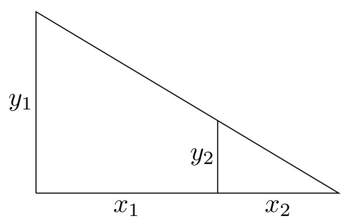
Review for Math 124 based on Math 124 Assessment

Here you will find review material, examples and practice problems corresponding to each problem on the Math 124 Assessment. If you were unable to solve any of the problems on the assessment, look here for suggestions for how to approach such problems.

1. Assessment problem: *A person is walking directly away from a light on an 18-foot tall pole. At this instant, the person casts a shadow* 14 *feet long. If they walk* 10 *feet farther from the pole, they will cast a shadow* 20 *feet long. How tall is the person?*

   Problems of this sort require you to draw a careful figure (sometimes drawing more than one figure is helpful), introduce two variables, apply the concept of **similar triangles** to create two equations, and solve those equations for the values of the variables.

   Consider the right triangular figure below.



   Similar triangles tells us that

$$\frac{y_1}{x_1 + x_2} = \frac{y_2}{x_2}.$$

   If we know all but one of the quantities x_1, x_2, y_1, and y_2, we can solve the above equation for that unknown value.

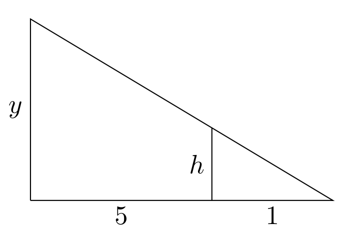
   If we know two of the quantities x_1, x_2, y_1, and y_2, this equation will be an equation in two unknowns. By creating another figure of this type involving those same unknowns (based on information presented in the problem), we will get another equation in those two unknowns. Thus, we will be in a two-equations-in-two-unknowns situation, and can solve for the unknown quantities.

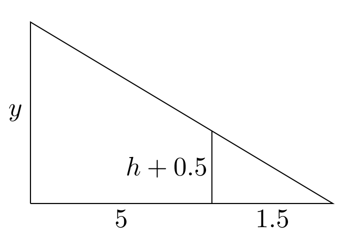
   The same goes for the situation involving three unknowns: we use similar triangles to find *three* equations in those three variables and solve for them.

   Example: A person casts a shadow 1 meter long when they are 5 meters from a light pole. The person then puts on a hat that makes them appear 0.5 meters taller and their shadow becomes 1.5 meters long. How tall is the light pole?

   We first notice that we do not know how tall the person is, and we do not know how tall the light pole is. So let's introduce variables for these quantities: let y be the light pole's height, and let h be the person's height.

   Next, we draw two pictures: one for the initial situation, and one for when the person is wearing the hat. Both will incorporate y and h.





Notice that it is not necessary to try to draw the drawings "to scale". As long as the drawing helps us apply Similar Triangles, then the drawing has done its job.

From the figures, using Similar Triangles, we have

$$\frac{y}{5+1} = \frac{h}{1} \text{ and } \frac{y}{6.5} = \frac{h+0.5}{1.5}$$

from which we get

$$y = 6h \text{ and } \frac{y}{13} = \frac{h+0.5}{3}$$

so

$$18h = 13h + 6.5 \text{ so } h = 1.3 \text{ meters}$$

and thus $y = 6h = 7.8$ meters. So the light pole is 7.8 meters tall.

Practice problems

(a) A person casts a shadow 4 meters long when they are 5 meters from a light pole. The pole is 3 meters taller than the person. How tall is the person? (Ans.: 2.75 meters.)

(b) A person casts a shadow as long as they are tall when they are 6 meters from a light pole. They walk two meters closer to the pole and find that their shadow is one meter shorter than they are tall. How tall is the pole? (Ans.: 9 meters.)

(c) A wire extends from the top of a pole to the ground. A five-foot tall person and a six-foot tall person both stand under the wire so that it just touches their heads. The stand two feet apart. How tall is the tower? (Ans.: 10 feet.)

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2. Assessment problem: *Line A passes through the points* $(1, 5)$ *and* $(6, 3)$. *Line B is perpendicular to line A and passes through the point* $(4, 4)$. *At what point do Line A and Line B intersect?*

The find the equation of a line through two given points (x_1, y_1) and (x_2, y_2), begin by finding the **slope** m of the line via the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

This works perfectly unless $x_1 = x_2$ in which case the line is vertical; in that case, the equation of the line is $x = x_1$

Then use the **point-slope** formula for a line

$$y = m(x - x_1) + y_1.$$

We can use this form any time we know the slope of a line and a point that it passes through.

The other thing you need to know is that if a line has slope m_1 and another line has slope m_2, then the lines are perpendicular when $m_1 m_2 = -1$, or, equivalently,

$$m_2 = \frac{-1}{m_1}.$$

Suppose we have the equations of two in the form $y = f_1(x)$ and $y = f_2(x)$ and we wish to find he point of intersection of the lines. We can do this by solving the equation $f_1(x) = f_2(x)$. This will yield the x-coordinate of the point of intersection, and the y coordinate can be found by substituting the x-coordinate into $y = f_1(x)$.

Example: Line A passes through $(5, 4)$ and $(10, 1)$. Line B is perpendicular to the line $y = 5x + 8$ and passes through the point $(0, 6)$. Where do line A and line B intersect?

We first find the equation of line A.

Since it passes through $(5, 4)$ and $(10, 1)$, the line has slope

$$\frac{1-4}{10-5} = -\frac{3}{5}.$$

Hence, it has equation $y = -\frac{3}{5}(x-5)+4$.

Line B is perpendicular to $y = 5x+8$ which has slope 5. Hence, line B has slope $-\frac{1}{5}$ and equation

$$y = -\frac{1}{5}(x-0)+6 = -\frac{1}{5}x+6.$$

To find the x-coordinate of their point of intersection, we solve

$$-\frac{1}{5}x+6 = -\frac{3}{5}(x-5)+4$$

and find

$$x = \frac{5}{2}$$

and $y = -\frac{1}{5}\left(\frac{5}{2}\right)+6 = \frac{11}{2}$. Thus the point of intersection if $(5/2, 11/2)$.

Practice problems:

(a) Let line A be the line $y = 5(x-1)+3$. Let B be the line that is perpendicular to line A and passes through line A's y-intersept. (Ans.: $y = -\frac{1}{5}x - 2$.)

(b) Let line A be the line $y = 2x+3$. Find the intersection of line A with a line perpendicular to it that passes through the origin. (Ans.: $(-6/5, 3/5)$)

(c) Let line A be the line through $(7,0)$ and $(1,5)$. Let line B be the line perpendicular to line A which passes through the point $(10,0)$. Find the point of intersection of lines A and B. (Ans.: $(535/61, -90/61)$)

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3. Assessment problem: *The graph of a quadratic function f passes through the points $(-2,4)$, $(1,1)$ and $(5,0)$. What is the y-intercept of the graph of f?*

To find the equation of a quadratic function given through points on its graph, we begin by noting that the general form for a quadratic function is

$$f(x) = ax^2+bx+c$$

where a, b and c are constants.

If we know the points $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) are three points on the graph of this function, then we know three things:

$$\begin{aligned} y_1 &= ax_1^2+bx_1+c \\ y_2 &= ax_2^2+bx_2+c \text{ and} \\ y_3 &= ax_3^2+bx_3+c \end{aligned}$$

This gives us **three equations in three unknowns**: a good algebraic situation! We can chew on these equations and solve for a, b and c. One note: if you subtract the first equation from the second, and the second from the third, you will eliminate c and have two equations in two unknowns (a and b): an even better algebraic situation!

Example: Let's find the quadratic function whose graph passes through the points $(2,1),(3,2)$ and $(5,1)$. Say our function is $f(x) = ax^2+bx+c$. Then we have three equations

$$1 = a(2)^2+b(2)+c = 4a+2b+c \quad (1)$$
$$2 = a(3)^2+b(3)+c = 9a+3b+c \quad (2)$$
$$1 = a(5)^2+b(5)+c = 25a+5b+c. \quad (3)$$

Subtracting the first equation from the second, and the second from the third yields the equations

$$1 = 5a+b \quad (4)$$
$$-1 = 16a+2b \quad (5)$$

Subtracting twice the first equation from the second eliminates b and we have $-3 = 6a$ and so $a = -\frac{1}{2}$. Plugging this in to $1 = 5a + b$, we find $b = \frac{7}{2}$. Plugging our values for a and b into $1 = 4a + 2b + c$, we find $c = -4$.

Thus, the function sought is $f(x) = -\dfrac{1}{2}x^2 + \dfrac{7}{2} - 4$.

Practice problems: Find the quadratic function whose graph passes through each set of points:

(a) $(0,0), (1,1)$ and $(3,-1)$ (Ans.: $y = -\frac{2}{3}x^2 + \frac{5}{3}x$)

(b) $(-1,1), (1,-2)$ and $(3,4)$ (Ans.: $y = 1.125x^2 - 1.5x - 1.625$)

(c) $(0,1), (1,1)$ and $(1,3)$ (Ans.: No solution.)

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4. Assessment problem: *Let $f(x) = 2x^2 - 3x + 5$. Let $g(x) = f(6-x)$. What is the lowest point on the graph of g?*

Given a quadratic function $f(x)$ and a linear function $h(x)$, the composite function $f(h(x))$ will be a quadratic function.

All quadratic functions have an extreme point on their graph; this point is known as the **vertex** of the graph. Given any quadratic function, we can use the method of **completing the square** to put the function in the form

$$f(x) = a(x-h)^2 + k.$$

Once in this form, we can read off the vertex: it is the point (h,k).

Alternatively, the vertex of the function $f(x) = ax^2 + bx + c$ is the point

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

The vertex will be the highest point on the graph if $a < 0$ and it will be the lowest point on the graph when $a > 0$.

Example: Given that $f(x) = x^2 + 3x + 4$ and $g(x) = f(x-1)$, find the lowest point on the graph of g.

One approach is to find the quadratic expression for $g(x)$ and then find the vertex. We have

$$g(x) = f(x-1) = (x-1)^2 + 3(x-1) + 4 = x^2 + x + 2 = \left(x + \frac{1}{2}\right)^2 + \frac{7}{4} = \left(x - \left(-\frac{1}{2}\right)\right)^2 + \frac{7}{4}.$$

after substituting and completing the square. We can thus see that the vertex, which is the lowest point on the graph of g, is the point $\left(-\dfrac{1}{2}, \dfrac{7}{4}\right)$.

Another approach is to note that replacing x by $x-1$ in $f(x)$ results in a **horizontal shift** of the graph of $f(x)$; it is shifted one unit to the right. That is, the graph of $g(x)$ is the graph of $f(x)$ after shifting it one unit to the right. We can find the vertex of f to be

$$\left(\frac{-3}{2(1)}, f\left(\frac{-3}{2(1)}\right)\right) = \left(-\frac{3}{2}, \frac{7}{4}\right).$$

Then, since the graph of g is a horizontal shift by one unit of the graph of f, the vertex of g is the vertex of f shifted one unit to the right. Hence, the vertex of g is the point $\left(-\dfrac{3}{2} + 1, \dfrac{7}{4}\right) = \left(-\dfrac{1}{2}, \dfrac{7}{4}\right)$.

Practice problems:

(a) Let $f(x) = x^2 + x + 1$ and $g(x) = f(x+3)$. Find the vertex of g. (Ans.: $(-7/2, 3/4)$)

(b) Let $f(x) = -2x^2 - 5x + 6$ and $g(x) = f(1-x)$. Find the vertex of g. (Ans.: $(9/4, 73/8)$)

(c) Let $f(x) = 5x^2 + 4x - 11$ and $g(x) = f(x-2)$. Find the vertex of g. (Ans.: $(8/5, -59/5)$)

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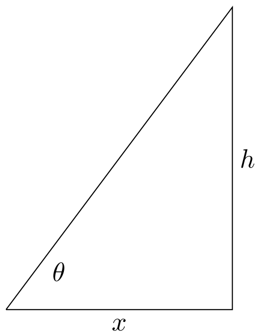
5. Assessment problem: *You are walking along a flat, horizontal road toward a mountain in the distance. You measure the mountain's angle of elevation and find it to be* $4°$. *You then walk 2 kilometers closer to the mountain and measure the angle again: this time the angle is* $4.5°$. *How tall is the mountain?*

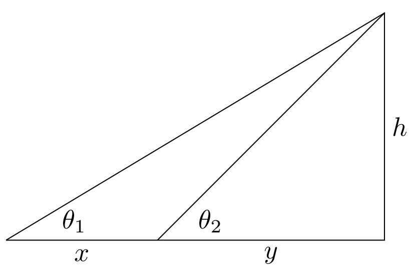
   Many trigonometry problems involve the use of the tangent function to express relationships between horizontal and vertical lengths and an angle. The common basic situation looks like this:



   The basic arrangement is that we have a horizontal distance (x), and vertical distance or height (h) and an angle θ. The language used here is often that an object of height h makes an angle θ with the horizontal. Frequently, θ is called the *angle of elevation* of the object with height h. Common scenarios involve a measurement of this angle for a distance mountain, building, tree, etc.

   Trigonometry comes in since $\tan\theta = \dfrac{h}{x}$. This is an equation in three variables, so if we know two of the variables, we can solve this equation for the other two.

   More elaborate problems might be diagrammed like this, for example:



   Here we have five variables. Applying trigonometry, we have two equations

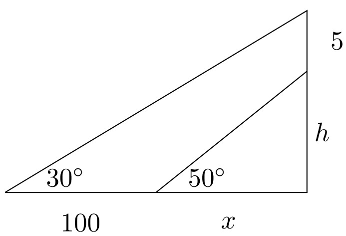
$$\tan\theta_1 = \frac{h}{x+y} \text{ and } \tan\theta_2 = \frac{h}{y}.$$

   In a problem like this, we might be given the two angles, and one of x and y. So the only unknowns would be h and one of x and y; since we have two equations above and only two unknowns, we are in good algebraic situation and can solve for those unknowns.

   Be sure to begin a problem of this sort by carefully drawing a correct figure.

   Example: Godzilla is attacking! You measure the angle from your location to the top of Godzilla's head and find it to be $50°$. You run to a location that is 100 meters farther from Godzilla and measure again: now the angle is $30°$. Between the two measurements, Godzilla grew 5 meters taller! How tall is Godzilla at the second measurement?

   Let x be the horizontal distance between our first measurement and Godzilla. Let h be Godzilla's height at the time of the first measurement. Then we can draw a figure to represent the situation.



Using trigonometry, we create two equations in our two unknowns:

$$\tan 50^\circ = \frac{h}{x} \quad \text{and} \quad \tan 30^\circ = \frac{h+5}{100+x}$$

Solving these two equations, we find $x = 85.831$ meters, and Godzilla's initial height was $h = 102.289$ meters.

Practice problems:

(a) Fred was measuring the height of a building. He stood standing at some distance from the building and measured an angle of 38 degrees to the top of the building. He then walked 100 feet further away from the building and measured an angle of 34 degrees to the top. How tall is the building? (Ans.: 493.54 feet)

(b) You measure the angle to the top of a building and find it to be 66°. A floor is added to the building, making it 10 feet taller. You measure again from the same spot and find the angle is now 68°. How tall is the building now? (Ans.: 108.058 feet tall.)

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6. Assessment problem: *The population of town A grows exponentially and doubles every 7 years. Currently, town A has a population of* 4000. *The population of town B increases at the constant rate of* 12% *per year. Five years from now, town B will have a population that is half that of town A. When, in years from now, will the populations of the two towns be equal?*

A quantity Q growing exponentially can be represented as $Q = Q_0 b^t$ where t is the amount of time since $Q = Q_0$. There are two parameters that determine the quantity over all time: the so-called initial value Q_0 and the growth constant b. So, to nail down the function that represents an exponentially growing quantity we will generally need two pieces of information, such as the value of the quantity at two different points in time.

Example: The population of Pinedale, Wyoming, is growing exponentially. In 1990, the population was 860 and in 1995 it was 1210. What was the population in 2010?

We may begin by letting t represent years after 1990. Then we know the population $P = 860b^t$ for some value of b. Since $P = 1210$ when $t = 5$, we have

$$1210 = 860b^5$$

and so $b = 1.0706743138795$. Thus, the population function is

$$P = 860(1.0706743138795)^t.$$

Plugging in $t = 20$ for the year 2010, we find the population was

$$860(1.0706743138795)^{20} = 3370.126$$

in 2010.

Suppose further that the number of horses in Pinedale always increases by 11 percent each year. In the year 2000, there were twice as many people as horses in Pinedale. When will the number of people and the number of horses be equal?

So now we need to find the exponential function for the horse population. Let's let $H = H_0 c^t$ be the function, and we seek H_0 and c. Note here that we will use t in the same way as earlier: it is the years after 1990.

Since we know the horse population increases by 11 percent per year, we know that $c = 1.11$ (note that multiplying any quantity by 1.11 increases the quantity by 11 percent). We now need to find H_0. The other piece of information given was that in the year 2000 ($t = 10$), the population of people and the population of horses were equal. We find the population of people to be

$$860(1.0706743138795)^{10} = 1702.441860$$

people and so $H = 851.22093$ when $t = 10$. We can thus solve for H_0 in the equation

$$851.2209302 = H_0(1.11)^{10}$$

and find $H_0 = 299.7867996$.

Thus the population of horses at time t (years after 1990) is

$$299.7867996(1.11)^t.$$

To find when the populations of people and horses are equal to equate

$$299.7867996(1.11)^t = 860(1.0706743138795)^t$$

and solve for t by first applying the natural logarithm:

$$\ln 299.7867996 + t\ln(1.11) = \ln 860 + t\ln(1.0706743138795)$$

Note that the natural logarithm puts the equation into linear form, and then we solve for t and find $t = 29.215995$. Thus, 29.215995 years after 1990, the populations of horses and people in Pinedale will be equal.

Practice problems:

(a) In the year 1980, the city of Gumbolantenden had a population of 20,000. The population of Gumbolantenden grows at the rate of 1.07% each year.

The population of the city of Attalioto is growing exponentially. In the year 2000, the population was 40,000. In the year 2005, the population was 62,000.

When will the populations of the two cities be the same? (Ans.: 13.763 years after 1980)

(b) The population of Aarb and the population of Bullm are each growing exponentially. In the year 2000, Aarb's population was 5000. In the year 2005, Bullm's population was 7000. Bullm's population doubles every 17 years. In the year 2011, there will be twice as many people in Aarb as in Bullm.

When will Aarb's population hit 40000? (Ans.: 17.9507 years after the year 2000)

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7. Assessment problem: *Find all solutions to the equation* $\sin x = -0.7$ *with* $-1 \le x \le 10$.

If $-1 \le a \le 1$, then the equation $\sin x = a$ has infinitely many solutions. We can find *one* of them via the inverse sine function. Let $P = sin^{-1}a$. Then $\sin P = a$ so P is a solution to the equation $\sin x = a$. P is often called the principal solution.

An important feature of the graph $y = \sin x$ is that it is symmetric about the vertical line $x = \dfrac{\pi}{2}$. As a result, if P is a solution to $\sin x = a$, then we will also have a **symmetric** solution, S, given by

$$S = \frac{\pi}{2} + \left(\frac{\pi}{2} - P\right).$$

That is, there will be a solution as far to the right of $\frac{\pi}{2}$ as P is the left of $\frac{\pi}{2}$.

Due to the nature of $y = \sin x$, all other solutions are shifts of P and S by multiples of 2π, the period of sine.

Example: Find all solutions to $\sin x = 0.8$ with $-5 \le x \le 5$.

We begin by finding the principal solution: $P = \sin^{-1} 0.8 = 0.9272952180$. The symmetry solution is then $S = \frac{\pi}{2} + \frac{\pi}{2} - P = 2.2142974355$. If we add 2π to P, we get a value greater than 5. If we subtract 2π from P we get a value less then -5. If we add 2π to S, we get a value greater than 5. If we subtract 2π from S, we get -4.06888787159, which is in our desired interval. Subtracting 2π again, though, would give a value less than -5.

Thus, our solutions are $0.9272952180, 2.2142974355$ and -4.06888787159.

Practice problems:

(a) Find all solutions to $\sin x = 0.3$ with $5 \leq x \leq 10$. (Ans.: There is only one solution: $x = 9.12008530675$.)

(b) Find the smallest value of $x > 100$ such that $\sin x = 0.6$. (Ans.: $x = 101.1744660236$.)

8. Assessment problem: *Find the equation of one of the tangent lines to the circle $x^2 + y^2 = 5^2$ that pass through the point $\left(\frac{25}{3}, 0\right)$.*

Suppose a line is tangent to the circle $x^2 + y^2 = r^2$ at the point (a, b). The radius from the center of the circle to (a, b) has slope $\frac{b}{a}$. The tangent line is perpendicular to this, and so has slope $-\frac{a}{b}$.

So, given information about a tangent line to a circle, work to find the point of tangency. Introduce (a, b) for the point of tangency and then set up two different equations involving a and b; you can then solve for a and b. Then you can use this information to find whatever the problem asks you to find.

Example: Find the equations of the tangent lines to the circle $x^2 + y^2 = 4$ that pass through the point $(3, 1)$.

First, draw a picture of the situation. Convince yourself that there are two such lines. Introducing variables for a point of tangency: let's call it (a, b). So we have two variables: if we can get two equations involving these variables, we will be in a good algebraic situation.

One equation comes from the circle: the point (a, b) is on the circle, so $a^2 + b^2 = 4$.

Now, we need to use the fact that the tangent line passes through $(3, 1)$ and is perpendicular to the radius (that is, the line connecting $(0, 0)$ and (a, b)). We may write down the slope of tangent line in two ways.

First, since the line passes through (a, b) and $(3, 1)$, the slope is

$$\frac{1-b}{3-a}.$$

Second, since the line is perpendicular to the radius, which has slope $\frac{b}{a}$, the tangent line has slope $-\frac{a}{b}$.

These two expressions must be equal, since they are both the slope of the line, and so we have out second equation:

$$-\frac{a}{b} = \frac{1-b}{3-a}.$$

We can chew on this equation a bit and rearrange it to

$$b + 3a = a^2 + b^2.$$

Hooray! We know $a^2 + b^2 = 4$, so now we know $b + 3a = 4$, hence $b = 4 - 3a$.

Thus,

$$a^2 + b^2 = a^2 + (4 - 3a)^2 = 4$$
$$10a^2 - 24a + 12 = 0.$$

Using the quadratic formula, we can find two possible values of (a, b):

$$(0.710102, 1.869694) \qquad \text{and} \qquad (1.689898, -1.069693)$$

which yield the tangent lines with approximate equations

$$y = -0.37980x + 2.1394 \qquad \text{and} \qquad y = 1.5798x - 3.7394.$$

Practice problems

(a) Find the equations of the tangent lines to the circle $x^2 + y^2 = 1$ which pass through the point $(-4, 0)$. (Ans.: $y = \pm \left(\frac{1}{\sqrt{15}} \left(x + \frac{1}{4} \right) + \frac{\sqrt{15}}{4} \right)$).

(b) Find the equations of the tangent lines to the circle $x^2 + y^2 = 9$ which pass through the point $(-7, 2)$. (Ans.: Using decimal approximations, the lines are approximately $y = -0.847493x - 3.932456$ and $0.147493x + 3.032456$.)

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9. Assessment problem: *Let* $f(x) = \dfrac{2x+4}{5x+2}$. *Find the equations of the horizontal and vertical asymptotes of* $y = f(1 - 3x)$.

Given a definition for a function f, if you want to find an expression for $f(ax + b)$, you need to replace every x that appears in the definition of f with $ax + b$.

More generally, if you have an expression for $f(g(x))$, where $g(x)$ is some expression in x, you need to replace every x in the definition of $f(x)$ with $g(x)$.

For example, if $f(x) = x^2 - 3x + |x|$, then $f(2x - 1) = (2x - 1)^2 - 3(2x - 1) + |2x - 1| = 4x^2 - 10x + 4 + |2x - 1|$.

For a rational function with numerator and denominator of degree one (i.e., linear numerator and denominators), the vertical asymptote will occur where the function is undefined, which is at the zero of the denominator.

The horizontal asymptote can be found by dividing the numerator and denominator by x, and considering what the value of the function approaches as x gets very, very large.

For example, suppose $f(x) = \dfrac{2x+3}{3x+5}$. If we divide the numerator and denonominator by x, we find

$$f(x) = \frac{2 + \frac{3}{x}}{3 + \frac{5}{x}}.$$

As x gets very large, $\frac{3}{x}$ and $\frac{5}{x}$ get very small (i.e., become close to zero), and this is only more true the larger x becomes. As a result, we see that $f(x)$ becomes very close to $\frac{2}{3}$ as x gets very large, and, again, this is only more true the larger x becomes. As a result, the graph $y = f(x)$ will approach the horizontal line $y = \frac{2}{3}$ as we get farther and farther from the origin. We say $y = \frac{2}{3}$ is the horizontal asymptote of $y = f(x)$.

For the vertical asymptote, we note that $3x + 5 = 0$ when $x = -\frac{5}{3}$. When x is close to $-\frac{5}{3}$, the denominator of $f(x)$ is close to zero (while the numerator is not), and so $f(x)$ will be large in absolute value. The closer x gets to $-\frac{5}{3}$, the closer the denominator will be, and so the larger $f(x)$ will become. As a result, we say that $x = -\frac{5}{3}$ is the vertical asymptote of $y = f(x)$.

Practice problems

(a) Let $f(x) = \frac{x}{4x-2}$. Find the asymptotes of $y = f(5x - 1)$. (Ans.: $y = \frac{1}{4}$ and $x = \frac{3}{10}$.

(b) Let $f(x) = \frac{1-x}{x+1}$. Find the asymptotes of $y = f(2x + 3)$.(Ans.: $y = -1$ and $x = -2$.)

(c) Let $f(x) = \frac{3}{5x+4}$ and $g(x) = 2x + 8$. Find the asymptotes of $y = f(g(x))$. (Ans.: $y = 0$ and $x = -\frac{22}{5}$.)

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10. Assessment problem: *Two numbers sum to* 100. *How small can the sum of one number and the square of the other number be?*

For a problem like this, the first key step is to introduce variables for any unknowns. Then use the information in the problem to express relationships between the variables, and express quantities we are interested in (e.g., "the sum of one number and the square of the other number") in terms of those variables. If we are trying to maximize or minimize a quantity, we will need to express that quantity as a function of a single variable, and then use our knowledge of quadratic functions to find the optimal value.

Example: Two numbers sum to 10. How small can the sum of the squares of the two numbers be?

We start by letting a and b be the two numbers. Then we know $a + b = 10$, which we can write as $b = 10 - a$. We are interested in the sum of the squares of the numbers, which we may write as $a^2 + b^2$. To study this quantity, it is useful to replace b by $10 - a$ so that it is expressed in terms of just one variable, a: the sum of the squares of the numbers is

$$a^2 + b^2 = a^2 + (10 - a)^2 = 2x^2 - 20a + 100 = 2(a - 5)^2 + 50.$$

By completing the square and writing the sum in vertex form, we can see that the sum is ≥ 50 with equality achieved when $a = 5$. We conclude that the smallest the sum could be is 50.

Practice problems

(a) The sum of two numbers is 20. How large can the product of the two numbers be? (Ans.: 100)

(b) Suppose you wish to enclose three sides of a rectangular region with 100 meters of fencing. How large a region can you enclose? (Ans.: 1250 square meters.)