,T]; H^{m-k}(N)), \qquad \|w\|_Z^2 := \sup_{t \in [0,T]} \sum_{k=0}^{m} \|\partial_t^k w(t)\|_{H^{m-k}}^2 \leq R^2.$$

We can write $F(x, \widetilde{u}, h) = \mathscr{F} + G(x, \widetilde{u}, h)\widetilde{u}$ where $\mathscr{F} = -\Box_g h - H(x, h)$ and

$$G(x, \widetilde{u}, h) = -\int_0^1 \partial_z H(x, h + \tau \widetilde{u}) \mathrm{d}\tau.$$

Since $H(x, z)$ is smooth in z, we have

$$\sup_{t \in [0,T]} \sum_{k=0}^{m-1} \|\partial_t^k \mathscr{F}(t)\|_{H^{m-k-1}} \leq C \sup_{t \in [0,T]} \sum_{k=0}^{m-1} \|\partial_t^k \mathscr{F}(t)\|_{\mathcal{C}^{m-k-1}} \leq C'\epsilon_0.$$

Moreover, $\partial_z H(x, z)$ vanishes linearly in z, hence we have

$$G(x, \widetilde{u}, h) \in \bigcap_{k=0}^{m} W^{k,\infty}([0,T]; H^{m-k}(N)), \quad \|G(x, \widetilde{u}, h)\|_Z \leq C(\|h\|_Z + \|\widetilde{u}\|_Z) \leq C'(\epsilon_0 + \|\widetilde{u}\|_Z)$$

for $\widetilde{u} \in Z(\rho_0, T)$ with ρ_0 small enough.

Given $\widetilde{w} \in Z(\rho_0, T)$, consider first the *linear* initial boundary value problem

$$
\begin{aligned}
\Box_g \widetilde{u} - G(x, \widetilde{w}, h)\widetilde{w} &= \mathscr{F}(x, h), & t &\in (0, T), \\
\partial_\nu \widetilde{u} &= 0, & t &\in (0, T), \\
\widetilde{u}(0) = \frac{\partial \widetilde{u}}{\partial t}(0) &= 0.
\end{aligned}
$$

(3)

By [6, Theorem 3.1], there exists a unique solution $\widetilde{u} \in \bigcap_{k=0}^{m} \mathcal{C}^k([0, T]; H^{m-k}(N))$ to (3), and it satisfies the estimate

$$
\|\widetilde{u}\|_Z \le C(\epsilon_0 + \epsilon_0 \|\widetilde{w}\|_Z + \|\widetilde{w}\|_Z^2) e^{KT},
$$

where C, K are positive constants depending on the coefficients of the equation. Denote $\mathscr{T}$ to be the map which maps $\widetilde{w} \in Z(\rho_0, T)$ to the solution $\widetilde{u}$ of (3). Notice that we can take ρ_0 small enough and $\epsilon_0 = \frac{e^{-KT}}{2C}\rho_0$ such that

$$
C(\epsilon_0 + \epsilon_0 \rho_0 + \rho_0^2) e^{KT} < \rho_0.
$$

Then $\mathscr{T}$ maps $Z(\rho_0, T)$ to itself.

Now assume $\widetilde{u}_j$, $j = 1, 2$, solve the equation

$$
\begin{aligned}
\Box_g \widetilde{u}_j - G(x, \widetilde{w}_j, h)\widetilde{w}_j &= \mathscr{F}(x, h), & t \in (0, T) \\
\widetilde{u}_j(0) = \frac{\partial \widetilde{u}_j}{\partial t}(0) &= 0.
\end{aligned}
$$

We have $\widetilde{u}_j = \mathscr{T}\widetilde{w}_j$, $j = 1, 2$ and

$$
\Box_g(\widetilde{u}_1 - \widetilde{u}_2) = -\left(\int_0^1 \partial_z H(x, h + \widetilde{w}_2 + \tau(\widetilde{w}_1 - \widetilde{w}_2))\mathrm{d}\tau \right)(\widetilde{w}_1 - \widetilde{w}_2).
$$

Then

$$
\|\mathscr{T}\widetilde{w}_1 - \mathscr{T}\widetilde{w}_2\|_Z = \|\widetilde{u}_1 - \widetilde{u}_2\|_Z \le C(\epsilon_0 + \rho_0)e^{KT}\|\widetilde{w}_1 - \widetilde{w}_2\|_Z.
$$

Choosing ρ_0 small enough such that $C(\epsilon_0 + \rho_0)e^{KT} < 1$, the map $\mathscr{T}$ is a contraction. Consequently, the equation (2) has a unique solution $\widetilde{u}$ in $Z(\rho_0, T)$. Using [6, Theorem 3.1] again, we have

$$
\widetilde{u} \in \bigcap_{k=0}^{m} \mathcal{C}^k([0, T]; H^{m-k}(N)).
$$

In summary, we have shown:

Theorem 2. *Let $T > 0$ be fixed. Assume that $f \in \mathcal{C}^{m+1}([0, T) \times \partial N)$, $m \ge 5$, and $f = \partial_t f = 0$ at $t = 0$. Then there exists $\epsilon_0 > 0$ such that for $\|f\|_{\mathcal{C}^m} \le \epsilon_0$, there exists a unique solution*

$$
u \in \bigcap_{k=0}^{m} \mathcal{C}^k([0, T]; H^{m-k}(N))
$$

of equation (1). *It satisfies the estimate*

$$
\sup_{t \in [0, T]} \|\partial_t^{m-k} u(t)\|_{H^{m-k}(N)} \le C\|f\|_{\mathcal{C}^{m+1}([0,T]\times\partial N)},
$$

where $C > 0$ is independent of f.

3. Recovery using distorted plane waves

In this section we will show how to recover h_k, $k = 1, 2, \ldots$ by using the nonlinear interaction of distorted plane waves. First we extend the metric g on M smoothly to a slightly larger open manifold $\widetilde{M} = \mathbb{R}_t \times \widetilde{N}$ such that N is contained in the interior of $\widetilde{N}$, and thus M is contained in the interior of $\widetilde{M}$.

3.1. **Notations and preliminaries.** For $p \in \widetilde{M}$, denote the set of light-like vectors at p by

$$L_p\widetilde{M} = \{\zeta \in T_p\widetilde{M} \setminus \{0\} : g(\zeta, \zeta) = 0\}.$$

The set of light-like covectors at p is denoted by $L_p^*\widetilde{M}$. The sets of future and past light-like vectors (covectors) are denoted by $L_p^+\widetilde{M}$ and $L_p^-\widetilde{M}$ ($L_p^{*,+}\widetilde{M}$ and $L_p^{*,+}\widetilde{M}$). Define the future directed light-cone emanating from p by

$$\mathcal{L}^+(p) = \{\gamma_{p,\zeta}(t) \in \widetilde{M} : \zeta \in L_p^+\widetilde{M}, t \geq 0\} \subset \widetilde{M}.$$

Distorted plane waves have singularities conormal to a submanifold of $\widetilde{M}$ and can be viewed as Lagrangian distributions. We review them briefly, closely following the notation used in [30]. Recall that $T^*\widetilde{M}$ is a symplectic manifold with canonical 2-form, given in local coordinates by $\omega = \sum_{j=1}^4 d\xi_j \wedge dx^j$. A submanifold $\Lambda \subset T^*\widetilde{M}$ is called Lagrangian if $\dim \Lambda = 4$ and ω vanishes on Λ. For K a smooth submanifold of $\widetilde{M}$, its conormal bundle

$$N^*K = \{(x, \zeta) \in T^*\widetilde{M} : x \in K, \langle \zeta, \theta \rangle = 0, \theta \in T_xK\}$$

is a Lagrangian submanifold of $T^*\widetilde{M}$.

Let Λ be a smooth conic Lagrangian submanifold of $T^*\widetilde{M} \setminus 0$. We denote by $\mathcal{I}^\mu(\Lambda)$ the space of Lagrangian distributions of order μ associated with Λ. Let $\Lambda_0, \Lambda_1 \subset T^*\widetilde{M} \setminus 0$ be two Lagrangian submanifolds intersecting cleanly, i.e.,

$$T_p\Lambda_0 \cap T_p\Lambda_1 = T_p(\Lambda_0 \cap \Lambda_1) \quad \forall p \in \Lambda_0 \cap \Lambda_1.$$

We denote the space of paired Lagrangian distributions associated with (Λ_0, Λ_1) by $\mathcal{I}^{p,l}(\Lambda_0, \Lambda_1)$. For more details, we refer to [31, 15].

Fix a Riemannian metric g^+ on $\widetilde{M}$. Given $x_0 \in \widetilde{M} \setminus M$, $\zeta_0 \in L_{x_0}^+\widetilde{M}$, and $s_0 > 0$, put

$$\mathcal{W}_{x_0,\zeta_0,s_0} = \{\eta \in L_{x_0}^+M : \|\eta - \zeta_0\|_{g^+} < s_0\},$$
$$K(x_0, \zeta_0, s_0) = \{\gamma_{x_0,\eta}(s) \in M : \eta \in \mathcal{W}_{x_0,\zeta_0,s_0}, s \in (0, \infty)\},$$
$$\Lambda(x_0, \zeta_0, s_0) = \{(\gamma_{x_0,\eta}(s), r\dot{\gamma}_{x_0,\eta}(s)^\flat) \in T^*M; \eta \in \mathcal{W}_{x_0,\zeta_0,s_0}, s \in (0, \infty), r > 0\}.$$

Notice that $K(x_0, \zeta_0, s_0)$ is a subset of codimension 1 of the light cone $\mathcal{L}^+(x_0)$, and

$$N^*K(x_0, \zeta_0, s_0) = \Lambda(x_0, \zeta_0, s_0).$$

By [24, Lemma 3.1], one can construct distributions $u_0 \in \mathcal{I}^\mu(\widetilde{M}, \Lambda(x_0, \zeta_0, s_0))$ which on M satisfy $\Box_g u_0 \in \mathcal{C}^\infty(M)$, and whose principal symbol is nonzero on $(\gamma_{x_0,\zeta_0}(s), \dot{\gamma}_{x_0,\zeta_0}(s)^\flat)$. Thus, u_0 is a nontrivial distorted plane wave propagating on the surface $K(x_0, \zeta_0, s_0)$.

We consider four distorted plane waves

$$u_j \in \mathcal{I}^\mu(\widetilde{M}, \Lambda(x_j, \xi_j, s_0)), \quad j = 1, 2, 3, 4,$$

which are approximate solutions of the linearized wave equation in M, that is, $\Box_g u_j \in \mathcal{C}^\infty(M)$. Let

(4) $$K_j = K(x_j, \xi_j, s_0), \qquad \Lambda_j = \Lambda(x_j, \xi_j, s_0) = N^* K_j.$$

Assume that

(1) K_i, K_j, $i \neq j$, intersect transversally at a co-dimension 2 submanifold $K_{ij} \subset \widetilde{M}$;
(2) K_i, K_j, K_k, i, j, k distinct, intersect at a co-dimension 3 submanifold $K_{ijk} \subset \widetilde{M}$;
(3) K_1, K_2, K_3, K_4 intersect at a point $q_0 \in M$.

We use the notations

$$\Lambda_{ij} = N^* K_{ij}, \quad \Lambda_{ijk} = N^* K_{ijk}, \qquad \Lambda_{q_0} = T^*_{q_0} M \setminus 0;$$

which are all Lagrangian submanifolds in $T^* M$. For any $\Gamma \subset T^* M$, we denote by Γ^g the flow-out of $\Gamma \cap L^{*,+} \widetilde{M}$ under the null-geodesic flow of g lifted to $T^* \widetilde{M}$.

We assume $x_j \in (0, T) \times \widetilde{N}$; we can take s_0 small enough so that u_j is smooth near $t = 0$. Denote $f_i = \partial_\nu u_i|_{\partial M}$; then the solutions v_i of the linear equations

$$\begin{aligned}
\Box_g v_i(x) &= 0, & &\text{on } M, \\
\partial_\nu v_i(x) &= f_i(x), & &\text{on } \partial M, \\
v_i(t, y) &= 0, & &t < 0,
\end{aligned}$$

are equal to u_i modulo $\mathcal{C}^\infty(M)$. For $N = 3$ or 4, consider then

(5) $$f = \sum_{i=1}^N \epsilon_i f_i,$$

and denote $v = \sum_{i=1}^N \epsilon_i v_i$. We write $w = Q_g(F)$ if w solves the linear wave equation

$$\begin{aligned}
\Box_g w(x) &= F, & &\text{on } M, \\
\partial_\nu w(x) &= 0, & &\text{on } \partial M, \\
w &= 0, & &t < 0.
\end{aligned}$$

The solution u to (1) is then given by the asymptotic expansion [30, (2.9)]

(6) $$\begin{aligned}
u = {}& v - Q_g(h_2 v^2) + 2Q_g(h_2 v Q_g(h_2 v^2) - 4Q_g(h_2 v Q_g(h_2 v Q_g(h_2 v^2)))) \\
& - Q_g(h_2 Q_g(h_2 v^2) Q_g(h_2 v^2)) + 2Q_g(h_2 v Q_g(h_3 v^3)) - Q_g(h_3 v^3) + 3Q_g(h_3 v^2 Q_g(h_2 v^2)) \\
& - Q_g(h_4 v^4) + \text{higher order terms in } \epsilon_1, \dots, \epsilon_N.
\end{aligned}$$

We will use the singularities from the terms in (6) to recover the coefficients of (1). Notice that those terms involve nonlinear interactions of distorted plane waves v_j, $j = 1, \dots, N$, and thus new singularities can be created. Recovery of a Lorentzian metric from the source-to-solution map using those newly generated singularities was first carried out in [24]. For recovery of the coefficients of nonlinear terms, we refer to [30, 8].

3.2. **Nonlinear interactions of three waves and recovery of h_2^2 and h_3.** First, we will first use three distorted plane waves, i.e. taking $N = 3$ in (5) and using Neumann data

$$f = \sum_{i=1}^3 \epsilon_i f_i$$

with $\epsilon_i > 0$, $i = 1, 2, 3$, small parameters. We will construct suitable sources f_i, $i = 1, 2, 3$, and denote by v_i the corresponding distorted plane wave.

For any $p \in M$ and $\xi \in L_p^{*,+}M$ define $\gamma(s) = \gamma_{p,\xi}(s)$ to be the geodesic such that $\gamma(0) = p$ and $\dot\gamma(0) = \xi^\sharp$. Define

$$s^+(p, \xi) = \inf\{s > 0 : \gamma(s) \in \partial M\}, \quad s^-(p, \xi) = \sup\{s < 0 : \gamma(s) \in \partial M\}.$$

Fix a point $q_0 \in \mathbb{U}$. There exist $\xi^{(0)}, \xi^{(1)} \in L_{q_0}^{*,+}M$ such that

$$(7) \qquad x^- = \gamma_{q_0, \xi^{(1)}}(s^-(q_0, \xi^{(1)})) \in (0, T) \times \partial N, \quad x_0 = \gamma_{q_0, \xi^{(0)}}(s^+(q_0, \xi^{(0)})) \in (0, T) \times \partial N.$$

Put $\gamma^{(j)} = \gamma_{q_0, \xi^{(j)}}$, $j = 0, 1$ and denote $x_1 = \gamma^{(1)}(s^-(q_0, \xi^{(1)}) - \epsilon)$ for $\epsilon > 0$ small; thus, $x_1 \in \widetilde{M} \setminus M$ lies just barely outside of M.

Choose local coordinates so that g coincides with the Minkowski metric at q_0. Without loss of generality, one can assume

$$\xi^{(0)} = (-1, -\sqrt{1 - r_0^2}, r_0, 0), \qquad \xi^{(1)} = (-1, 1, 0, 0),$$

for some $r_0 \in [-1, 1]$. Take a small parameter $\varsigma > 0$ and introduce two perturbations of $\xi^{(1)}$

$$\xi^{(2)} = (-1, \sqrt{1 - \varsigma^2}, \varsigma, 0), \qquad \xi^{(3)} = (-1, \sqrt{1 - \varsigma^2}, -\varsigma, 0).$$

Notice $\xi^{(2)}, \xi^{(3)} \in L_p^{*,+}M$. One can then write $\xi^{(0)}$ as a linear combination of $\xi^{(1)}, \xi^{(2)}, \xi^{(3)}$,

$$\xi^{(0)} = \alpha_1 \xi^{(1)} + \alpha_2 \xi^{(2)} + \alpha_3 \xi^{(3)},$$

with

$$\alpha_1 = \frac{-\sqrt{1 - \varsigma^2} - \sqrt{1 - r_0^2}}{1 - \sqrt{1 - \varsigma^2}}, \quad \alpha_2 = \frac{1 + \sqrt{1 - r_0^2}}{2(1 - \sqrt{1 - \varsigma^2})} + \frac{r_0}{2\varsigma}, \quad \alpha_3 = \frac{1 + \sqrt{1 - r_0^2}}{2(1 - \sqrt{1 - \varsigma^2})} - \frac{r_0}{2\varsigma}.$$

Denote $b(r_0) = 1 + \sqrt{1 - r_0^2}$. By direct calculation, and using the asymptotics $\sqrt{1 - \varsigma^2} = 1 - \frac{1}{2}\varsigma^2 + \mathcal{O}(\varsigma^4)$, we obtain

$$|\alpha_1 \xi^{(1)} + \alpha_2 \xi^{(2)}|_g^2 = 2b(r_0)^2 \varsigma^{-2} + \mathcal{O}(\varsigma^{-1}),$$
$$|\alpha_1 \xi^{(1)} + \alpha_3 \xi^{(3)}|_g^2 = 2b(r_0)^2 \varsigma^{-2} + \mathcal{O}(\varsigma^{-1}),$$
$$|\alpha_2 \xi^{(2)} + \alpha_3 \xi^{(3)}|_g^2 = -4b(r_0)^2 \varsigma^{-2} + \mathcal{O}(\varsigma^{-1}).$$

Therefore,

$$|\alpha_1 \xi^{(1)} + \alpha_2 \xi^{(2)}|_g^{-2} + |\alpha_1 \xi^{(1)} + \alpha_3 \xi^{(3)}|_g^{-2} + |\alpha_2 \xi^{(2)} + \alpha_3 \xi^{(3)}|_g^{-2} = \frac{3}{4b(r_0)^2}\varsigma^2 + \mathcal{O}(\varsigma^3).$$

By taking ς small enough, the quantity

$$\sum_{\sigma \in \Sigma(3)} \left| \alpha_{\sigma(2)} \xi^{(\sigma(2))} + \alpha_{\sigma(3)} \xi^{(\sigma(3))} \right|_{g^*(q_0)}^{-2}$$

is nonvanishing; here, $\Sigma(3)$ denotes the permutation group of $\{1, 2, 3\}$.

For $j = 2, 3$, let $\gamma^{(j)} = \gamma_{q_0, \xi^{(j)}}$, and denote

$$x_j = \gamma^{(j)}(s^-(q_0, \xi^{(j)}) - \epsilon), \quad j = 2, 3,$$

for $\epsilon > 0$ small. Here, if we took $\epsilon = 0$, then we could choose ς small enough so that $x_j \in (0, T) \times \partial N$; fixing ς in this manner, we can then take $\epsilon > 0$ small enough so that $x_j \in \widetilde{M} \setminus M$ and $t > 0$ at x_j still. Now for $j = 1, 2, 3$ denote

$$\xi_j = \dot{\gamma}_{q_0, \xi^{(j)}}(s^-(q_0, \xi^{(j)}) - \epsilon)^\flat \in L_{x_j}^{*,+}M.$$

Use these (x_j, ξ_j), $j = 1, 2, 3$, in (4) and denote associated distorted plane waves by

$$u_j \in \mathcal{I}^\mu(\Lambda_j), \quad j = 1, 2, 3.$$

We note that $\xi^{(0)} \in N_p^* K_{123}$.

Let u denote the solution of (1) with $f = \sum_{i=1}^3 \epsilon_i f_i$, and put

$$\mathcal{U}^{(3)} = \partial_{\epsilon_1} \partial_{\epsilon_2} \partial_{\epsilon_3} u|_{\epsilon_1 = \epsilon_2 = \epsilon_3 = 0}.$$

We can then decompose

$$\mathcal{U}^{(3)} = \mathcal{U}_0^{(3)} + \mathcal{U}_1^{(3)}, \qquad \mathcal{U}_0^{(3)} := -6 Q_g(h_3 v_1 v_2 v_3), \quad \mathcal{U}_1^{(3)} := 2 \sum_{\sigma \in \Sigma(3)} Q_g(h_2 v_{\sigma(1)} Q(h_2 v_{\sigma(2)} v_{\sigma(3)})).$$

Denote by $\widetilde{Q}_g = \Box_g^{-1}$ causal (retarded) inverse of $\Box_g$ on $\widetilde{M}$ (no boundary!). Then

$$\mathcal{U}^{(3),\mathrm{inc}} := \mathcal{U}_0^{(3),\mathrm{inc}} + \mathcal{U}_1^{(3),\mathrm{inc}} = -6\widetilde{Q}_g(h_2 v_1 v_2 v_3) + 2 \sum_{\sigma \in \Sigma(3)} \widetilde{Q}_g(h_2 v_{\sigma(1)} \widetilde{Q}(h_2 v_{\sigma(2)} v_{\sigma(3)}))$$

is the incident wave before reflection on the boundary. By [30, Proposition 2.1, 3.7], we know that

$$\mathcal{U}^{(3),\mathrm{inc}} \in \mathcal{I}^{3\mu + \frac{1}{2}, -\frac{1}{2}}(\Lambda_{123}, \Lambda_{123}^g),$$

away from $\cup_{i=1}^3 \Lambda^{(i)}$. Its principal symbol is as follows: given $\zeta = \alpha \xi^{(0)}$ for some $\alpha \in \mathbb{R}$, there exists a unique decomposition $\zeta = \sum_{j=1}^3 \zeta_j$ with $\zeta_j \in N_q^* K_j$ (in fact, $\zeta_j = \alpha \alpha_j \xi^{(j)}$); if (y, η) lies along the forward null-bicharacteristic of $\Box_g$ starting at (q_0, ζ), we have
(8)

$$\sigma^{(p)}(\mathcal{U}^{(3),\mathrm{inc}})(y, \eta) = \sigma^{(p)}(\mathcal{U}_0^{(3),\mathrm{inc}})(y, \eta) = -6(2\pi)^{-2} \sigma^{(p)}(\widetilde{Q}_g)(y, \eta, q_0, \zeta) h_3(q_0) \prod_{j=1}^3 \sigma^{(p)}(v_j)(q_0, \zeta_j).$$

We are particularly interested in this expression for $y = x_0$.

Now, the solution $\mathcal{U}^{(3)}$ of the initial-boundary value problem can be written as the sum of the incident wave $\mathcal{U}^{(3),\mathrm{inc}}$ and wave $\mathcal{U}^{(3),\mathrm{ref}}$ arising from reflection at $\widetilde{M}$

$$\mathcal{U}^{(3)} = \mathcal{U}^{(3),\mathrm{inc}} + \mathcal{U}^{(3),\mathrm{ref}}.$$

The reflected wave vanishes prior to the intersection of supp $\mathcal{U}^{(3),\mathrm{inc}}$ with the boundary ∂M, and in a small neighborhood of y, satisfies $\Box_g \mathcal{U}^{(3),\mathrm{ref}} = 0$ with Neumann data $\partial_\nu \mathcal{U}^{(3),\mathrm{ref}} = -\partial_\nu \mathcal{U}^{(3),\mathrm{inc}}$. Since, near y, $\mathcal{U}^{(3),\mathrm{inc}}$ is a conormal distribution relative to the conormal bundle of a submanifold transversal to ∂M (due to the null-convexity assumption), so is $\mathcal{U}^{(3),\mathrm{ref}}$, and the principal symbols of their restrictions to ∂M agree due to the Neumann boundary condition. (Indeed, following [33], we have, microlocally near y, η, $\mathcal{U}^{(3),\bullet} = (2\pi)^{-3} \int e^{i\phi^\bullet(x,\theta)} a^\bullet(x, \theta) \, d\theta$ for $\bullet = \mathrm{inc}, \mathrm{ref}$ and suitable symbols $a^\bullet$, where the phase functions $\phi^\bullet$ solve the eikonal equation $|d\phi^\bullet|_{g^*}^2 = 0$ with boundary conditions $\phi^\bullet(x, \theta) = x \cdot \theta$, $x \in \partial M$, and $\partial_\nu \phi^\mathrm{ref} = -\partial_\nu \phi^\mathrm{inc}$. The Neumann boundary condition $\partial_\nu \mathcal{U}^{(3)}|_{\partial M} = 0$ implies $(\partial_\nu \phi^\mathrm{inc}) a^\mathrm{inc} + (\partial_\nu \phi^\mathrm{ref}) a^\mathrm{ref} = 0$, thus $a^\mathrm{inc} = a^\mathrm{ref}$ at ∂M, as claimed.)

Denote $\mathcal{R}(\mathcal{U}^{(3),\text{inc}})$ to be the trace of $\mathcal{U}^{(3),\text{inc}}$ on ∂M; this is an FIO of order $\frac{1}{4}$ with canonical relation

$$\Gamma_{\mathcal{R}} = \{(y_|, \eta_|, y, \eta) \in (T^*(\partial M) \times T^*M) \setminus 0; y_| = y, \eta_| = \eta|_{T_y(\partial M)}\}$$

For any $(y_|, \eta_|) \in T^*(\partial M)$, there exists at most one outward pointing $\eta \in L_y^* M$ such that $\eta_| = \eta|_{T_y(\partial M)}$. For such $(y_|, \eta_|, y, \eta)$, the principal symbol $\sigma^{(p)}(\mathcal{R})(y_|, \eta_|, y, \eta)$ is nonzero (cf. [10]). Then

$$\frac{1}{2}\sigma^{(p)} \left(\partial_{\epsilon_1}\partial_{\epsilon_2}\partial_{\epsilon_3}\Lambda \left(\epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3\right)|_{\epsilon_1=\epsilon_2=\epsilon_3=0}\right)(y_|, \eta_|) = \sigma^{(p)}(\mathcal{R})(y_|, \eta_|, y, \eta)\sigma^{(p)}(\mathcal{U}_0^{(3),\text{inc}})(y, \eta).$$

We now show how to use this to recover h_3 from the principal symbol of $\mathcal{U}_0^{(3),\text{inc}}$: for $j = 1, 2$, let $u^{(j)}$ solve the equation (1) with $H = H^{(j)}$ and $\partial_\nu u^{(j)} = f$. Set

$$\mathcal{U}^{(3),\text{inc},j} = \mathcal{U}_0^{(3),\text{inc},j} + \mathcal{U}_1^{(3),\text{inc},j},$$

By assumption, we have

$$\partial_{\epsilon_1}\partial_{\epsilon_2}\partial_{\epsilon_3}\Lambda^{(1)}(f)|_{\epsilon_1=\epsilon_2=\epsilon_3=0} = \partial_{\epsilon_1}\partial_{\epsilon_2}\partial_{\epsilon_3}\Lambda^{(2)}(f)|_{\epsilon_1=\epsilon_2=\epsilon_3=0};$$

the above expression shows that this implies

$$\sigma^{(p)}(\mathcal{U}_0^{(3),\text{inc},1})(y, \eta) = \sigma^{(p)}(\mathcal{U}_0^{(3),\text{inc},2})(y, \eta)$$

(in particular for $y = x_0$). By the explicit formula for $\sigma^{(p)}(\mathcal{U}_0^{(3),\text{inc},j})(y, \eta)$ given by (8), we get

$$h_3^{(1)}(q_0) = h_3^{(2)}(q_0).$$

Now we analyze

$$\mathcal{U}_1^{(3)} := 2 \sum_{\sigma \in \Sigma(3)} Q_g(h_2 v_{\sigma(1)} Q_g(h_2 v_{\sigma(2)} v_{\sigma(3)})).$$

Since h_3 has already been recovered, we can subtract its contribution to $\mathcal{U}^{(3)}$; we can thus determine $\mathcal{U}_1^{(3)}|_{\partial M}$. Similarly to before, we write $\mathcal{U}_1^{(3)} = \mathcal{U}_1^{(3),\text{inc}} + \mathcal{U}_1^{(3),\text{ref}}$, where

$$\mathcal{U}_1^{(3),\text{inc}} = 2 \sum_{\sigma \in \Sigma(3)} \widetilde{Q}_g(h_2 v_{\sigma(1)} \widetilde{Q}_g(h_2 v_{\sigma(2)} v_{\sigma(3)}))$$

is the incident wave and $\mathcal{U}_1^{(3),\text{ref}}$ is the reflected wave. By [30, Lemma 3.3, 3.4], we have

$$\widetilde{Q}_g(h_2 v_i v_j) \in \mathcal{I}^{\mu-1,\mu}(\Lambda_{ij}, \Lambda_i) + \mathcal{I}^{\mu-1,\mu}(\Lambda_{ij}, \Lambda_j).$$

Then by [30, Lemma 3.6 and Proposition 2.1]

$$\mathcal{U}_1^{(3),\text{inc}} \in \mathcal{I}^{3\mu-\frac{3}{2},-\frac{1}{2}}(\Lambda_{123}, \Lambda_{123}^g),$$

away from $\cup_{i=1}^3 \Lambda_i$. By the calculation in [30], we have

$$\sigma^{(p)}(\mathcal{U}_1^{(3),\text{inc}})(y, \eta) = 2(2\pi)^{-2}\sigma^{(p)}(\widetilde{Q}_g)(y, \eta, q_0, \zeta)h_2(q_0)^2 \left(\sum_{\sigma \in \Sigma(3)} \left|\zeta_{\sigma(2)} + \zeta_{\sigma(3)}\right|_{g^*(p)}^{-2}\right)$$

$$\times \prod_{j=1}^3 \sigma^{(p)}(v_j)(q_0, \zeta_j).$$

Therefore $\Lambda^{(1)} = \Lambda^{(2)}$ implies $\sigma^{(p)}(\mathcal{U}_1^{(3),\text{inc},1})(y,\eta) = \sigma^{(p)}(\mathcal{U}_1^{(3),\text{inc},2})(y,\eta)$. As shown above, the sum $\sum_{\sigma \in \Sigma(3)} \left| \zeta_{\sigma(2)} + \zeta_{\sigma(3)} \right|_{g^*(p)}^{-2}$ appearing here is nonvanishing; therefore,

$$(h_2^{(1)}(p))^2 = (h_2^{(2)}(p))^2.$$

3.3. **Nonlinear interactions of four waves and recovery of h_2 and h_4.** In this section, we use nonlinear interaction of four distorted plane waves. Thus, we take $N = 4$ in (5) and consider Neumann data

$$f = \sum_{i=1}^4 \epsilon_i f_i.$$

Take $x_1, x_2, x_3, x_4 \in \widetilde{M} \setminus M$ in a neighborhood of x_-, where x_- is as in (7) for some point $q_0 \in \mathbb{U}$; suppose γ_{x_j,ξ_j} joins x_j to q_0. Take $u_i \in \mathcal{I}^\mu(\Lambda(x_i,\xi_i,s_0))$ and let $f_i = \partial_\nu u_i|_{\partial M}$ for $i = 1,2,3,4$. One can ensure that $\Lambda_i = \Lambda(x_i,\xi_i,s_0)$, $i = 1,2,3,4$ satisfy the assumptions in Section 3.1.

In this section, we will use the notations

$$\Theta^{(1)} = \cup_{i=1}^4 \Lambda_i; \quad \Theta^{(2)} = \cup_{i,j=1}^4 \Lambda_{ij}; \quad \Theta^{(3)} = \cup_{i,j,k=1}^4 \Lambda_{ijk};$$

$$K^{(1)} = \cup_{i=1}^4 K_i; \quad K^{(2)} = \cup_{i,j=1}^4 K_{ij}; \quad K^{(3)} = \cup_{i,j,k=1}^4 K_{ijk},$$

$$\Xi = \Theta^{(1)} \cup \Theta^{(3),g} \cup \Lambda_{q_0}.$$

Write

$$\begin{aligned}
\mathcal{V}^{(4)} =& \partial_{\epsilon_1} \partial_{\epsilon_2} \partial_{\epsilon_3} \partial_{\epsilon_4} u|_{\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=0} \\
=& -4 \sum_{\sigma \in \Sigma} Q_g(h_2 v_{\sigma(1)} Q_g(h_2 v_{\sigma(2)} Q_g(h_2 v_{\sigma(3)} v_{\sigma(4)}))) \\
& - \sum_{\sigma \in \Sigma} Q_g(h_2 Q_g(h_2 v_{\sigma(1)} v_{\sigma(2)}) Q_g(h_2 v_{\sigma(3)} v_{\sigma(4)})) \\
& + 2 \sum_{\sigma \in \Sigma} Q_g(h_2 v_{\sigma(1)} Q_g(h_3 v_{\sigma(2)} v_{\sigma(3)} v_{\sigma(4)})) + 3 \sum_{\sigma \in \Sigma} Q_g(h_3 v_{\sigma(1)} v_{\sigma(2)} Q_g(h_2 v_{\sigma(3)} v_{\sigma(4)})) \\
& - 24 Q_g(h_4 v_1 v_2 v_3 v_4).
\end{aligned}$$

Assume $\mathcal{V}^{(4)} = \mathcal{V}^{(4),\text{inc}} + \mathcal{V}^{(4),\text{ref}}$, where $\mathcal{V}^{(4),\text{inc}}$ is the incident wave, and $\mathcal{V}^{(4),\text{ref}}$ is the reflected wave. By [30, Proposition 3.11,3.12], we have

$$\mathcal{V}^{(4),\text{inc}} \in \mathcal{I}^{4\mu+\frac{3}{2}}(\Lambda_{q_0}^g \setminus \Xi)$$

with principal symbol

(9) $$\sigma^{(p)}(\mathcal{V}^{(4),\text{inc}})(y,\eta) = -24(2\pi)^{-3} \sigma^{(p)}(\widetilde{Q}_g)(y,\eta,q_0,\zeta) h_4(q_0) \prod_{j=1}^4 \sigma^{(p)}(v_j)(q_0,\zeta_j),$$

for $(y,\eta) \in \Lambda_{q_0}^g \setminus \Xi$ and $y \in \partial M$. Here (y,η) is joined with (q_0,ζ) by a null-bicharacteristic of $\Box_g$, and $\zeta \in L_{q_0}^{*,+}M$ has the unique decomposition $\zeta = \sum_{i=4}^4 \zeta_i$ with $\zeta_i \in N^*K_i$. Assume $h_3^{(1)}, h_4^{(2)} \neq 0$ at q_0. Denote $\mathcal{K}^{(3)} = \pi(\Theta^{(3),g}) \subset M$. By taking $s_0 \to 0$, the set $K^{(1)} \cup \mathcal{K}^{(3)}$ tends to a set of Hausdorff dimension 2. Thus we can choose s_0 small enough such that there exists $\zeta \in \Lambda_{q_0} \setminus (\Theta^{(1)} \cup \Theta^{(3)})$ such that $y \in (0,T) \times \partial N$. But then

$$\partial_{\epsilon_1} \partial_{\epsilon_2} \partial_{\epsilon_3} \partial_{\epsilon_4} \Lambda^{(1)}(f)|_{\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=0} = \partial_{\epsilon_1} \partial_{\epsilon_2} \partial_{\epsilon_3} \partial_{\epsilon_4} \Lambda^{(2)}(f)|_{\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=0}$$

implies
$$\sigma^{(p)}(\mathcal{V}^{(4),\text{inc},1})(y,\eta) = \sigma^{(p)}(\mathcal{V}^{(4),\text{inc},2})(y,\eta).$$
By the explicit expression for $\sigma^{(p)}(\mathcal{V}^{(4),\text{inc},j})(y,\eta)$ given in (9), we obtain
$$h_4^{(1)}(q_0) = h_4^{(2)}(q_0).$$

With h_4 thus recovered in $\mathbb{U}$, we can determine
$$\mathcal{V}_1^{(4)} = \mathcal{V}^{(4)} + 24Q_g(h_4 v_1 v_2 v_3 v_4).$$
at the boundary $(0,T) \times \partial N$. Here we use the fact that, by the finite speed of propagation, $Q_g(h_4 v_1 v_2 v_3 v_4)|_{(0,T)\times\partial N}$ depends only on the value of $h_4 v_1 v_2 v_3 v_4$ in $J^-((0,T)\times\partial N)$ and v_j vanishes on $M \setminus J^+((0,T) \times \partial N)$. Similar as the previous section, we can write $\mathcal{V}_1^{(4)} = \mathcal{V}_1^{(4),\text{inc}} + \mathcal{V}_1^{(4),\text{ref}}$, which is the sum of the incident wave and reflected wave. The microlocal property of $\mathcal{V}_1^{(4),\text{inc}}$ is analyzed carefully in the proofs of [30, Proposition 3.11, 3.12]. We summarize the results that we need in the following proposition.

Proposition 1. *Assume $(y,\eta) \in \Lambda_{q_0}^g \setminus \Xi$ is joined from $(q_0,\zeta) \in \Lambda_{q_0}$ by a null-bicharacteristic.*

(1) *If $h_3(q_0) \neq 0$, we have $\mathcal{V}_1^{(4),\text{inc}} \in \mathcal{I}^{4\mu-\frac{1}{2}}(\Lambda_{q_0}^g \setminus \Xi)$ with principal symbol*
$$\sigma^{(p)}(\mathcal{V}_1^{(4),\text{inc}})(y,\eta) = (2\pi)^{-3}h_2(q_0)h_3(q_0)\mathcal{G}_2(\zeta)\sigma^{(p)}(Q_g)(y,\eta,q_0,\zeta)\prod_{j=1}^{4}\sigma^{(p)}(v_j)(q_0,\zeta_j),$$

where
$$\mathcal{G}_2(\zeta) = \sum_{\sigma\in\Sigma(4)}\left(\frac{3}{|\zeta_{\sigma(1)} + \zeta_{\sigma(2)}|_{g^*(q_0)}^2} + \frac{2}{|\zeta_{\sigma(2)} + \zeta_{\sigma(3)} + \zeta_{\sigma(4)}|_{g^*(q_0)}^2}\right).$$

(2) *If $h_3 = 0$ in a neighborhood of q_0, we have $\mathcal{V}_1^{(4),\text{inc}} \in \mathcal{I}^{4\mu-\frac{5}{2}}(\Lambda_{q_0}^g \setminus \Xi)$ with principal symbol*
$$\sigma^{(p)}(\mathcal{V}_1^{(4),\text{inc}})(y,\eta) = (2\pi)^{-3}h_2(q_0)^3\mathcal{G}_3(\zeta)\sigma^{(p)}(Q_g)(y,\eta,q_0,\zeta)\prod_{j=1}^{4}\sigma^{(p)}(v_j)(q_0,\zeta_j),$$

where
$$\mathcal{G}_3(\zeta) = \sum_{\sigma\in\Sigma(4)}\left(\frac{4}{|\zeta_{\sigma(2)} + \zeta_{\sigma(3)} + \zeta_{\sigma(4)}|_{g^*(q_0)}^2} + \frac{1}{|\zeta_{\sigma(1)} + \zeta_{\sigma(2)}|_{g^*(q_0)}^2}\right)\frac{1}{|\zeta_{\sigma(3)} + \zeta_{\sigma(4)}|_{g^*(q_0)}^2}.$$

Now $\Lambda^{(1)} = \Lambda^{(2)}$ implies
$$\sigma^{(p)}(\mathcal{V}_1^{(4),\text{inc},1})(y,\eta) = \sigma^{(p)}(\mathcal{V}_1^{(4),\text{inc},2})(y,\eta).$$
Using Proposition 1, and the (generic) nonvanishing of $\mathcal{G}_2$ and $\mathcal{G}_3$ ([30, Proposition 3.12]), we now have
$$h_2^{(1)}(q_0)h_3^{(1)}(q_0) = h_2^{(2)}(q_0)h_3^{(2)}(q_0)$$
if $h_3^{(j)}(q_0) \neq 0$ or
$$(10) \qquad\qquad h_2^{(1)}(q_0)^3 = h_2^{(2)}(q_0)^3.$$
if $h_3^{(j)}$ vanishes near q_0. For either case, we can obtain
$$h_2^{(1)}(q_0) = h_2^{(2)}(q_0),$$

invoking the facts $h_2^{(1)}(q_0)^2 = h_2^{(2)}(q_0)^2$ and $h_3^{(1)}(q_0) = h_3^{(2)}(q_0)$. If $h_3^{(j)}$ vanishes at q_0 but not nearby, then we are in case (10) at a sequence of points tending to q_0, hence obtaining the equality $h_3^{(1)}(q_0) = h_3^{(2)}(q_0) = 0$ by continuity.

3.4. **Recovery of h_k, $k \geq 5$.** Finally, we recover h_k for $k = 5, 6, \ldots$, using the interaction of three waves. The coefficients h_2, h_3, h_4 have already been determined above. Inductively, assume that all h_k, $k \leq N - 1$ ($N \geq 5$), have already be recovered; we proceed to recover h_N. Denote

$$\mathcal{U}^{(N)} = \partial_{\epsilon_1}^{N-2} \partial_{\epsilon_2} \partial_{\epsilon_3} u|_{\epsilon_1 = \epsilon_2 = \epsilon_3 = 0},$$

where u is the solution to (1) with $f = \sum_{i=1}^{3} \epsilon_i f_i$. We observe that

$$\mathcal{U}^{(N)} = -N! Q_g(h_N v_1^{N-2} v_2 v_3) + R_N(v_1, v_2, v_3; h_2, \ldots, h_{N-1}),$$

where $R_N(v_1, v_2, v_3; h_2, \ldots, h_{N-1})$ depends on v_1, v_2, v_3 and $h_2, \ldots, h_{N-1}$ only. We note here that the singularities in R_N are very complicated. The Sobolev regularity of R_N was analyzed in [30, Section 5] on boundaryless Lorentzian manifolds. We avoid the complication by using the inductive procedure.

Now, $h_2, \ldots, h_{N-1}$ have already been recovered in $\mathbb{U}$; moreoever, v_1, v_2, v_3 (which vanish on $M \setminus J^+((0, T) \times \partial N)$) are known; hence, R_N is known on $(0, T) \times \partial N$ by finite speed of propagation. Thus we can recover

$$\mathcal{U}_0^{(N)} = -N! Q_g(h_N v_1^{N-2} v_2 v_3)$$

on the boundary $(0, T) \times \partial N$ from Λ. Assume $\mathcal{U}_0^{(N)} = \mathcal{U}_0^{(N),\text{inc}} + \mathcal{U}_0^{(N),\text{ref}}$, where

$$\mathcal{U}_0^{(N),\text{inc}} = -N! \widetilde{Q}_g(h_N v_1^{N-2} v_2 v_3).$$

By [30, Lemma 5.1], we have $v_1^{N-2} \in \mathcal{I}^{\mu + (N-3)(\mu + \frac{3}{2})}(K_1)$, with

$$\sigma^{(p)}(v_1^{N-2}) = (2\pi)^{-\frac{N-3}{2}} \underbrace{\sigma^{(p)}(v_1) * \sigma^{(p)}(v_1) * \cdots * \sigma^{(p)}(v_1)}_{N-2 \text{ factors, } N-3 \text{ convolutions}} =: (2\pi)^{-\frac{N-3}{2}} A_1^{(N-2)}.$$

By [30, Lemma 3.3], $v_2 v_3 \in \mathcal{I}^{\mu, \mu+1}(\Lambda_{23}, \Lambda_2) + \mathcal{I}^{\mu, \mu+1}(\Lambda_{23}, \Lambda_3)$, and then by [30, Lemma 3.6]

$$v_1^{N-2} v_2 v_3 \in \mathcal{I}^{3\mu + (N-3)(\mu + \frac{3}{2})}(\Lambda_{123}) \quad \text{away from } \cup_{i=1}^{3} \Lambda_i.$$

By [30, Proposition 2.1], we have

$$\mathcal{U}_0^{(N),\text{inc}} \in \mathcal{I}^{3\mu + (N-3)(\mu + \frac{3}{2}) + \frac{1}{2}, -\frac{1}{2}}(\Lambda_{123}, \Lambda_{123}^g) \quad \text{away from } \cup_{i=1}^{3} \Lambda_i,$$

with principal symbol

(11)
$$\sigma^{(p)}(\mathcal{U}_0^{(N),\text{inc}})(y, \eta)$$
$$= -N!(2\pi)^{-2 - \frac{N-3}{2}} \sigma^{(p)}(\widetilde{Q}_g)(y, \eta, q_0, \zeta) h_N(q_0) A_1^{(N-2)}(q_0, \zeta_1) \prod_{j=2}^{3} \sigma^{(p)}(v_j)(q_0, \zeta_j).$$

Similarly to before (and using the same notation),

$$\partial_{\epsilon_1}^{N-2} \partial_{\epsilon_2} \partial_{\epsilon_3} \Lambda^{(1)}(f)|_{\epsilon_1 = \epsilon_2 = \epsilon_3 = 0} = \partial_{\epsilon_1}^{N-2} \partial_{\epsilon_2} \partial_{\epsilon_3} \Lambda^{(2)}(f)|_{\epsilon_1 = \epsilon_2 = \epsilon_3 = 0}$$

thus implies

$$\sigma^{(p)}(\mathcal{U}_0^{(N),\text{inc},1})(y, \eta) = \sigma^{(p)}(\mathcal{U}_0^{(N),\text{inc},2})(y, \eta).$$

By the explicit formula for $\sigma^{(p)}(\mathcal{U}_0^{(N),\mathrm{inc},j})(y,\eta)$ given by (11), we get

$$h_N^{(1)}(p) = h_N^{(2)}(p).$$

This completes the proof of Theorem 1.

4. Recovery using Gaussian beams

In this section, we give an alternative approach to recover H, assuming h_2 is *a priori* known, using Gaussian beam solutions to the linear wave equation. Such approach for nonlinear wave equations have been undertaken in [23, 14, 32]. We note here that Gaussian beams have also been used for various inverse problems [2, 3, 9, 19].

We still use higher order linearization of the Neumann-to-Dirichlet map Λ, but will obtain an integral identity and use it to recover the parameters. Gaussian beams will be used in the integral identity. A similar technique was applied to a nonlinear elastic wave equation in [36]. Higher order linearizations of the Dirichlet-to-Neumann map and the resulting integral identities for semilinear and quasilinear elliptic equations have been used in [34, 18, 1, 4, 26, 27, 14, 21, 20].

Let v_j, $j = 1, 2, \ldots$, solve

$$(12) \qquad \begin{aligned} \Box_g v_j &= 0 && \text{in } (0,T) \times N, \\ \partial_\nu v_j &= f_j && \text{on } (0,T) \times \partial N, \\ v_j &= \partial_t v_j = 0 && \text{on } \{t = 0\}. \end{aligned}$$

Let v_0 be the solution to the backward wave equation

$$(13) \qquad \begin{aligned} \Box v_0 &= 0 && \text{in } (0,T) \times N, \\ \partial_\nu v_0 &= f_0 && \text{on } (0,T) \times \partial N, \\ v_0 &= \partial_t v_0 = 0 && \text{on } \{t = T\}. \end{aligned}$$

First let us recover h_3. Take $f = \epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3$, and let u solve (1). Denote $\mathcal{U}^{(123)} = \frac{\partial^3}{\partial\epsilon_1 \partial\epsilon_2 \partial\epsilon_3} u|_{\epsilon_1=\epsilon_2=\epsilon_3=0}$, $\mathcal{U}^{(ij)} = \frac{\partial^2}{\partial\epsilon_i \partial\epsilon_j} u|_{\epsilon_i=\epsilon_j=0}$. Notice $\frac{\partial}{\partial\epsilon_i} u|_{\epsilon_i=0} = v_i$ and $\mathcal{U}^{(ij)}$ solves

$$\begin{aligned} \Box \mathcal{U}^{(ij)} + h_2(x) v_i v_j &= 0 && \text{in } (0,T) \times N \\ \partial_\nu \mathcal{U}^{(ij)} &= 0 && \text{on } (0,T) \times \partial N, \\ \mathcal{U}^{(ij)} = \partial_t \mathcal{U}^{(ij)} &= 0 && \text{on } \{t = 0\}. \end{aligned}$$

Applying $\frac{\partial^3}{\partial\epsilon_1 \partial\epsilon_2 \partial\epsilon_3}$ to (1) evaluated at at $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, we get

$$\Box \mathcal{U}^{(123)} + h_2(x) \sum_{\sigma \in \Sigma(3)} \mathcal{U}^{(\sigma(1)\sigma(2))} v_{\sigma(3)} + 6 h_3(x) v_1 v_2 v_3 = 0$$

Integration by parts gives

$$\begin{aligned} (14) \qquad &\int_{\partial M} \frac{\partial^3}{\partial\epsilon_1 \partial\epsilon_2 \partial\epsilon_3}\Big|_{\epsilon_1=\epsilon_2=\epsilon_3=0} \Lambda(\epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3) f_0 \, dV_g \\ &= \int_M h_3 v_1 v_2 v_3 v_0 \, dV_g + \int_M h_2(x) \sum_{\sigma \in \Sigma(3)} \mathcal{U}^{(\sigma(1)\sigma(2))} v_{\sigma(3)} v_0 \, dV_g. \end{aligned}$$

we note here that by finite speed of propagation for solutions of the wave equation, the functions v_i, v_j and thus also $\mathcal{U}^{(ij)}$ vanish in $M \setminus J^+((0,T) \times \partial N)$, $i, j = 1, 2, 3$, and likewise v_0 vanishes

in $M \setminus J^-((0, T) \times \partial N)$; therefore, our knowledge of h_2 in $\mathbb{U}$ is sufficient to compute the second summand in (14). Therefore, we can recover

$$(15) \qquad \int_M h_3 v_1 v_2 v_3 v_0 \, dV_g.$$

We will use special solutions v_1, v_2, v_3, v_0 in the above identity and thereby recover the coefficient h_3. Concretely, we shall use Gaussian beam solutions for the wave equation $\Box_g v = 0$ on $\widetilde{M}$ of the form

$$v(x) = e^{i\rho\varphi(x)} \mathfrak{a}_\rho(x) + R_\rho(x),$$

with a large parameter ρ. The phase function φ is complex-valued. The principal term $e^{i\rho\varphi(x)} \mathfrak{a}(x)$ is concentrated near a null geodesic γ in the manifold $\mathbb{R} \times N$. The remainder term R_ρ will vanish rapidly as $\rho \to +\infty$.

Fermi coordinates on $\widetilde{M}$. Assume γ passes through a point $p \in M$ and joins two points $\gamma(\tau_-)$ and $\gamma(\tau_+)$ on the boundary $\mathbb{R} \times \partial N$. We will use the Fermi coordinates Φ on $\widetilde{M}$ in a neighborhood of $\gamma([\tau_-, \tau_+])$, denoted by $(z^0 := \tau, z^1, z^2, z^3)$, such that $\Phi(\gamma(\tau)) = (\tau, 0)$ (cf. [13, Lemma 1]). In the Fermi coordinates the metric g has the form

$$g|_\gamma = 2d\tau dz^1 + \sum_{\alpha=2}^{3} dz^\alpha \otimes dz^\alpha,$$

and $\partial_i g_{jk}|_\gamma = 0$ for $i, j, k = 0, 1, 2, 3$.

Construction of Gaussian beams. We will construct asymptotic solutions of the form $u_\rho = \mathfrak{a}_\rho e^{i\rho\varphi}$ on $\widetilde{M}$ with

$$\varphi = \sum_{k=0}^{N} \varphi_k(\tau, z'), \quad \mathfrak{a}_\rho(\tau, z') = \chi\left(\frac{|z'|}{\delta}\right) \sum_{k=0}^{N} \rho^{-k} a_k(\tau, z'), \quad a_k(\tau, z') = \sum_{j=0}^{N} a_{k,j}(\tau, z')$$

in a neighborhood of γ,

$$(16) \qquad \mathcal{V} = \left\{ (\tau, z') \in \widetilde{M} : \tau \in \left[\tau_- - \tfrac{\epsilon}{\sqrt{2}}, \tau_+ + \tfrac{\epsilon}{\sqrt{2}}\right], |z'| < \delta \right\}.$$

Here for each j, φ_j and $a_{k,j}$ are a complex valued homogeneous polynomials of degree j with respect to the variables z^i, $i = 1, 2, 3$, and $\delta > 0$ is a small parameter. The smooth function $\chi : \mathbb{R} \to [0, +\infty)$ satisfies $\chi(t) = 1$ for $|t| \leq \frac{1}{4}$ and $\chi(t) = 0$ for $|t| \geq \frac{1}{2}$. The parameter δ is small enough to ensure that $\mathfrak{a}_\rho = 0$ near $\{t = 0\}$.

We have

$$(17) \qquad \begin{aligned} \Box_g(\mathfrak{a}_\rho e^{i\rho\varphi}) &= e^{i\rho\varphi}(\rho^2(\mathcal{S}\varphi)\mathfrak{a}_\rho - i\rho\mathcal{T}\mathfrak{a}_\rho + \Box_g\mathfrak{a}_\rho), \\ \mathcal{S}\varphi &= \langle d\varphi, d\varphi \rangle_g, \\ \mathcal{T}a &= 2\langle d\varphi, da \rangle_g - (\Box_g\varphi)a. \end{aligned}$$

We need to construct φ and $\mathfrak{a}_\rho$ such that

$$(18) \qquad \frac{\partial^\Theta}{\partial z^\Theta}(\mathcal{S}\varphi)(\tau, 0) = 0, \quad \frac{\partial^\Theta}{\partial z^\Theta}(\mathcal{T}a_0)(\tau, 0) = 0, \quad \frac{\partial^\Theta}{\partial z^\Theta}(-i\mathcal{T}a_k + \Box_g a_{k-1})(\tau, 0) = 0$$

for $\Theta = (0, \Theta_1, \Theta_2, \Theta_3)$ with $|\Theta| \leq N$. For more details we refer to [13]. Following [11], we take

$$\varphi_0 = 0, \quad \varphi_1 = z^1, \quad \varphi_2(\tau, z) = \sum_{1 \leq i,j \leq 3} H_{ij}(\tau) z^i z^j.$$

Here H is a symmetric matrix with $\Im H(\tau) > 0$; the matrix H satisfies a Riccati ODE,

$$(19) \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}H + HCH + D = 0, \quad \tau \in \left(\tau_- - \tfrac{\epsilon}{2}, \tau_+ + \tfrac{\epsilon}{2}\right), \quad H(0) = H_0, \text{ with } \Im H_0 > 0,$$

where C, D are matrices with $C_{11} = 0$, $C_{ii} = 2$, $i = 2, 3$, $C_{ij} = 0$, $i \neq j$ and $D_{ij} = \frac{1}{4}(\partial_{ij}^2 g^{11})$.

Lemma 1 ([11, Lemma 3.2])**.** *The Riccati equation* (19) *has a unique solution. Moreover the solution H is symmetric and $\Im(H(\tau)) > 0$ for all $\tau \in (\tau_- - \frac{\delta}{2}, \tau_+ + \frac{\delta}{2})$. For solving the above Riccati equation, one has $H(\tau) = Z(\tau)Y(\tau)^{-1}$, where $Y(\tau)$ and $Z(\tau)$ solve the ODEs*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau) = CZ(\tau), \quad Y(0) = Y_0,$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau) = -D(\tau)Y(\tau), \quad Z(0) = Y_1 = H_0 Y_0.$$

In addition, $Y(\tau)$ is nondegenerate.

Lemma 2 ([11, Lemma 3.3])**.** *The following identity holds:*

$$\det(\Im(H(\tau))|\det(Y(\tau))|^2 = c_0$$

with c_0 independent of τ.

We see that the matrix $Y(\tau)$ satisfies

$$(20) \qquad \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}Y + CDY = 0, \quad Y(0) = Y_0, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}Y(0) = CY_1.$$

As in [13], we have the following estimate by the construction of u_ρ (cf. (18))

$$(21) \qquad \|\Box_g u_\rho\|_{H^k(M)} \leq C\rho^{-K}, \qquad K = \frac{N+1-k}{2} - 1.$$

Consider a point $p \in \mathbb{U}$, let x_j, $j = 0, 1, 2, 3$ be the points on $(0, T) \times N$ chosen in Section 3.2, and $\gamma^{(j)}$ the null-geodesics passing through x_j and q_0. Also $\xi^{(j)} \in L_{q_0}^{*,+}M$ is the cotangent vector to $\gamma^{(j)}$ at q_0. By the discussions in Section 3.2, there exits constant κ_j, $j = 0, 1, 2, 3$ such that

$$(22) \qquad \kappa_0 \xi^{(0)} + \kappa_1 \xi^{(1)} + \kappa_2 \xi^{(2)} + \kappa_3 \xi^{(3)} = 0.$$

We construct Gaussian beams $u_\rho^{(j)}$, $j = 0, 1, 2, 3$ as above of the form

$$u_\rho^{(j)} = e^{i\kappa_j \rho \varphi^{(j)}} \mathfrak{a}_{\kappa_j \rho}^{(j)},$$

which is compactly supported in the neighborhood $\mathcal{V}$ of the null-geodeisc $\gamma^{(j)}$ (cf. (16)). The parameter δ can be taken small enough such that $u_\rho^{(j)} = \partial_t u_\rho^{(j)} = 0$ at $t = 0$ for $j = 1, 2, 3$ and $u_\rho^{(0)} = \partial_t u_\rho^{(0)} = 0$ at $t = T$.

For $j = 1, 2, 3$, we can construct a solution v_j for the initial boundary value problem (12) of the form $v_j = u_\rho^{(j)} + R_\rho^{(j)}$, where the remainder term $R_\rho^{(1)}$ is a solution of

$$\Box_g R_\rho^{(j)} = -\Box_g u_\rho^{(1)} \quad \text{on } \partial N \times (0, T),$$

$$\partial_\nu R_\rho^{(j)} = 0 \quad \text{on } \partial N \times (0, T),$$

$$R_\rho^{(j)} = \partial_t R_\rho^{(j)} = 0 \quad \text{on } \{t = 0\}.$$

We note here that $v_j = u_\rho^{(j)} + R_\rho^{(j)}$ is the solution to (12) with boundary value $f_j = \partial_\nu u_\rho^{(j)}|_{\partial M}$. Invoking (21), the solution $R_\rho^{(j)}$ satisfies the estimate

$$\|R_\rho^{(j)}\|_{H^{k+1}(M)} \leq C\rho^{-K}.$$

Using Sobolev embedding, we can choose K large enough such that

$$(23) \qquad\qquad \|R_\rho^{(j)}\|_{C(M)} \leq C\rho^{-\frac{n+1}{2}-2}.$$

Similarly, we can construct a solution to (13) of the form $v_0 = u_\rho^{(0)} + R_\rho^{(0)}$. We only need to take the remainder term $R_\rho^{(0)}$ to be the solution to the initial value problem

$$\Box_g R_\rho^{(0)} = -\Box_g u_\rho^{(0)}$$
$$\partial_\nu R_\rho^{(0)} = 0 \text{ on } \partial N \times (0, T),$$
$$R_\rho^{(0)} = \partial_t R_\rho^{(0)} = 0 \text{ on } \{t = 0\}.$$

Now v_0 is the solution to (13) with $g = \partial_\nu u_\rho^{(0)}|_{\partial M}$.

Then by the estimate (23), the Neumann-to-Dirichlet map determines

$$(24) \qquad \begin{aligned} \mathcal{I} &= \rho^{\frac{n+1}{2}} \int_M h_3 v_1 v_2 v_3 v_0 \, dV_g \\ &= \rho^{\frac{n+1}{2}} \int_M h_3 e^{i\rho(\kappa_0\varphi^{(0)} + \kappa_1\varphi^{(1)} + \kappa_2\varphi^{(2)} + \kappa_3\varphi^{(3)})} \mathfrak{a}_{\kappa_0\rho}^{(0)} \mathfrak{a}_{\kappa_1\rho}^{(1)} \mathfrak{a}_{\kappa_3\rho}^{(2)} \mathfrak{a}_{\kappa_3\rho}^{(3)} \, dV_g + \mathcal{O}(\rho^{-1}). \end{aligned}$$

Lemma 3 ([13, Lemma 5]). *The function*

$$S := \kappa_0\varphi^{(0)} + \kappa_1\varphi^{(1)} + \kappa_2\varphi^{(2)} + \kappa_3\varphi^{(3)}$$

is well-defined in a neighborhood of q_0 and

(1) $S(q_0) = 0$;
(2) $\nabla S(q_0) = 0$;
(3) $\Im S(q) \geq c d(q, q_0)^2$ *for q in a neighborhood of q_0, where $c > 0$ is a constant.*

The product $\mathfrak{a}_{\kappa_0\rho}^{(0)} \mathfrak{a}_{\kappa_1\rho}^{(1)} \mathfrak{a}_{\kappa_3\rho}^{(2)} \mathfrak{a}_{\kappa_3\rho}^{(3)}$ is supported in a neighborhood of p. By the above lemma, and applying stationary phase (cf., for example, [17, Theorem 7.7.5]) to (24), we have

$$c\mathcal{I} = h_3(p)a_0^{(0)}(p)a_0^{(1)}(p)a_0^{(2)}(p)a_0^{(3)}(p) + \mathcal{O}(\rho^{-1}),$$

for some explicit constant $c \neq 0$. Hence the Neumann-to-Dirichlet map Λ determines $h_3(p)$.

Next we recover the higher order coefficients h_k, $k = 4, 5, \ldots$. Recursively, assume we have already recovered $h_3, \ldots, h_{N-1}$, $N \geq 4$, in $\mathbb{U}$. To recover h_N, take $f = \sum_{k=1}^N \epsilon_k f_k$ and apply $\frac{\partial^N}{\partial\epsilon_1 \cdots \partial\epsilon_N}$ to (1) evaluated at at $\epsilon_1 = \cdots = \epsilon_N = 0$, we get the equation for $\mathcal{U}^{(12\cdots N)} = \frac{\partial^N}{\partial\epsilon_1 \cdots \partial\epsilon_N} u$

$$\Box \mathcal{U}^{(12\cdots N)} + R_N(v_1, \ldots, v_N; h_1, \ldots, h_{N-1}) + N! h_N \prod_{k=1}^N v_k = 0 \text{ in } N \times (0, T),$$
$$\partial_\nu \mathcal{U}^{(12\cdots N)} = 0 \text{ on } \partial N \times (0, T).$$

By the recursive assumption, $R_N(v_1, \ldots, v_N, h_1, \ldots, h_{N-1})$ is already known. By integration by parts, we have

$$\int_{\partial M} \frac{\partial^N}{\partial \epsilon_1 \cdots \partial \epsilon_N}\Big|_{\epsilon_1 = \cdots = \epsilon_N = 0} \Lambda \left(\sum_{k=1}^{N} \epsilon_k f_k \right) g \, \mathrm{d}S_g$$

$$= \int_M N! h_N v_1 \cdots v_N v_0 \, \mathrm{d}V_g + \int_M R_N(v_1, \ldots, v_N; h_1, \ldots, h_{N-1}) v_0 \, \mathrm{d}V_g.$$

Thus, we can recover

$$\tag{25} \int_M h_N v_0 v_1 \cdots v_N \, \mathrm{d}V_g.$$

Take

$$u_\rho^{(0)} = e^{\mathrm{i}\kappa_0 \rho \varphi^{(0)}} \mathfrak{a}_{\kappa_0 \rho}^{(0)},$$

$$u_\rho^{(j)} = e^{\mathrm{i}\kappa_j \rho \varphi^{(j)}} \mathfrak{a}_{\kappa_j \rho}^{(j)}, \quad j = 1, 2,$$

$$u_\rho^{(j)} = e^{\mathrm{i}\frac{\kappa_3}{N-2}\rho \varphi^{(3)}} \mathfrak{a}_{\frac{\kappa_3}{N-2}\rho}^{(3)}, \quad j = 3, \ldots, N.$$

Take $f_j = \partial_\nu v^{(j)}|_{\partial M}$, $j = 1, \ldots, N$, $g = \partial_\nu v_\rho^{(0)}|_{\partial M}$ this time. Then we can recover

$$\rho^{\frac{n+1}{2}} \int_M h_N e^{\mathrm{i}\rho S_0} \mathfrak{a}_{\kappa_0 \rho}^{(0)} \mathfrak{a}_{\kappa_1 \rho}^{(1)} \mathfrak{a}_{\kappa_2 \rho}^{(2)} (\mathfrak{a}_{\frac{\kappa_3}{N-2}\rho}^{(3)})^{N-2} \, \mathrm{d}V_g + \mathcal{O}(\rho^{-1}).$$

Again applying stationary phase, we can recover $h_N(p)$.

5. Discussion

We can see that h_2 is more difficult to recover than h_k, $k = 3, 4, \ldots$. Indeed, we need to exploit the interaction of four waves (associated with four future light-like vectors) in Section 3; three light-like vectors are not sufficient. (And certainly not two: as pointed out in [30], the interaction of two conormal waves does not produce new propagating singularities.)

The use of Gaussian beams avoids some involved microlocal analysis and simplifies the proof substantially. In our problem, we are however unable to recover h_2 using Gaussian beams. This suggests that the usage of distorted plane waves might be more powerful for certain types of problems. Despite their difference, the two approaches recover h_k for $k \geq 3$ in a very similar way. They both choose solutions $v_1, \ldots, v_k$ such that $v_1 v_2 \cdots v_k$ is supported in a neighborhood of a single point $q_0 \in \mathbb{U}$ at which one wishes to determine $h_k(q_0)$.

Distorted plane waves and Gaussian beams can be constructed even when conjugate points exist. In this paper, we assume that conjugate points do not exist for the sake of simplicity of exposition. Since we prove that local recovery is possible, a layer stripping strategy can be applied if there are conjugate points.

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA (phintz@mit.edu)

Department of Mathematics, University of Washington, Seattle, WA 98195, USA; Institute for Advanced Study, The Hong Kong University of Science and Technology, Kowloon, Hong Kong, China (gunther@math.washington.edu)

Institute for Advanced Study, The Hong Kong University of Science and Technology, Kowloon, Hong Kong, China (iasjzhai@ust.hk).