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A starshaped polyhedron with no net

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Abstract. A simple construction is given for a polyhedron which establishes the following theorem of A. S. Tarasov: There exist polyhedra of spherical type, with convex faces, which admit no planar nets.

The *net* of a polyhedron P consists of a connected collection of polygons, congruent to the faces of the polyhedron, arranged in one plane (conveniently thought of as a sheet of cardboard) without overlaps, in such a way that some of the polygons share edges in pairs, and that, if the net is cut out and folded along the shared edges, and the remaining edges joined in pairs in a suitable manner, a model of the polyhedron P is obtained. Nets of convex polyhedra have been considered at least since Dürer's famous work [3] was published in 1525.

From the definition it follows that the union of the faces of a net of a convex polyhedron P is a simple polygon N (usually not convex) in the plane, subdivided into convex polygons that correspond to, and are congruent to the faces of the polyhedron. (It may be observed that, as a planar graph, this partition of N is *outerplanar*, that is, all its vertices are on the boundary.)

Some of the history of nets, and some problems about them, were discussed in the first volume of GEOMBINATORICS (see [5], [6]). To the early literature on nets we may add Gergonne [4], who asked for a characterization of those planar polygons which admit a partition that is a net for some convex 3-polytope. This received an answer of sorts in the work of Alexandrow [1], although not in quite the sense that Gergonne probably meant, and not relevant to the present discussion. However, it is very interesting to note that Gergonne introduced his question by stating unequivocally:

> "Every convex polyhedron has as a plane net a convex or nonconvex polygon, subdivided into polygonal compartments".

This statement reflects the naïve spirit of early 19th century geometers — jumping to conclusions based on the outcome in very simple situations. It seems that the question of existence of nets of arbitrary convex polyhedra was explicitly posed only much later, by

Shephard [8]; he examined several related problems as well, and provided most of the few known answers to some of them. The question was repeated by Croft et al. [2] and Ziegler [10], together with many related problems; it seems to have been posed in various sessions at the Oberwolfach institute as well (see references in [9]). The question is unresolved to this day, and led me to formulate (in [6]) the following

Conjecture 1. Every convex polyhedron has a net.

Considering nonconvex polyhedra, the concept of a net clearly carries over, but Conjecture 1 does not hold even if one assumes the polyhedron to be starshaped. An obvious counter-example is given by any right prism over a suitable 5-pointed star (see Figure 1). As mentioned by Tarasov [9], N. G. Dolbilin seems to have been the first to ask whether every polyhedral sphere with convex faces has a net. This has been answered in the negative by Tarasov, who established:

Theorem (Tarasov). There exists a nonconvex polyhedron homeomorphic to a sphere and having convex faces, which admits no net.

Tarasov obtains the polyhedron establishing the Theorem from a cube truncated a little at each vertex, by adding sufficiently tall pyramids on the resulting triangular faces. His example requires various quantitative estimates, and well as a detailed analysis of the possible spanning trees that might determine a net.

The aim of this note is to show that by going to a somewhat larger polyhedron, the idea used in Tarasov's construction may be more easily shown to yield a starshaped polyhedron with convex faces and with no net.

Here is the construction. We start with the regular dodecahedron, cut off a small tetrahedron at each vertex, and replace it by a tetrahedron with isosceles triangles whose base angles are greater than 72°. An example of the resulting "spiky dodecahedron" is

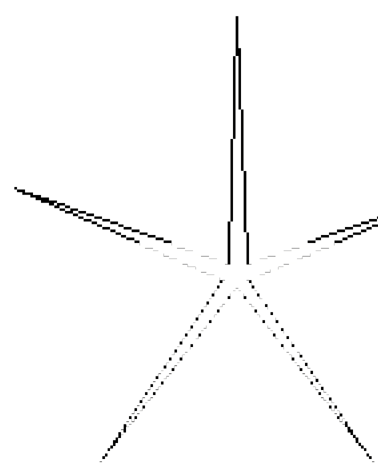

Figure 1. Any right prism over this starshaped polygon has no net.

shown in Figure 2; here about a sixth of each edge was cut off near each vertex, and the base angles are 75°. At each vertex of the spiky dodecahedron which is not the point of a spike, we have two angles of 144° and two angles of more than 72° each. For a net, if a triangle is attached to a decagon, no other decagon may be attached to any of the triangles attached to that triangle, see Figure 3. Hence, the 12 decagons must be directly connected to each other. On the other hand, if two decagons are adjacent to each other (along a long edge E) then no triangle may be attached to a short edge of a decagon if that edge is adjacent to E (see Figure 4). It follows that no decagon may be adjacent to two other decagons along neighboring long sides (see Figure 5), since then the three triangles that are adjacent to them

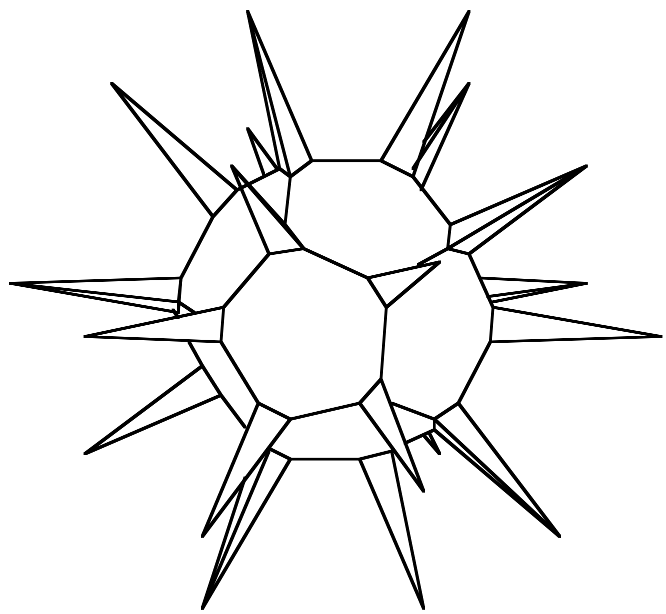

Figure 2. A polyhedron with no net, with convex faces and homeomorphic to the sphere.

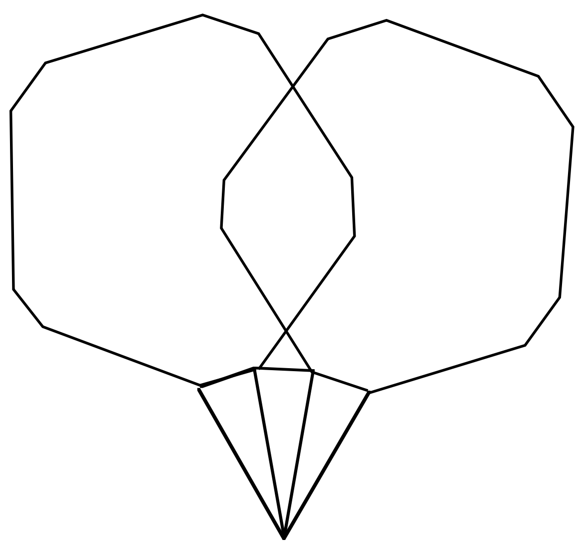

Figure 3. One type of obstruction to forming a net for the spiky dodecahedron.

along the three short sides could not be accommodated. Hence each decagon can be connected to at most two other decagons, and moreover, if it is connected to two, they are attached along non-adjacent long edges (Figure 6). Hence all decagons must form a simple chain. But it is easy to verify that a chain of twelve decagons with this property does not exist on the truncated dodecahedron.

This completes the proof of Tarasov's theorem. ◊

Alternatively, the proof can be completed by observing that there are 20 triplets of triangles, which would have to be accommodated at the free short sides of the chain of 12 decagons. But this is impossible, since each of the end-decagons could accommodate at

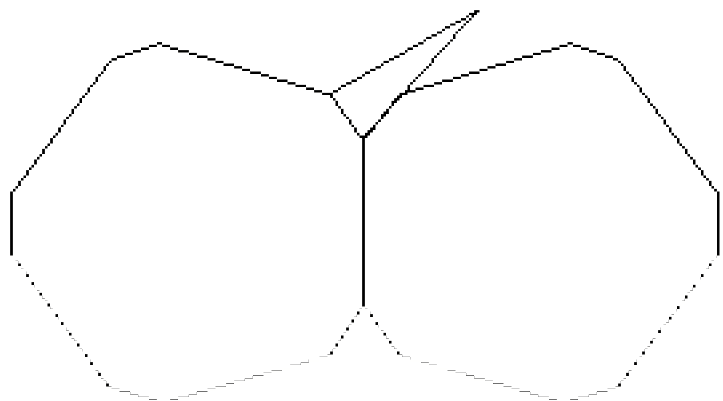

Figure 4. Another obstruction.

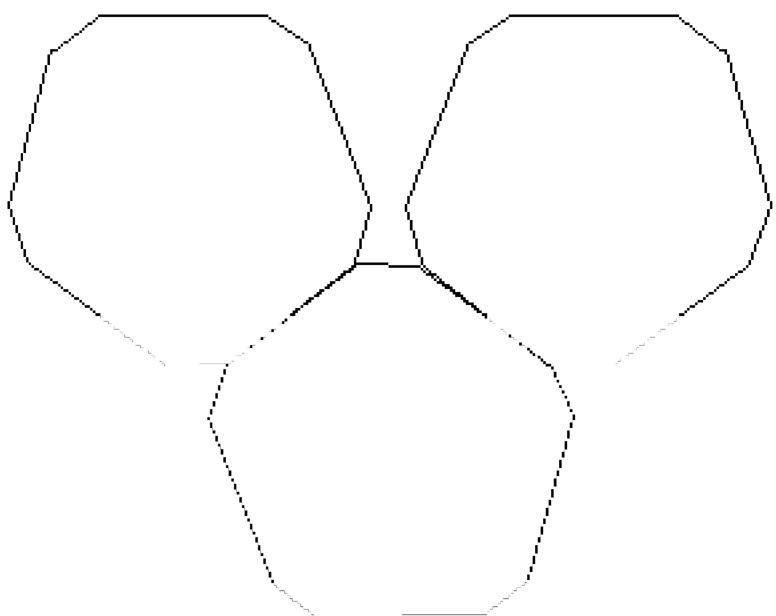

Figure 5. Still another obstruction.

most three triplets, and each of the other ten decagons only one triplet — but $3 + 10 + 3 = 16 < 20$. ◊

It seems that the need to use convex polygons with many sides in the construction is a shortcoming of the method. We venture:

Conjecture 2. There exist simplicial (that is, triangle-faced) polyhedra homeomorphic to the sphere, which have no net.

As an update of [5] we should mention the following result of M. Namiki, presented by Schlickenrieder [7]: There is a tetrahedron for which one particular spanning tree does not determine a net. This provides a negative answer to question (B) posed in [5]. A "would-be net" of Namiki's tetrahedron is shown in Figure 7. On the other hand, regular tetrahedra and cubes, 3-sided and 5-sided Archimedean prisms, as well as many other polyhedra have the property that all trees lead to nets. We venture the following

Conjecture 3. Every combinatorial type of convex polyhedra admits a representative for which all spanning trees yield nets.

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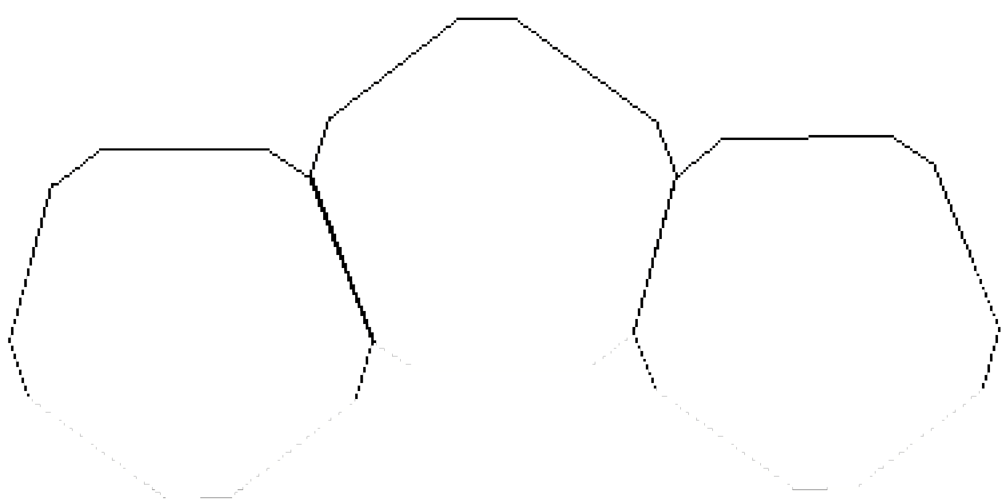

Figure 6. The only way one decagon can be adjacent to two other decagons.

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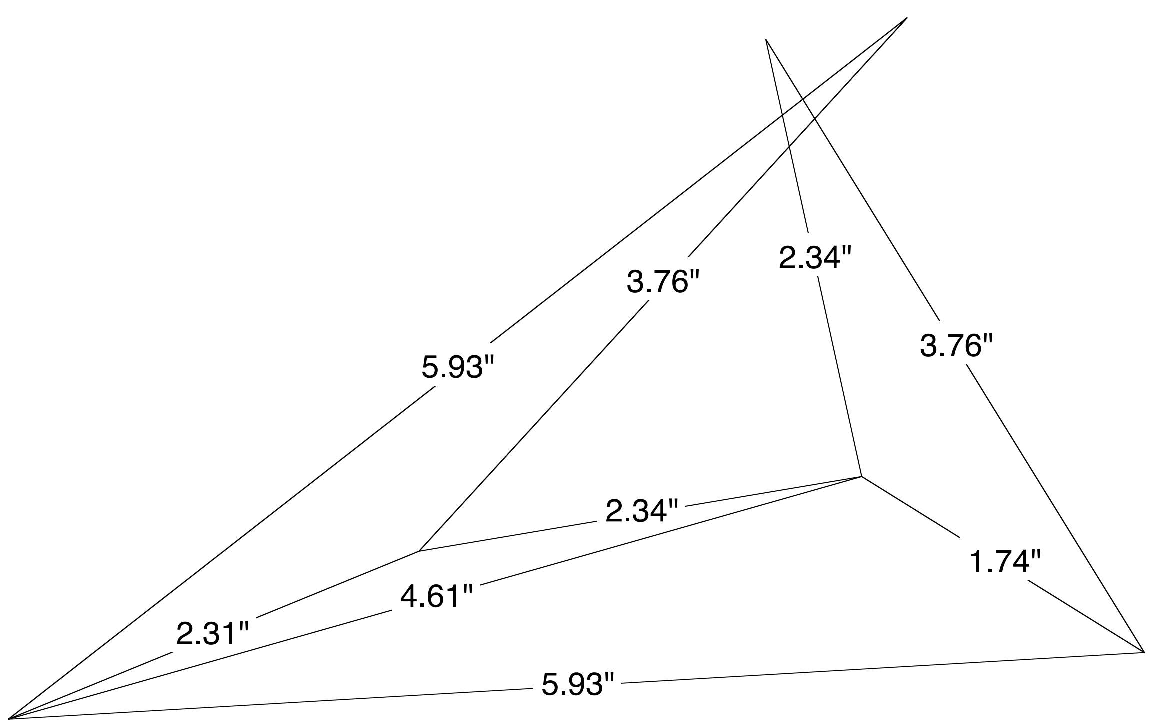

Figure 7. A would-be net for Namiki's tetrahedron.

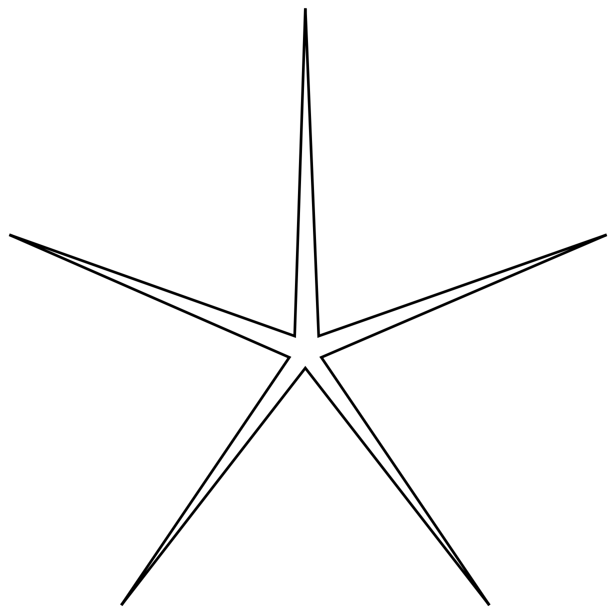

Figure 1.

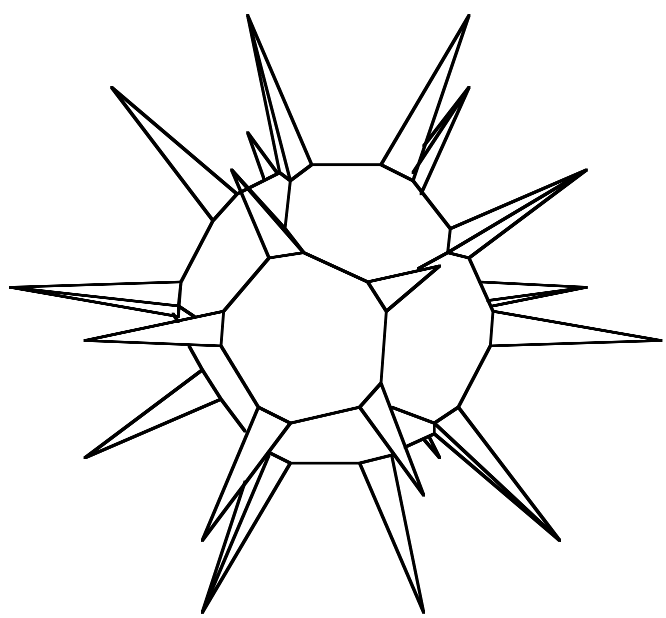

Figure 2.

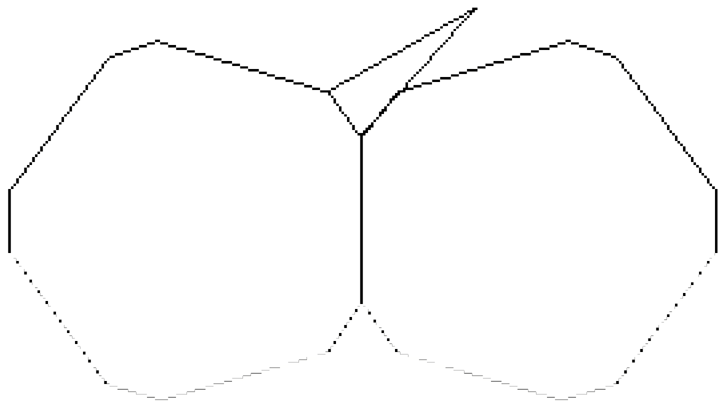

Figure 3.

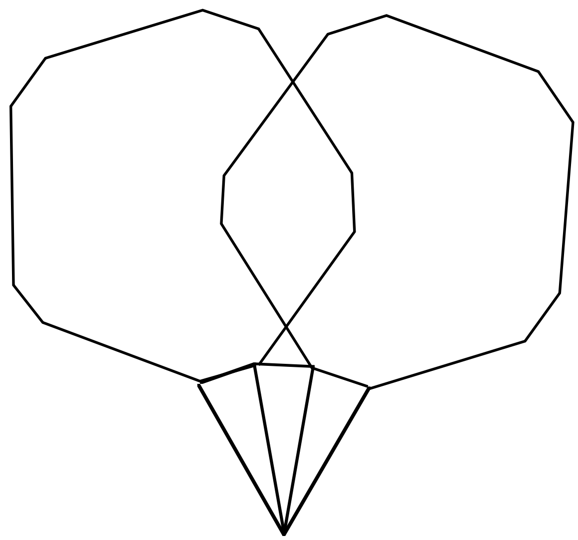

Figure 4.

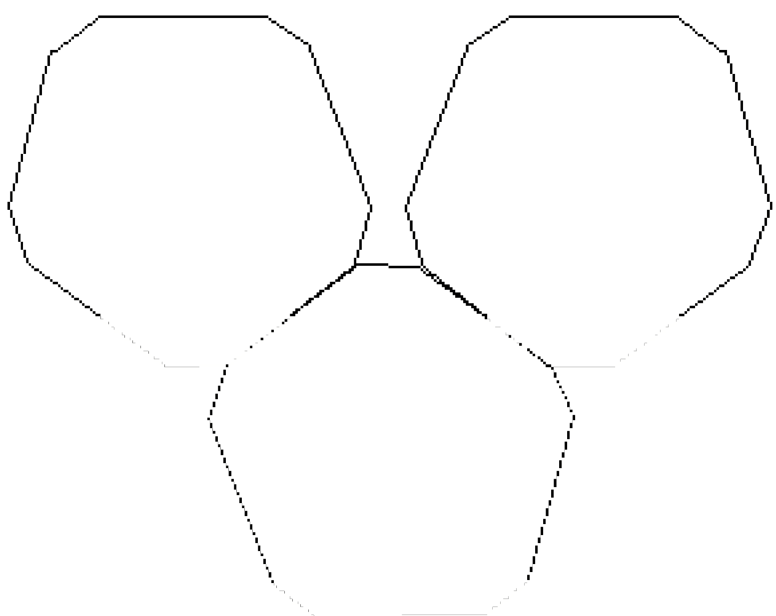

Figure 5.

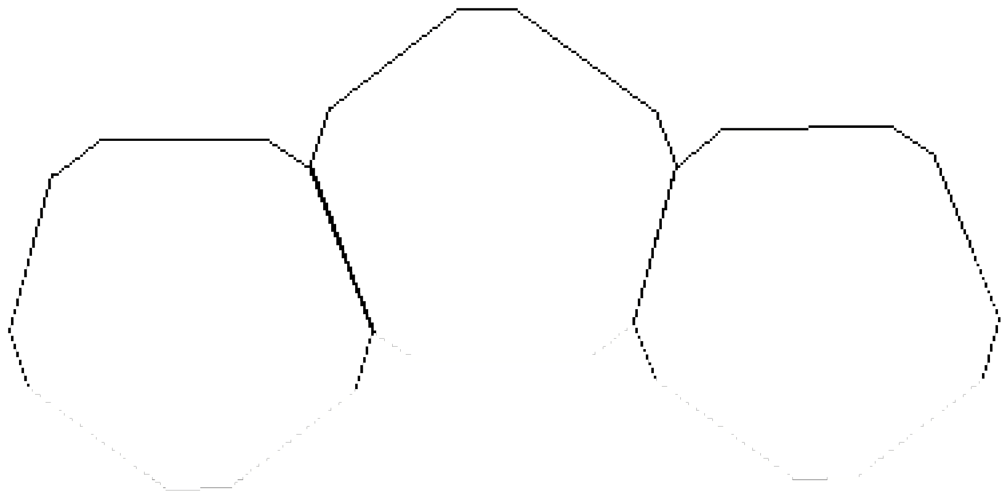

Figure 6.

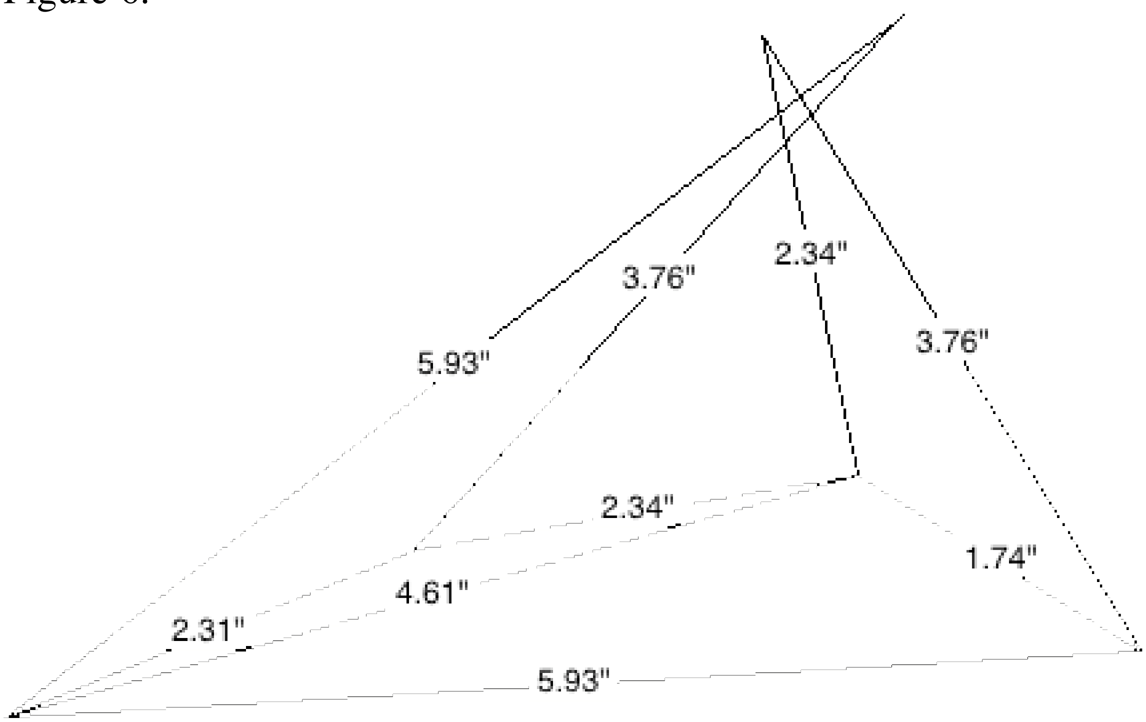

Figure 7.