Gradient Estimate of Solutions of Parabolic Operator with Discontinuous Coefficients

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Introduction

Background/Motivation

Li-Vogelius (2000), Li-Nirenberg (2003) showed a gradient estimate of solution to elliptic equations with piecewise smooth coefficients.

Our aims

1) To show the parabolic version of [LV], [LN], that is, to obtain a gradient est of sol to parabolic eq with piecewise smooth coefficients.

2) To show an application and further possible application to thermography, that is, to dynamical probe method.
Content of Grad. Estimate

- Results of [LV], [LN].

Why piecewise smooth coefficients?
  - Babuška’s conjecture

Applications of their results to inverse problems.

- Main result for parabolic equations.
- Application to the gradient estimate of fundamental solution.
- Outline of the proof.
The case for elliptic eq

\( D \subset \mathbb{R}^n \): b’dd domain, \( \partial D \): smooth.

\( A(x) = (A_{jk}(x)) \): defined a.e. on \( D \), symm, pos. def.

\[
\lambda |\xi|^2 \leq \sum A_{jk}(x)\xi_j \xi_k \leq \Lambda |\xi|^2.
\]

We first consider the elliptic eq (E):

\[
\nabla \cdot (A\nabla u) = g + \nabla \cdot f \text{ in } D.
\]

with \( g \in L^\infty(D) \), \( f \in C^\mu(D_m) \) (\( 0 < \mu < 1 \)) for each \( m \).

(See below for \( D_m \).)

For simplicity, we assume the right hand side is zero.
The case for elliptic eq

In [LV] & [LN], they considered the following cond:

Let $D = \left( \bigcup_{m=1}^{L} \overline{D_m} \right) \setminus \partial D$.

$A^{(m)} \in C^{\mu}(\overline{D_m}) \ (0 < \mu < 1), \ A(x) = A^{(m)}(x) \ (x \in D_m)$.

Each separated $D_m$ is of $C^{1,\alpha}$ smooth with $0 < \alpha \leq 1$ and non-separated one is the limit of the separated one.
**Thm [LV], [LN]**

Let $D' \subseteq D$. Any solution $u$ to the elliptic equation $(E) \nabla \cdot (A \nabla u) = 0$ in $D$ has the following interior regularity estimate: 

$$\|u\|_{C^{1,\alpha'}(D' \cap D_m)} \leq C \|u\|_{L^2(D)},$$

where $C$ is independent of the distance between inclusions and $0 < \alpha' \leq \min(\mu, \frac{\alpha}{2(\alpha+1)})$. 
The case for elliptic eq

Why ‘piecewise smooth’? How about $L^\infty$?

- De Giorgi-Nash-Moser theorem:
  - Let $A_{jk} \in L^\infty$, $D' \Subset D$.
  - Then sol $u$ is Hölder conti: $\exists \alpha \in (0,1]: \|u\|_{C^\alpha(D')} \leq C\|u\|_{L^2(D)}$.

- Piccinini-Spagnolo (1971)(Meyers (1963))
  - An example that we cannot take $\alpha = 1$. 
Babuška’s conj

- Babuška et al (1999) numerically observed that the gradient est of sol is indep of the distances between inclusions, that is $|\nabla u|$ is bounded indep’ly of the distances between inclusions.

- The proof of this conjecture was given by [LV] & [LN].
Application to Inverse Problems

Application of the results of [LV] & [LN]:

PDEs with piecewise smooth coeffs often appear when we consider inverse probs for identifying/reconstructing inclusions.

For that the type of estimate: \[ \| u \|_{C^{1,\alpha'}(\overline{D'\cap D_m})} \leq C' \| u \|_{L^2(D)} \] is useful.

ex.) Nagayasu-Uhlmann-Wang (preprint) (2D), Yoshida (3D) gave reconstruction of inclusions by the enclosure method.
Main result (The case for parabolic eq)

\[ D \subset \mathbb{R}^n : \text{b'/dd domain with inclusions } D_m \text{ inside as before, } \partial D : \text{smooth} \]
\[ A(x) = (A_{jk}(x)) : \text{defined a.e. on } D, \text{ sym, pos def.} \]
\[ \lambda |\xi|^2 \leq \sum A_{j,k}(x)\xi_j\xi_k \leq \Lambda |\xi|^2 \]

We consider the parabolic eq (P):
\[ \partial_t u - \nabla \cdot (A \nabla u) = 0 \text{ in } D \times (0, T). \]
with

\[
D = \left( \bigcup_{m=1}^{L} \overline{D_m} \right) \setminus \partial D,
\]

\[
A^{(m)} \in C^\mu(\overline{D_m}) \ (0 < \mu < 1) : A(x) = A^{(m)}(x) (x \in D_m).
\]
Main result

Thm [FKNN]
Let $D' \subseteq D$, $0 < t_0 < T$. Any sol $u$ to (P): $\partial_t u - \nabla \cdot (A \nabla u) = 0$ in $D \times (0, T)$ has the following interior regularity est:

$$\sup_{t_0 < t < T} \| u(\cdot, t) \|_{C^{1, \alpha}(\overline{D' \cap D_m})} \leq C \| u \|_{L^2(D \times (0,T))},$$

where $C$ is indep of the dist between inclusions.
Remarks

• We can obtain a similar estimate for non-homog parabolic eq:
\[ \partial_t u - \nabla \cdot (A \nabla u) = g + \nabla \cdot f. \]

• By applying our main theorem and a scaling argument, we obtain for a pointwise \(0 < t - s < T\) gradient est:
\[
|\nabla_x \Gamma(x, t; y, s)| \leq \frac{C_T}{(t - s)^\frac{n+1}{2}} \exp \left( -\frac{c|x - y|^2}{t - s} \right)
\]
of fund sol \(\Gamma(x, t; y, s)\) for \(\partial_t - \nabla \cdot (A \nabla \cdot)\).
Idea of Proof

Idea of proof:

- Some interior estimates (Lemma).
- (ref. Ladyzenskaja-Rivkind-Uralceva)
- Apply [LN] to (P).
Proof

**Lem** (Ref. Ladyzenskaja-Rivkind-Uralceva)

Let $\bar{D} \subseteq D$, $0 < t_0 < T$. A solution $u$ to

$$\partial_t u - \nabla \cdot (A\nabla u) = 0 \text{ in } D \times (0, T) =: Q$$

has the following estimates:

$$\sup_{t_0 < t < T} \| u(\cdot, t) \|_{L^2(\bar{D})} \leq C \| u \|_{L^2(Q)} \text{ (standard)},$$

$$\| u \|_{L^\infty(\bar{D} \times (t_0, T))} \leq C \| u \|_{L^2(Q)} \text{ (Di Giorgi’s arg.)},$$

$$\| u_t \|_{L^2(\bar{D} \times (t_0, T))} \leq C \| u \|_{L^2(Q)} \text{ ([LRU]).}$$

**Rem.** This lemma holds for $A \in L^\infty$. 
Proof

Let $\tilde{D}_3 \subseteq \tilde{D}_2 \subseteq \tilde{D}_1 \subseteq \tilde{D}_0 := D$, $0 < \delta_1 < \delta_2 < T$. Then

\[(*) \sup_{\delta_2 < t < T} \| u(\cdot, t) \|_{L^2(\tilde{D}_2)} \leq C \| u \|_{L^2(Q)},\]
\[(**) \| u_t \|_{L^2(\tilde{D}_1 \times (\delta_1, T))} \leq C \| u \|_{L^2(Q)}. \]

Since $\partial_t u_t - \nabla \cdot (A \nabla u_t) = 0$, we have

\[(* * *) \| u_t \|_{L^\infty(\tilde{D}_2 \times (\delta_2, T))} \leq C \| u_t \|_{L^2(\tilde{D}_1 \times (\delta_1, T))}. \]

Now we fix $t \in (\delta_2, T)$: $\nabla \cdot (A \nabla u) = u_t \in L^\infty(\tilde{D}_2)$.


Proof

Then by [LN], we have

$$\|u(\cdot, t)\|_{C^{1,\alpha'}(\bar{D}_m \cap \tilde{D}_3)}$$

$$\leq C \left( \|u(\cdot, t)\|_{L^2(\tilde{D}_2)} + \|u_t(\cdot, t)\|_{L^\infty(\tilde{D}_2)} \right).$$

Taking sup\(_{\delta_2 < t < T}\), we have by (\*), (\**), (\***),

$$\sup_{\delta_2 < t < T} \|u(\cdot, t)\|_{C^{1,\alpha'}(\bar{D}_m \cap \tilde{D}_3)}$$

$$\leq C \left( \sup_{\delta_2 < t < T} \|u(\cdot, t)\|_{L^2(\tilde{D}_2)} + \|u_t\|_{L^\infty(\tilde{D}_2 \times (\delta_2, T))} \right)$$

$$\leq C \|u\|_{L^2(Q)}.$$
Thermography and Dynamical Probe Method
Active Thermography

\[ \partial_{\nu} u|_{\partial \Omega} = f \]
Principle of active thermography
Mixed problem (set up)

\( \Omega \subset \mathbb{R}^n \ (1 \leq n \leq 3) \) : bounded domain, \( \partial \Omega : C^2 \ (n = 2, 3) \)

\( D \subset \Omega \) : open set (inclusion(s)), \( \overline{D} \subset \Omega, \Omega \setminus \overline{D} \) : connected

\( \partial D : C^{1,\alpha} \ (0 < \alpha \leq 1) \)

Heat conductivity:

\( \gamma(x) = 1 + (k - 1)\chi_D \) (for simplicity),

where \( 0 < k (\neq 1) \) is a constant.

\( H^p(\partial \Omega), H^{p,q}(\Omega \times (0, T)) \): usual Sobolev space

\[ (p, q \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \text{ or } p = \frac{1}{2} ) \]

\( L^2((0, T); H^p(\partial \Omega)) := \{ f ; \int_0^T \| f(\cdot, t) \|^2_{H^p(\partial \Omega)} dt < \infty \} \)
Mixed problem (forward problem)

Given \( f \in L^2((0, T) : (H^{1/2}(\partial \Omega))^*) \),

(\?) \exists! weak solution
\[ u = u(f) \in W(\Omega_T) := \{ u \in H^{1,0}(\Omega_T), \partial_t u \in L^2((0, T); H^1(\Omega)^*) \} \]

such that
\[
\begin{aligned}
P_D u(x, t) &:= \partial_t u(x, t) - \text{div}_x(\gamma(x)\nabla_x u(x, t)) = 0 \quad \text{in } \Omega_T \\
\partial_{\nu} u(x, t) &:= \nu \cdot \nabla u(x, t) = f(x, t) \quad \text{in } \partial \Omega_T, \quad u(x, 0) = 0 \quad \text{for } x \in \Omega,
\end{aligned}
\]

where \( \nu \) is the outer unit normal of \( \partial \Omega \),

\[ \Omega_T = \Omega_{(0,T)} := \Omega \times (0, T), \quad \partial \Omega_T = \partial \Omega_{(0,T)} := \partial \Omega \times (0, T). \]

(cylindrical sets)

This is a well-posed problem.

\[ P_{\emptyset} = \partial_t - \Delta_x, \quad P_{\emptyset}^* = -\partial_t - \Delta_x. \]
**Measured data**

*Neummann-to-Dirichlet map $\Lambda_D$:*

$$\Lambda_D : L^2((0,T); (H^{1/2}(\partial\Omega))^*) \to L^2((0,T); H^{1/2}(\partial\Omega))$$

$$f \mapsto u(f)|_{\partial\Omega_T}.$$ 

**Inverse boundary value problem**

Reconstruct the unknown inclusion $D$ from $\Lambda_D$. 
Known results I


* M. Di Cristo and S. Vessella (2010): Optimal stability estimate (i.e. log type stability estimate) even for time dependent inclusions.


Known results II


Dynamical probe method (fundamental solutions)

For \((y, s), (y', s') \in \Omega_T \setminus D\) with \((y, s) \neq (y', s')\), let \(\Gamma(x, t; y, s)\) be

\[
\Gamma(x, t; y, s) = \begin{cases} 
\frac{1}{[4\pi(t-s)]^{n/2}} \exp\left[-\frac{|x-y|^2}{4(t-s)}\right], & t > s, \\
0, & t \leq s,
\end{cases}
\]

and \(\Gamma^*(x, t; y', s') = \Gamma(x, s'; y', t)\) for \((x, t) \in \Omega_T\).

Then,

\[
P_0\Gamma(x, t; y, s) = 0 \quad \text{if} \ (x, t) \neq (y, s)
\]

\[
P_0^*\Gamma^*(x, t; y', s') = 0 \quad \text{if} \ (x, t) \neq (y', s').
\]
Dynamical probe method (Runge’s approximation functions)

\[ \exists \{ v^j_{(y,s)} \}, \{ \varphi^j_{(y',s')} \} \in H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)}) \text{ for } \forall \varepsilon > 0 \text{ s.t.} \]

\[
\begin{aligned}
\mathcal{P}_0 v^j_{(y,s)} &= 0 & \text{in } \Omega_{(-\varepsilon,T+\varepsilon)}, \\
v^j_{(y,s)}(x,t) &= 0 & \text{if } -\varepsilon < t \leq 0, \\
v^j_{(y,s)} &\to \Gamma(\cdot,\cdot;y,s) & \text{in } H^{2,1}(U \times (-\varepsilon',T+\varepsilon')) \text{ as } j \to \infty,
\end{aligned}
\]

\[
\begin{aligned}
\mathcal{P}_0^* \varphi^j_{(y',s')} &= 0 & \text{in } \Omega_{(-\varepsilon,T+\varepsilon)}, \\
\varphi^j_{(y',s')}(x,t) &= 0 & \text{if } T \leq t < T + \varepsilon, \\
\varphi^j_{(y',s')} &\to \Gamma^*(\cdot,\cdot;y',s') & \text{in } H^{2,1}(U \times (-\varepsilon',T+\varepsilon')) \text{ as } j \to \infty
\end{aligned}
\]

for \( 0 < \forall \varepsilon' < \varepsilon, \forall U \subset \Omega : \text{open} \) s.t.

\( \overline{U} \subset \Omega, \, \Omega \setminus \overline{U} : \text{connected}, \partial U : \text{Lipschitz}, \overline{U} \not\ni y, y', \)

and \( -\varepsilon < s, s' < T + \varepsilon. \)

\{ v^j_{(y,s)} \}, \{ \varphi^j_{(y',s')} \} : \text{Runge’s approximation functions}
**Pre-indicator function**

**Definition 1**

\[(y, s), (y', s') \in \Omega_T\]

\[\{v^j(y,s)\}, \{\varphi^j(y',s')\} \subset H^{2,1}(\Omega(-\varepsilon,T+\varepsilon)) : Runge's approximation functions\]

**Pre-indicator function**:

\[I(y', s'; y, s)\]

\[= \lim_{j \to \infty} \int_{\partial \Omega_T} \left[ \partial_{\nu} v^j(y,s) |_{\partial \Omega_T} \varphi^j(y',s') |_{\partial \Omega_T} - \Lambda_D(\partial_{\nu} v^j(y,s)) |_{\partial \Omega_T} \partial_{\nu} \varphi^j(y',s') |_{\partial \Omega_T} \right]\]

whenever the limit exists.
Reflected solution

Lemma 2

\[ y \notin \overline{D}, \ -\varepsilon < s < T + \varepsilon, \ \{v^j_{(y,s)}\} \subset H^{2,1}(\Omega(-\varepsilon, T+\varepsilon)) : \text{Runge's approximation functions}, \]

\[ w^j_{(y,s)} := u(\partial_\nu v^j_{(y,s)}|_{\partial\Omega_T}), \ w^j_{(y,s)} := w^j_{(y,s)} - v^j_{(y,s)} \]

Then, \( w^j_{(y,s)} \) has a limit \( w_{(y,s)} \in W(\Omega_T) \) satisfying

\[
\begin{aligned}
\mathcal{P}_D w_{(y,s)} &= (k - 1)\text{div}_x(\chi_D \nabla_x \Gamma(\cdot, \cdot; y, s)) \quad \text{in } \Omega_T, \\
\partial_\nu w_{(y,s)} &= 0 \quad \text{on } \partial\Omega_T, \quad w_{(y,s)}(x,0) = 0 \quad \text{for } x \in \Omega.
\end{aligned}
\]

\( w_{(y,s)} \) : reflected solution
Representation formula

Theorem 3

For \( y, y' \notin D, -\varepsilon < s, s' < T + \varepsilon \) such that \((y, s) \neq (y', s')\), the pre-indicator function \( I(y', s'; y, s) \) has the representation formula in terms of the reflected solution \( w(y, x) \):

\[
I(y', s'; y, s) = -w(y, s)(y', s') - \int_{\partial \Omega_T} w(y, s) \partial_n \Gamma^*(\cdot, \cdot; y', s') d\sigma dt
\]
Main result \textbf{(indicator function)}

\textbf{Definition 4}

\[ \mathcal{C} := \{ c(\lambda) ; 0 \leq \lambda \leq 1 \} : \text{non-selfintersecting } C^1 \text{ curve in } \overline{\Omega}, \]
\[ c(0), c(1) \in \partial \Omega \text{ (We call this } \mathcal{C} \text{ a needle.)} \]

Then, for each \( c(\lambda) \in \Omega \) and each fixed \( s \in (0, T) \),

\textbf{Indicator function \textbf{(mathematical testing machine)}}

\[ J(c(\lambda), s) := \lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} \sup |I(c(\lambda - \delta), s + \epsilon^2; c(\lambda - \delta), s)| \]

whenever the limit exists.
Figure 1: Domains $\Omega$, $D$, and a curve $C$
Seperated inclusions case result (theorem) [IKN]

**Theorem 5**

Let $D$ consist of separated inclusions, and $C$, $c(\lambda)$ be as in the definition above.

Fix $s \in (0, T)$.

(i) $C \subset \Omega \setminus \overline{D}$ except $c(0)$ and $c(1)$

$\implies J(c(\lambda), s) < \infty$ for all $\lambda$, $0 \leq \lambda \leq 1$

(ii) $C \cap \overline{D} \neq \emptyset$

$\lambda_s \ (0 < \lambda_s < 1) \ s.t. \ c(\lambda_s) \in \partial D, \ c(\lambda) \in \Omega \setminus \overline{D} \ (0 < \lambda < \lambda_s)$

$\implies$

$\lambda_s = \sup \{ 0 < \lambda < 1 ; \ J(c(\lambda'), s) < \infty \ \text{for any} \ 0 < \lambda' < \lambda \}$. 
Remark:

(i) We can numerical realize this reconstruction scheme.

(ii) The theorem is also true for mixed type boundary condition (and hence we can guarantee the exponential decay of the temperature after the experiment), inhomogeneous conductors, time dependent conductors and anisotropic conductors.
Proof of Theorem:

Consider only the case $n = 3$ in the rest of the arguments.

First, we recall the previous two facts.

(i) $w_{(y,s)} \in W(\Omega_T)$ : solution to
\[
\begin{cases}
\mathcal{P}_D w_{(y,s)} = (k - 1) \text{div}_x (\chi_D(t) \nabla_x \Gamma(\cdot, \cdot; y, s)) & \text{in } \Omega_T, \\
\partial_{\nu} w_{(y,s)} = 0 & \text{on } \partial \Omega_T, \\
w_{(y,s)}(x,0) = 0 & \text{for } x \in \Omega.
\end{cases}
\]

(ii)
\[
I(y, s'; y, s) = -w_{(y,s)}(y, s') - \int_{\partial \Omega_T} w_{(y,s)} \partial_{\nu} \Gamma^* (\cdot, \cdot; y, s') d\sigma dt
\]

If $y = c(\lambda) \notin \partial D$, it is easy to see the indicator function is finite at $y$. So, let’s consider the case $y \in \partial D$. 

Setup

Note that

\[ \mathcal{P}_D w_{(y,s)} = (k - 1) \text{div}_x (\chi_D \nabla_x \Gamma(\cdot; \cdot; y, s)) \quad \text{in } \Omega_T \]

Hence,

\[ E(x, t; y, s) := w_{(y,s)}(x, t) + \Gamma(x, t; y, s) \]

(\Rightarrow \text{fundamental solution for } \mathcal{P}_D.)

Let \( P = c(\lambda_0) \in \partial D \) for some \( \lambda_0 \)

\[ x = y = c(\lambda_0 - \delta) \in C \setminus \overline{D} \text{ for } \delta > 0. \]

\( \Phi : \mathbb{R}^3 \to \mathbb{R}^3 \) with \( \Phi(P) = O \quad (C^{1,\alpha} \text{ diffeomorphism}, \ 0 < \alpha \leq 1), \)

\( \Phi(D) \subset \mathbb{R}_-^3 = \{ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3; \xi_3 < 0 \}, \)

Jacobi matrix of \( \Phi \) at \( P = \text{identity matrix}. \)
Outline of the proof

\[ \Gamma_- : \text{fundamental solution for the operator} \]

\[ \partial_t - \text{div}((1 + (k - 1)\chi_-) \nabla) \quad \text{in} \quad \mathbb{R}^4, \]

where \( \chi_- : \) the characteristic function of the space \( \mathbb{R}^3_- \).

**Main part of the proof**

Decompose \( w(y,s) \) as follows:

\[
 w(y,s)(x,t) = E(x, t; y, s) - \Gamma(x, t; y, s) \\
 = \{ E(x, t; y, s) - \Gamma_-(\Phi(x), t; \Phi(y), s) \} \\
 + \{ \Gamma_-(\Phi(x), t; \Phi(y), s) - \Gamma(\Phi(x), t; \Phi(y), s) \} \\
 + \{ \Gamma(\Phi(x), t; \Phi(y), s) - \Gamma(x, t; y, s) \}. 
\]

To show: \( |w(y,s)(y,t)| \to \infty \) as \( t = s' \to s, \ y \to \partial D \)

Let \( \xi = \eta = \Phi(x) = \Phi(y) \to O \ (\delta \downarrow 0) \).
Behavior of each term

1. \( \tilde{E}(\xi, t; \eta, s) := E(\Phi^{-1}(\xi), t; \Phi^{-1}(\eta), s) \)

   \[ \limsup_{\delta \downarrow 0} |(\tilde{E} - \Gamma_\delta)(\xi, s + \varepsilon^2; \eta, s)| = O(\varepsilon^{\alpha-3}) \quad \text{as} \quad \varepsilon \to 0 \quad (0 < \alpha \leq 1) \]

   (Here, we used a pointwise space gradient estimate for a fundamental solution of \( \mathcal{P}_D \).)

2. \[ \limsup_{\delta \downarrow 0} |\Gamma(\xi, t; \eta, s) - \Gamma(\Phi^{-1}(\xi), t; \Phi^{-1}(\eta), s)| = O(\varepsilon^{-2}) \]

   as \( \varepsilon \to 0 \) by explicit computation.

3. \( W(\xi, t; \eta, s) := \Gamma_\delta(\xi, t; \eta, s) - \Gamma(\xi, t; \eta, s), \]

   \( W^+(\xi, t; \eta, s) := W(\xi, t; \eta, s) \) for \( \xi_3 > 0 \)

   \[ \limsup_{\delta \downarrow 0} |W^+(\xi, s + \varepsilon^2; \eta, s)| \approx C\varepsilon^{-3} \quad \text{as} \quad \varepsilon \to 0 \quad \text{with a const.} \quad C > 0. \]
The outline of the proof of 3:

\( W \) satisfies

\[
\partial_t W(\xi, t) - \nabla_\xi \cdot ((1+(k-1)\chi_-)\nabla_\xi W(\xi, t)) = (k-1)\nabla_\xi \cdot (\chi_- \nabla_\xi \Gamma(\xi, t; \eta, s)).
\]

By using the Laplace transform and Fourier transform, we have

\[
W^+(\xi, t) = \frac{\sqrt{k-1}}{(2\pi)^2 4\pi} \int_0^1 \frac{\sqrt{r} + i\sqrt{k(1-r)}}{\sqrt{r}(\sqrt{r} - i\sqrt{k(1-r)})} J(r) dr,
\]

where

\[
J(r) = \frac{\sqrt{\pi}}{4A^2 \sqrt{A}} \int_0^\pi \left[ 2A - \{|\xi' - \eta'| \cos \theta - B\}^2 \right] \exp \left[ - \frac{\{|\xi' - \eta'| \cos \theta - B\}^2}{4A} \right] d\theta
\]

\[
+ \frac{\sqrt{\pi}}{4A^2 \sqrt{A}} \int_0^\pi \left[ 2A - \{|\xi' - \eta'| \cos \theta + B\}^2 \right] \exp \left[ - \frac{\{|\xi' - \eta'| \cos \theta + B\}^2}{4A} \right] d\theta
\]

and \( A = (t-s)(kr-r+1) \) and \( B = (\xi_3 + \eta_3)\sqrt{(k-1)r} \).
Remark for Non-separated inclusions (open question)

The previous proof for the separated inclusions case works well except the estimate for $W(\xi, t; \eta, s)$. 
Thank you for your attention.