Inverse problems for differential forms on Riemannian manifolds with boundary

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Motivation : Electrical Impedance Tomography

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $\gamma \in L^\infty(\Omega)$ be a positive function (conductivity).

Consider

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in} \quad \Omega,$$
$$u|_{\partial\Omega} = f.$$

The Dirichlet-to-Neumann map :

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad \Lambda_\gamma(f) = \nu \cdot \gamma \nabla u|_{\partial\Omega},$$

where ν is the unit outer normal to $\partial\Omega$.

EIT (Calderón's inverse problem) : Does Λ_γ determine γ in Ω ?

$n \geq 3$:

- Sylvester–Uhlmann, 1987 : $0 < \gamma_i \in C^2(\overline{\Omega})$, $i = 1, 2$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in Ω.
- Päivärinta–Panchenko–Uhlmann, 2003 : Extended to $\gamma \in C^{3/2}(\overline{\Omega})$.
- Greenleaf–Lassas–Uhlmann, 2003 : $\gamma \in C^{1+\varepsilon}(\overline{\Omega})$, γ conormal.
- ...

$n = 2$:

- Nachman, 1996 : $0 < \gamma \in W^{2,p}(\Omega)$, $p > 1$.
- ...
- Astala–Päivärinta, 2006 : $0 < \gamma \in L^\infty(\Omega)$.

Anisotropic conductivities

Anisotropic conductivities depend on direction.

Example. Muscle tissue in the human body. Cardiac muscle has a conductivity of 2.3 mho in the transverse direction and 6.3 in the longitudinal direction.

The conductivity is represented by a positive definite symmetric matrix $\gamma = (\gamma^{ij}(x))$ on Ω.

Is γ uniquely determined by Λ_γ ?

L. Tartar : Λ_γ does not determine γ uniquely.

Let $\psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism with $\psi|_{\partial\Omega} = Id$. We have

$$\Lambda_{\tilde{\gamma}} = \Lambda_\gamma, \quad \tilde{\gamma} = \left(\frac{(D\psi)^T \circ \gamma \circ (D\psi)}{|\det D\psi|} \right) \circ \psi^{-1}.$$

Thus, we have a large number of conductivities with the same DN map :

any change of variables of Ω that leaves the boundary fixed gives rise to a new conductivity with the same boundary measurements.

Is this the only obstruction to unique identifiability of the conductivity ?

The case $n = 2$

This is the only obstruction :

- Sylvester, 1990 : $\gamma \in C^3(\overline{\Omega})$, ($\gamma$ near 1).
- Nachman, 1996 : $\gamma \in C^3(\overline{\Omega})$.
- Sun–Uhlmann, 2003 : $\gamma \in W^{1,p}$.
- Astala–Lassas–Päivärinta, 2005 : $\gamma \in L^\infty(\Omega)$

The case $n \geq 3$

Lee–Uhlmann, 1989 : This is a problem of geometrical nature and makes sense for general Riemannian manifolds with boundary.

Let (M, g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to g is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x^j} \right),$$

$(g^{ij}) = (g_{ij})^{-1}$. The Dirichlet problem,

$$\begin{aligned} \Delta_g u &= 0 \quad \text{in} \quad M, \\ u|_{\partial M} &= f. \end{aligned}$$

The DN map is given by

$$\Lambda_g(f) = (\sqrt{\det g}\, \partial_\nu u)|_{\partial M},$$

$$\partial_\nu u = \sum_{j,k=1}^{n} g^{jk} \nu_k \frac{\partial u}{\partial x^j}, \quad \nu = (\nu_1, \dots, \nu_n) - \text{conormal to } \partial M.$$

The inverse problem is to recover g from Λ_g.

We have

$$\Lambda_{\psi^* g} = \Lambda_g, \qquad (1.1)$$

where ψ is any C^∞ diffeomorphism of M such that $\psi|_{\partial M} = Id$. Here $\psi^* g$ denotes the pullback of the metric g by ψ.

When M is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $n \geq 3$, then (Lee–Uhlmann, 1989)

$$\Lambda_g = \Lambda_\gamma,$$

where

$$(g_{ij}) = (\det \gamma)^{\frac{1}{n-2}} (\gamma^{ij})^{-1}, \quad (\gamma^{ij}) = (\det g)^{1/2} (g_{ij})^{-1}.$$

The case $n \geq 3$; M : compact, real-analytic, connected

Lassas–Uhlmann, 2001 : (1.1) is the only obstruction to unique identifiability of the conductivity for real-analytic manifolds.

Let Γ be an open subset of ∂M. We define for f, $\operatorname{supp} f \subset \Gamma$, the DN-map,

$$\Lambda_{g,\Gamma}(f) = \Lambda_g(f)|_\Gamma.$$

Theorem (Lassas–Uhlmann, 2001 ($n \geq 3$))

Let M_1 and M_2 be compact real-analytic connected Riemannian manifolds with smooth boundaries. Let $\Gamma_1 = \Gamma_2 = \Gamma$ be a non-empty open subset of ∂M_1 and ∂M_2, which is real-analytic. Moreover, let the metrics g_1 and g_2 be real-analytic up to Γ. If

$$\Lambda_{g_1,\Gamma} = \Lambda_{g_2,\Gamma},$$

then M_1 and M_2 are isometric.

Here the sets $\Gamma_1 \subset \partial M_1$ and $\Gamma_2 \subset \partial M_2$ are identified by a real-analytic diffeomorphism.

The case $n = 2$

In the case $n = 2$, there is an additional obstruction to unique identifiability of g, since the Laplace-Beltrami operator is conformally invariant, i.e.

$$\Delta_{\sigma g} = \sigma^{-1}\Delta_g, \quad \sigma \neq 0.$$

Therefore, for $n = 2$,

$$\Lambda_{\sigma\psi^* g} = \Lambda_g, \tag{1.2}$$

for any smooth function $\sigma \neq 0$ such that $\sigma|_{\partial M} = 1$.

The case $n = 2$; M : compact, smooth, connected

Lassas–Uhlmann, 2001 : (1.2) is the only obstruction to unique identifiability.

Theorem (Lassas–Uhlmann, 2001 ($n = 2$))

Let (M, g) be a compact connected smooth Riemannian surface with smooth boundary and $\Gamma \subset \partial M$ be a non-empty open subset. Then $\Lambda_{g,\Gamma}$ determines uniquely the conformal class of (M, g).

These results do not assume any condition on the topology of the manifolds.

Lee–Uhlmann, 1989 : (M, g) is strongly convex and simply connected, $\Gamma = \partial M$.

We are interested in the analogous inverse problems in the case of the Hodge-Laplacian on differential forms.

Notation

(M,g) : real-analytic orientable connected compact Riemannian manifold of dimension $n \geq 2$;

∂M : real-analytic boundary ;

g : real-analytic up to the boundary ;

T^*M : the cotangent bundle on M ;

$\Lambda^k T^*M$, $k = 0, 1, \dots, n$: the bundles of the exterior differential k-forms ;

$C^\infty(M, \Lambda^k T^*M)$: the space of smooth exterior differential forms of degree k. Here the smoothness is understood up to the boundary of M.

The metric tensor g induces the volume form $\mu = \mu_g \in C^\infty(M, \Lambda^n T^*M)$ and the Hodge star isomorphism

$$* : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{n-k} T^*M), \quad \omega \wedge *\eta = g(\omega, \eta)\mu.$$

Here in local coordinates, $\mu = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n$, provided that $dx^1, \dots, dx^n$ is a positive basis of T_x^*M.

$d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$: the exterior differential.

The codifferential operator :

$$\delta : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k-1} T^*M), \quad \delta\omega = (-1)^{nk+n+1} * d * \omega.$$

The Hodge-Laplace operator :

$$\Delta_g^{(k)} = \Delta^{(k)} : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^k T^*M), \quad \Delta^{(k)}\omega = (d\delta + \delta d)\omega.$$

$\Delta_g^{(k)}$ is a second order elliptic partial differential system, with a scalar principal symbol, which in local coordinates is given by

$$\Delta^{(k)} = -(g^{jk}(x)\partial_{x^j}\partial_{x^k}) \otimes I + B_j(x)\partial_{x^j} + C(x),$$

where $(g^{jk}) = (g_{jk})^{-1}$, I is $d \times d$ identity matrix, and $B_j(x)$, $C(x)$ are real-analytic functions with values in the set of $d \times d$-matrices, $d = \binom{n}{k}$.

The inclusion map : $i : \partial M \to M$.

Its pullback,

$$i^* : C^\infty(M, \Lambda^k T^* M) \to C^\infty(\partial M, \Lambda^k T^* M),$$

$$i^*\omega(y)(X_1, \dots, X_k) = \omega(y)(X_1^T, \cdots, X_k^T),$$

$$y \in \partial M, \quad X_j^T \text{ is tangential component.}$$

The tangential trace of a k-form :

$\mathbf{t} : C^\infty(M, \Lambda^k T^* M) \to C^\infty(\partial M, \Lambda^k T^* M),\ \mathbf{t}\omega = i^*\omega,\ k = 0, 1, \dots, n-1,$

The normal trace :

$\mathbf{n} : C^\infty(M, \Lambda^k T^* M) \to C^\infty(\partial M, \Lambda^{n-k} T^* M),\ \mathbf{n}\omega = i^*(*\omega),\ k = 1, 2, \dots, n.$

Near ∂M, consider the boundary normal coordinates : $(x^1, \ldots, x^n)$:

$x' = (x^1, \ldots, x^{n-1})$ is a chart on ∂M, $x^n \geq 0$ is the distance to ∂M.

k-form :

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Tangential component $\mathbf{t}\omega$: does not contain dx^n; Normal component $\mathbf{n}\omega$: has a common factor dx^n.

$$\omega|_{\partial M} = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k}|_{\partial M} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

$$\partial_{x^n}\omega|_{\partial M} = \sum_{i_1 < \cdots < i_k} \partial_{x^n}\omega_{i_1, \ldots, i_k}|_{\partial M} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Given the induced metric on ∂M and the first normal derivative of the metric on ∂M,

- $(\mathbf{t}\omega, \mathbf{n}\omega) \Rightarrow \omega|_{\partial M}$;
- $(\mathbf{t}\delta\omega, \mathbf{n}d\omega) \Rightarrow \partial_{x^n}\omega|_{\partial M}$.

The Dirichlet data for a harmonic k-form ω :

$$(\mathbf{t}\omega, \mathbf{n}\omega),$$

The Neumann data :

$$(\mathbf{n}d\omega, \mathbf{t}\delta\omega).$$

The set of the Cauchy data of harmonic k-forms :

$$\mathcal{C}_g^{(k)} = \{(\mathbf{t}\omega, \mathbf{n}\omega, \mathbf{n}d\omega, \mathbf{t}\delta\omega) : \omega \in C^\infty(M, \Lambda^k T^*M), \Delta^{(k)}\omega = 0\}.$$

$$\mathcal{C}_g^{(0)} = \{(\mathbf{t}\omega, \mathbf{n}d\omega) : \omega \in C^\infty(M, \Lambda^0 T^*M), \Delta^{(0)}\omega = 0\},$$

$$\mathcal{C}_g^{(n)} = \{(\mathbf{n}\omega, \mathbf{t}\delta\omega) : \omega \in C^\infty(M, \Lambda^n T^*M), \Delta^{(n)}\omega = 0\}.$$

The problem

$$\Delta^{(k)}\omega = 0 \quad \text{in} \quad M,$$
$$\mathbf{t}\omega = f_1 \quad \text{on} \quad \partial M,$$
$$\mathbf{n}\omega = f_2 \quad \text{on} \quad \partial M,$$

has a unique solution $\omega \in C^\infty(M, \Lambda^k T^*M)$ if $f_1 \in C^\infty(\partial M, \Lambda^k T^*M)$ and $f_2 \in C^\infty(\partial M, \Lambda^{n-k} T^*M)$.

The Dirichlet-to-Neumann map for the k-form Hodge Laplacian :

$$\Lambda_g^{(k)} : C^\infty(\partial M, \Lambda^k T^*M) \times C^\infty(\partial M, \Lambda^{n-k} T^*M)$$
$$\to C^\infty(\partial M, \Lambda^{n-k-1} T^*M) \times C^\infty(\partial M, \Lambda^{k-1} T^*M),$$
$$\Lambda_g^{(k)}(f_1, f_2) = (\mathbf{n}d\omega, \mathbf{t}\delta\omega).$$

$\mathcal{C}_g^{(k)}$ is equal to the graph of the Dirichlet-to-Neumann map $\Lambda_g^{(k)}$.

Previous works on inverse problems for differential forms

- Belishev–Sharafutdinov, 2008 : an explicit formula, which expresses the Betti numbers of the manifold M in terms of $\mathcal{C}_g^{(k)}$.
- Joshi–Lionheart, 2005 : when $n \geq 3$, the knowledge of $\Lambda_g^{(k)}$ determines the Taylor series at the boundary of the metric g in the boundary normal coordinates.
  (an extension of the result by Lee-Uhlmann, 1989, in the scalar case)

The case $n \geq 3$; M : compact, real-analytic

Theorem (K.–Lassas–Uhlmann)

Let M_1 and M_2 be compact real-analytic orientable connected Riemannian manifolds with real-analytic boundaries ∂M_1 and ∂M_2, respectively. Moreover, let the metrics g_1 and g_2 be real-analytic up to the boundary and $\dim M_1 = \dim M_2 = n \geq 3$. Let $\Gamma_1 = \Gamma_2 = \Gamma$ be a non-empty open subset of ∂M_1 and ∂M_2. If for some integer $1 \leq k \leq n$,

$$\mathcal{C}_{g_1}^{(k)}|_\Gamma = \mathcal{C}_{g_2}^{(k)}|_\Gamma,$$

then M_1 and M_2 are isometric.

Here the sets $\Gamma_1 \subset \partial M_1$ and $\Gamma_2 \subset \partial M_2$ are identified by a real-analytic diffeomorphism.

In other words, $\exists$ a real-analytic diffeomorphism $\psi : M_1 \to M_2$ such that $g_1 = \psi^* g_2$.

Idea of the proof

Lassas–Taylor–Uhlmann, 2003.

We would like to exhibit an isometry between M_1 and M_2.

Reconstruction near the boundary :

Joshi–Lionheart, 2005 : $\Longrightarrow$ extend g_j to a boundary collar $\Gamma \times (-r, 0]$, for $r > 0$ small enough, so that the extended metric remains real-analytic.

Introduce the real-analytic manifold $\tilde{M}_j$ obtained by attaching to M_j a boundary collar $\Gamma \times (-r, 0]$, equipped with this metric.

Denote $U = \tilde{M}_1 \setminus M_1 = \tilde{M}_2 \setminus M_2$. Strictly speaking, the manifolds $\tilde{M}_1 \setminus M_1$ and $\tilde{M}_2 \setminus M_2$ are identified by a real-analytic isometry.

For some $s < 1 - n/2$, define the maps

$$\mathcal{G}_j : \tilde{M}_j \to H^s(U, \Lambda^k T^* U), \quad x \mapsto G_j(x, \cdot).$$

A Green's form $G(x, y)$ is a double form of degree k, satisfying

$$\Delta_x^{(k)} G(x,y) = \delta_{x,y} \quad \text{in} \quad \tilde{M},$$
$$\mathbf{t}_x G(x,y) = 0 \quad \text{on} \quad \partial\tilde{M},$$
$$\mathbf{n}_x G(x,y) = 0 \quad \text{on} \quad \partial\tilde{M},$$

where $y \in \tilde{M} \setminus \partial\tilde{M}$ and

$$\delta_{x,y} = \sum_{\substack{i_1 < \cdots < i_k \\ j_1 < \cdots < j_k}} \delta(y - x) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \cdot dy^{j_1} \wedge \cdots \wedge dy^{j_k}$$

is the delta double current, supported at $x = y$.

By the properties of Green's forms :

- $\mathcal{G}_j$ are of class C^1.
- $\mathcal{G}_j$ is real-analytic on $\tilde{M}_j \setminus \overline{U}$.
- $\mathcal{G}_j$ is an injective immersion.

We prove that $\mathcal{G}_1(\tilde{M}_1) = \mathcal{G}_2(\tilde{M}_2)$, and $(\mathcal{G}_2)^{-1}\mathcal{G}_1 : \tilde{M}_1 \to \tilde{M}_2$ is an isometry.

The case of 0-forms.

The Hodge Laplacian on 0-forms is conformally invariant, i.e.

$$\Delta_{\tilde{g}}^{(0)} = e^{-2\varphi}\Delta_{g}^{(0)},$$

where

$$\tilde{g} = e^{2\varphi} g.$$

The case of 1-forms.

$\Delta^{(1)}$ is not conformally invariant.

Theorem (K.–Lassas–Uhlmann)

Let M_1 and M_2 be compact real-analytic orientable connected Riemannian manifolds of dimension two with real-analytic boundaries ∂M_1 and ∂M_2, respectively. Moreover, let the metrics g_1 and g_2 be real-analytic up to the boundary. Let $\Gamma_1 = \Gamma_2 = \Gamma$ be a non-empty open subset of ∂M_1 and ∂M_2. If

$$\mathcal{C}_{g_1}^{(1)}|_\Gamma = \mathcal{C}_{g_2}^{(1)}|_\Gamma,$$

then M_1 and M_2 are isometric.

The idea of the proof is the same as in the case $n \geq 3$.

Proposition

The knowledge of the subset of the set of Cauchy data on 1-forms given by

$$\{(\mathbf{t}\omega, \mathbf{n}d\omega) : \omega \in C^\infty(M, \Lambda^1 T^* M), \Delta^{(1)}\omega = 0, \mathbf{n}\omega = 0\},$$

determines the full Taylor series at the boundary of the metric g in the boundary normal coordinates.

Let (x^1, x^2) be the boundary normal coordinates defined locally near a point at the boundary. Here x^1 is a local coordinate for ∂M and $x^2 \geq 0$ is the distance to the boundary. In these coordinates, the metric has the following form

$$g = g_{11}(x^1, x^2)(dx^1)^2 + (dx^2)^2.$$

Let $\omega = \omega_1 dx^1 + \omega_2 dx^2$ be a 1-form.

An explicit computation shows that in boundary normal coordinates, the Hodge Laplacian on 1-forms has the form

$$\Delta^{(1)} = D_{x^2}^2 I + g^{11} D_{x^1}^2 I + iE(x) D_{x^2} + iF(x) D_{x^1} + Q(x),$$

where $D_{x^j} = \frac{1}{i}\partial_{x^j}$, $j = 1, 2$, and

$$E(x) = \frac{\partial_{x^2} g^{11}}{2g^{11}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$F(x) = \begin{pmatrix} -\frac{3}{2}\partial_{x^1} g^{11} & \frac{\partial_{x^2} g^{11}}{g^{11}} \\ -\partial_{x^2} g^{11} & -\frac{\partial_{x^1} g^{11}}{2} \end{pmatrix}, \quad Q(x) = \begin{pmatrix} -\frac{\partial_{x^1}^2 g^{11}}{2} & \frac{1}{2}\partial^2_{x^1 x^2} \log g^{11}, \\ -\frac{\partial^2_{x^1 x^2} g^{11}}{2} & \frac{1}{2}\partial^2_{x^2} \log g^{11} \end{pmatrix}.$$

Here I is the 2×2 identity matrix.

Similarly to Lee–Uhlmann, 1989, we show that there exists a matrix-valued pseudodifferential operator $A(x, D_{x^1})$ of order one in x^1 depending smoothly on x^2 such that

$$\Delta^{(1)} = (D_{x^2} I + iE(x) - iA(x, D_{x^1}))(D_{x^2} I + iA(x, D_{x^1})),$$

modulo a smoothing operator. Here $A(x, D_{x^1})$ is unique modulo a smoothing term, if we require that its principal symbol satisfies

$$\sigma(A((x^1, 0), D_{x^1})) = -\sqrt{g^{11}(x^1, 0)}|\xi_1| I.$$

The proposition is obtained by analyzing the full symbol of A.

The case of 2-forms.

$$\tilde{g} = e^{2\varphi} g.$$

The Hodge Laplacian on 2-forms,

$$\begin{aligned}
&\Delta_{\tilde{g}}^{(2)} \omega = \Delta_g^{(2)}(e^{-2\varphi}\omega), \quad \omega \in C^\infty(M, \Lambda^2 T^* M),\\
&\mathbf{n}_{\tilde{g}}\omega = \mathbf{n}_g(e^{-2\varphi}\omega),\\
&\mathbf{t}\delta_{\tilde{g}}\omega = \mathbf{t}\delta_g(e^{-2\varphi}\omega).
\end{aligned}$$

Hence,

$$\mathcal{C}_{\tilde{g}}^{(2)} = \mathcal{C}_g^{(2)}.$$

The case of a complete non-compact manifold

Let (M, g) be a complete non-compact real-analytic orientable connected Riemannian manifold with compact real-analytic boundary ∂M. Let the metric g be real-analytic up to the boundary and $n = \dim(M) \geq 2$.

Consider the Dirichlet problem for the Laplace-Beltrami operator,

$$\begin{aligned} \Delta_g u &= 0 \quad \text{on} \quad M, \\ u|_{\partial M} &= f, \end{aligned}$$

$f \in C(\partial M)$.

If M is compact, the solution is unique.

If M is non-compact, we specify the solution as follows. Let $M_j \subset\subset M_{j+1}$ be an exhaustion of M by compact manifolds.

Let u_j be the solution of the problem,

$$\begin{aligned} \Delta_g u_j &= 0 \quad \text{on} \quad M_j, \\ u_j|_{\partial M_j \cap \partial M} &= f, \\ u_j|_{\partial M_j \setminus (\partial M_j \cap \partial M)} &= 0. \end{aligned}$$

Let first $f \geq 0$. Then, by the maximum principle for harmonic functions, $0 \leq u_j \leq \sup_{\partial M} f$, and the sequence u_j is increasing.

Hence, the sequence u_j converges pointwise to the unique limit, $\mathrm{PI}(f) \in L^\infty(M)$.

The limit is independent of the choice of exhaustion of M, and it extends uniquely to a linear map $\mathrm{PI} : C(\partial M) \to L^\infty(M)$, which is the solution operator.

The Dirichlet-to-Neumann map,

$$\Lambda_{g,\Gamma} f = \partial_\nu \mathrm{PI} f|_\Gamma, \ \mathrm{supp}\, f \subset \Gamma \subset \partial M.$$

Theorem (Lassas–Taylor–Uhlmann, 2003)

Let M_1 and M_2 be complete real-analytic connected Riemannian manifolds, $\dim(M_1) = \dim(M_2) = n \geq 3$, with compact real-analytic boundaries ∂M_1 and ∂M_2, respectively. Let the metrics g_1 and g_2 be real-analytic up to the boundary, and let $\Gamma_1 = \Gamma_2 = \Gamma$ be a non-empty open subset of ∂M_1 and ∂M_2. If $\Lambda_{g_1,\Gamma} = \Lambda_{g_2,\Gamma}$, then M_1 and M_2 are isometric.

Example ($n = 2$), Lassas–Taylor–Uhlmann, 2003

Let $\overline{D(0,1)} \subset \mathbb{R}^2$ with the Euclidean metric $e = (\delta_{ij})$.
Consider a complete non-compact manifold,

$$\overline{D(0,1)} \setminus \{0\}, \quad g_{ij}(x) = \frac{1}{d(0,x)^2}\delta_{ij}(x).$$

For any $f \in C(\partial D(0,1))$,

$$\begin{aligned} \Delta_e u = 0 \quad &\text{in} \quad D(0,1), \\ u = f \quad &\text{on} \quad \partial D(0,1), \end{aligned}$$

has a unique solution u. By the maximum principle, u is bounded in $\overline{D(0,1)}$.

Since g and e are conformally equivalent on $\overline{D(0,1)} \setminus \{0\}$, u solves

$$\begin{aligned} \Delta_g u = 0 \quad &\text{in} \quad D(0,1) \setminus \{0\}, \\ u = f \quad &\text{on} \quad \partial D(0,1), \\ u \in L^\infty(&\overline{D(0,1)} \setminus \{0\}) \end{aligned}$$

The solution u is unique. Indeed, if w_1, w_2 are solutions, then $w = w_1 - w_2$ is harmonic in $D(0,1) \setminus \{0\}$, $w \in L^\infty(\overline{D(0,1)} \setminus \{0\})$, then w extends to a harmonic function on $D(0,1)$. Since $w|_{\partial D(0,1)} = 0$, $w = 0$.

$$\Lambda_e = \Lambda_g.$$

The Hodge Laplacian, non-compact complete manifold

Let $n \geq 2$ and ∂M is compact.

Let $f_1 \in C^\infty(\partial M, \Lambda^k T^* M)$ and $f_2 \in C^\infty(\partial M, \Lambda^{n-k} T^* M)$. Then consider the problem

$$\begin{aligned} \Delta^{(k)}\omega = 0 \quad &\text{in} \quad M, \\ \mathbf{t}\omega = f_1 \quad &\text{on} \quad \partial M, \\ \mathbf{n}\omega = f_2 \quad &\text{on} \quad \partial M. \end{aligned}$$

Solvability of the problem when M is non-compact ?

We use spectral theory.

Under some assumptions on the curvature of the manifold, we show that $\Delta^{(k)} = \Delta_F^{(k)}$, $k = 1, \dots, n$, equipped with the domain

$$\mathcal{D}(\Delta_F^{(k)}) = \mathcal{K} \cap \{\omega \in L^2(M, \Lambda^k T^* M) : \Delta^{(k)}\omega \in L^2(M, \Lambda^k T^* M)\},$$

where

$$\mathcal{K} = \{\omega \in H^1(M, \Lambda^k T^* M) : \mathbf{t}\omega = 0,\ \mathbf{n}\omega = 0\},$$

is a non-negative self-adjoint operator on $L^2(M, \Lambda^k T^* M)$.

The L^2-inner product of differential k-forms is given by

$$(\omega, \eta)_{L^2} = \int_M \omega \wedge *\bar{\eta}, \quad \omega, \eta \in C_0^\infty(M, \Lambda^k T^* M),$$

where $\bar{\eta}$ is the complex conjugate of η. The space $L^2(M, \Lambda^k T^* M)$ is defined as the completion of the space $C_0^\infty(M, \Lambda^k T^* M)$ of compactly supported differential k-forms on M in the corresponding L^2-norm.

Curvature assumptions

- (A1) Let $n \geq 3$.
  - If $k = 1$, then the Ricci curvature tensor is bounded from below.
  - If $k = 2, \dots, n$, then the Riemann curvature tensor is bounded on M.
- (A2) Let $n = 2$. Then the Gaussian curvature of M is bounded from below.

$\Delta^{(0)}$ has a non-negative self-adjoint extension, without any assumptions on the curvature of the manifold.

We have $0 \notin \mathrm{spec}_d(\Delta_F^{(k)})$.
Spectral assumption : $0 \notin \mathrm{spec}_{ess}(\Delta_F^{(k)})$.

Under the curvature assumptions, there exists a unique solution

$$\omega \in (L^2 \cap C^\infty)(M, \Lambda^k T^* M)$$

of the problem

$$\begin{aligned} \Delta^{(k)} \omega = 0 \quad & \text{in} \quad M, \\ \mathbf{t}\omega = f_1 \quad & \text{on} \quad \partial M, \\ \mathbf{n}\omega = f_2 \quad & \text{on} \quad \partial M, \end{aligned}$$

$f_1 \in C^\infty(\partial M, \Lambda^k T^* M)$, $f_2 \in C^\infty(\partial M, \Lambda^{n-k} T^* M)$.

The Dirichlet-to-Neumann map for the k-form Laplacian is determined by

$$\begin{aligned}\Lambda_g^{(k)} : C^\infty(\partial M, \Lambda^k T^* M) \times C^\infty(\partial M, \Lambda^{n-k} T^* M) \\ \to C^\infty(\partial M, \Lambda^{n-k-1} T^* M) \times C^\infty(\partial M, \Lambda^{k-1} T^* M), \\ \Lambda_g^{(k)}(f_1, f_2) = (\mathbf{n} d\omega, \mathbf{t}\delta\omega),\end{aligned}$$

where $\omega \in (L^2 \cap C^\infty)(M, \Lambda^k T^* M)$ is the solution.

Let Γ be a non-empty open subset of ∂M and $f_1 \in C^\infty(\partial M, \Lambda^k T^* M)$, $f_2 \in C^\infty(\partial M, \Lambda^{n-k} T^* M)$ be supported on Γ. Then we define the Dirichlet-to-Neumann map on Γ by

$$\Lambda_{g,\Gamma}^{(k)}(f_1, f_2) = (\mathbf{n} d\omega|_\Gamma, \mathbf{t}\delta\omega|_\Gamma).$$

Theorem (K.–Lassas–Ulhmann)

Let M_1 and M_2 be complete real-analytic orientable connected Riemannian manifolds, $\dim(M_1) = \dim(M_2) = n \geq 2$, *with compact real-analytic boundaries ∂M_1 and ∂M_2, respectively. Let the metrics g_1 and g_2 be real-analytic up to the boundary, and let $\Gamma_1 = \Gamma_2 = \Gamma$ be a non-empty open subset of ∂M_1 and ∂M_2.*

- *If $n \geq 3$, assume that (A1) holds. Suppose also that $0 \notin spec_{ess}(\Delta^{(k)}_{F,M_1})$ and $0 \notin spec_{ess}(\Delta^{(k)}_{F,M_2})$, and that for some integer $1 \leq k \leq n$,*
$$\Lambda^{(k)}_{g_1,\Gamma} = \Lambda^{(k)}_{g_2,\Gamma}.$$
- *If $n = 2$, assume that (A2) holds. Suppose also that $0 \notin spec_{ess}(\Delta^{(1)}_{F,M_1})$ and $0 \notin spec_{ess}(\Delta^{(1)}_{F,M_2})$, and that*
$$\Lambda^{(1)}_{g_1,\Gamma} = \Lambda^{(1)}_{g_2,\Gamma}.$$

Then M_1 and M_2 are isometric.

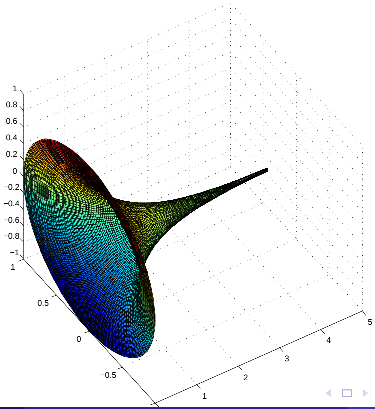
Example. Let $M = S^1_\theta \times [0, +\infty)_z \subset \mathbb{R}^2$ be an infinite cylinder.

Consider two complete metrics on M, belonging to the same conformal class,

$$g_1 = e^{-2z}(d\theta)^2 + (dz)^2,$$

and

$$g_2 = (1 + f(z))g_1, \quad f(z) = ze^{-z}.$$



The Gaussian curvature of the manifold (M, g_1) is $K_{g_1} = -1$.

The Gaussian curvature of the manifold (M, g_2),

$$K_{g_2} = \frac{e^{-2z}(z + 1 - 3z^2) + e^{-z}(3 - 6z) - 2}{2(1 + ze^{-z})^3}.$$

$|K_{g_2}|$ is uniformly bounded on M.

Let us verify that $0 \notin \mathrm{spec}(\Delta^{(1)}_{g_1,F})$.

1-form $\omega(\theta,z) = \omega_1(\theta,z)d\theta + \omega_2(\theta,z)dz$

$$\Delta^{(1)}_{g_1} = \begin{pmatrix} -\partial_z^2 - e^{2z}\partial_\theta^2 - \partial_z & 2\partial_\theta \\ -2e^{2z}\partial_\theta & -\partial_z^2 - e^{2z}\partial_\theta^2 + \partial_z \end{pmatrix}.$$

This operator is considered in the Hilbert space

$$\mathcal{H} = L^2(S^1 \times \mathbb{R}_+; e^z d\theta dz) \times L^2(S^1 \times \mathbb{R}_+; e^{-z} d\theta dz),$$

equipped with the inner product

$$\langle \omega, \eta \rangle_{L^2} = \int_0^{2\pi} \int_0^{+\infty} (\omega_1 \bar{\eta}_1 e^z + \omega_2 \bar{\eta}_2 e^{-z}) d\theta dz.$$

In order to pass to the unweighted space $(L^2(S^1 \times \mathbb{R}_+; d\theta dz))^2$, we perform a conjugation by the unitary operator

$$U = \begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix}.$$

Taking a Fourier decomposition with respect to the variable $\theta \in S^1$, we write

$$U\Delta_{g_1}^{(1)}U^{-1} = \bigoplus_{k\in\mathbb{Z}} P_k, \quad P_k = \begin{pmatrix} -\partial_z^2 + 1/4 + e^{2z}k^2 & 2ike^z \\ -2ike^z & -\partial_z^2 + 1/4 + e^{2z}k^2 \end{pmatrix}.$$

Hence, the contribution to the continuous spectrum of the operator $\Delta_{g_1}^{(1)}$ with the Dirichlet boundary conditions comes only from $k = 0$.

The spectrum of P_0 is $[1/4, +\infty)$ and is purely absolutely continuous.

$$0 \notin \mathrm{spec}(\Delta_{g_1,F}^{(1)}), \quad 0 \notin \mathrm{spec}(\Delta_{g_2,F}^{(1)}).$$

Our theorem shows that the set of the Cauchy data for harmonic 1-forms allows us to distinguish between the complete Riemannian manifolds (M, g_1) and (M, g_2).

However, (M, g_1) and (M, g_2) are undistinguishable on the level of the Cauchy data for harmonic 0-forms.

$$\Delta_{g_1}^{(0)} = -e^{2z}\partial_\theta^2 - \partial_z^2 + \partial_z.$$

$$\Delta_{g_2}^{(0)} = (1 + f(z))^{-1}\Delta_{g_1}^{(0)}.$$

$$0 \notin \mathrm{spec}(\Delta_{g_1,F}^{(0)}), \quad 0 \notin \mathrm{spec}(\Delta_{g_2,F}^{(0)}).$$

Given $v \in C^\infty(\partial M)$, the boundary value problem

$$\Delta_{g_j}^{(0)} w_j = 0, \quad \text{on } M,$$
$$\mathbf{t} w_j = v,$$

has a unique solution $w_j \in L^2(M, \mu_{g_j})$, $j = 1, 2$.

Since $\mu_{g_2} = (1 + f(z))\mu_{g_1}$, we have $L^2(M, \mu_{g_2}) = L^2(M, \mu_{g_1})$ as spaces, with equivalent norms.

Since the Laplacians are conformally invariant, we get $w_1 = w_2 = w$.

$\mathbf{n}_{g_2} dw = \mathbf{n}_{g_1} dw$, since $*_{g_2} = *_{g_1}$ on 1-forms.

$$\mathcal{C}_{g_1}^{(0)} = \mathcal{C}_{g_2}^{(0)}.$$