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A convex polyhedron which is not equifacettable

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1. Introduction. Can every triangle-faced convex polyhedron be deformed to a polyhedron all triangles of which are congruent ? We shall show that such a deformation is not always possible, even if the deformed polyhedron is not required to be convex; in this note, only polyhedra without selfintersections are considered.

This result was first presented, in sketchy form, in a course on polyhedra I gave in 1996. The motivation to present the details now was provided by the interesting paper [2] by Malkevitch, which appeared in the preceding issue of Geombinatorics. Here is the background.

A famous theorem due to E. Steinitz states, in one of its formulations, that every planar (or, equivalently, every spherical) graph can be realized as the graph of edges and vertices of a convex polyhedron in Euclidean 3-space (see, for example, Grünbaum [1, Section 13.1] or Ziegler [3, Chapter 4]). This representation is possible in many different ways, but in all of them the circuits that bound faces of the polyhedron are the same. Malkevitch considered the question whether, in case the graph is a triangulation of the sphere (or, equivalently, the graph of a triangle-faced convex polyhedron), one can insist that the polyhedron in Steinitz's theorem has congruent isosceles triangles as faces. He shows by an elegant example that the answer is negative, and discusses various other questions.

To simplify our exposition, we need some definitions. In this note, by *polyhedron* we mean a collection of planar polygons such that the union of all these polygons is homeomeorphic to a sphere; the image of the polyhedron under this homeomorphism determines a cell-complex decomposition of the sphere. Note that this does not require the polyhedron to be convex, but it does preclude self-intersections of any kind. Such polyhedra are called *acoptic* (from the Greek for "not cut"). All polyhedra realizing the same cell-complex are said to be *combinatorially equivalent*. A polyhedron P is said to be *monohedral*, or *equifacetted*, with a polygon F as *protoface*, provided each face of P is congruent to F.

2. The result. The purpose of this note is to show that there exist cell complexes which are triangulated spheres, but are not realizable by any acoptic monohedral polyhedron. More precisely, we shall establish the following result:

Theorem. There exists no acoptic polyhedron with all faces congruent, that is combinatorially equivalent to the triangle-faced polyhedron indicated by the Schlegel diagram in Figure 1.

Proof of the Theorem. We start by explaining the terms used in the theorem, and by describing the kind of polyhedra illustrated by the example shown in Figure 1. Steinitz's theorem mentioned above implies that in a topological sense there is no difference between triangulations of the sphere and triangle-faced convex polyhedra. Any convex polyhedron can be represented by a *Schlegel diagram*, which is a projection of the boundary of the polyhedron into one of its faces, from a center of projection located outside the polyhedron but sufficiently close to an interior point of a face, so that all other faces project into the chosen face. The polyhedron illustrated in Figure 1 can be considered as a 3-sided pyramid (a *tetrahedron*), onto some faces of which additional tetrahedra have been repeatedly added. We call this a *stacked polyhedron*, and call *height of the stack* the number of tetrahedra successively added onto each of three faces of the starting tetrahedron. Thus, Figure 1 shows a stacked polyhedron P with height of stack equal 4.

If we are concerned with realizations of P by a polyhedron equifacetted with a scalene triangle F as protoface, we note that if we designate the three lengths of edges of F by labels a, b, c, then

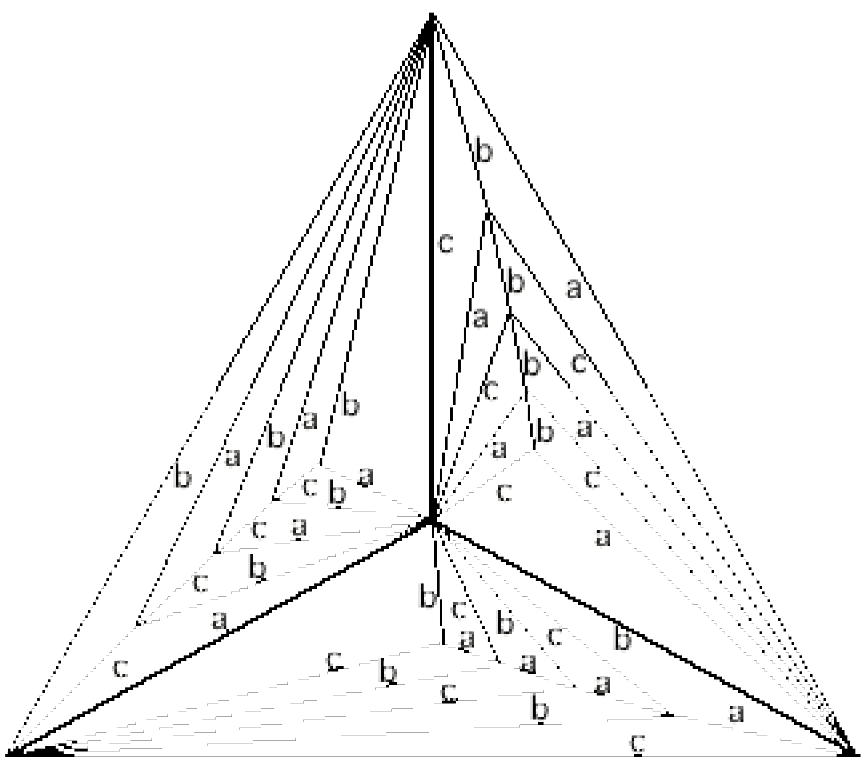


Figure 1. A triangulated sphere, represented by Schlegel diagram of a triangle-faced polyhedron. The meaning of the labels and the heavy lines is explained in the text.

up to the names of the labels the only possible labeling is the one shown in Figure 1. Recalling a result that goes back to Euclid –– namely, that the sum of face angles a any one vertex of a convex polyhedron is less than 360° — we see that P cannot be realized by a convex polyhedron equifacetted with a scalene triangle since the sum of face angles at the central vertex would be 5(□ + □ + □) = 5·180° > 360°. The same reasoning shows that, in fact, no stack polyhedron of any height of stack ≥ 1 can be convexly realized with a scalene protoface, or with an equilateral protoface. Similar (but slightly longer) arguments show that, at least for stack heights ≥ 4, no stack polyhedron has a monohedral convex realization with an isosceles prototile. Thus, stack polyhedra (of height ≥ 4) provide one strengthening of Malkevitch's result, since they are not monohedrally realizable by a convex polyhedron with *any* protoface.

However, our aim is to strengthen this more — by excluding the possibility of realization of P by any *acoptic* equifacetted polyhedron Q. For this purpose Euclid's theorem is not sufficient, since non-convex polyhedra can have arbitrarily large sums of the face angles at a vertex. The tool we can use instead are dihedral angles. Instead, we consider the monohedral acoptic polyhedron Q that supposedly realizes P as built up by stacking tetrahedra, and adding the dihedral angles they have at common edges. Clearly, at no edge of Q can these angles add up to more than 360°.

In case of a scalene protoface the situation is again governed by the labeling in Figure 1. If the dihedral angles at the edges a, b, c of a tetrahedron monohedral with the same scalene triangle as Q are □a, □b, □c, then the sums of the dihedral angles at the heavily drawn edges in Figure 1 are 6□a, 6□b, 6□c. Since at least one of the angles □a, □b, □c is at least as large as the angles of the regular tetrahedron (that is, 70.52878...°), at least one of these sums exceeds 360°.

Considering now the case of an isosceles protoface, with sides labeled a, a, b, there are just two essentially different possibilities to be considered. They depend on whether the vertical heavily drawn edge is labeled a or b; see Figures 2 and 3, where it is assumed that the outer triangle has the bottom edge labeled b. In both cases, each of the labels x, y, u, v, w, z could be either a or b, but the labeling of the other edges is completely determined. We shall again consider the polyhedron Q as built up from tetrahedra, but now these constituent tetrahedra can be of two different shapes. These tetrahedra T_1 and T_2 are illustrated in Figure 4, which also provides the coordinates of their vertices that are used in the calculations.

In order to prove the theorem, we have to show that regardless of the sizes of a and b and of the labeling proposed, the monohedral polyhedron cannot be acoptic. From the coordinates in Figure 4, it is easy to calculate that $b = 2p$ for both tetrahedra, and that $a^2 = 2p^2 + 4q^2$ for T_1, and $a^2 = 4p^2/3 + r^2$ for T_2. We shall denote by $\square_a$ and $\square_b$ the dihedral angles of the monohedral tetrahedron T_1

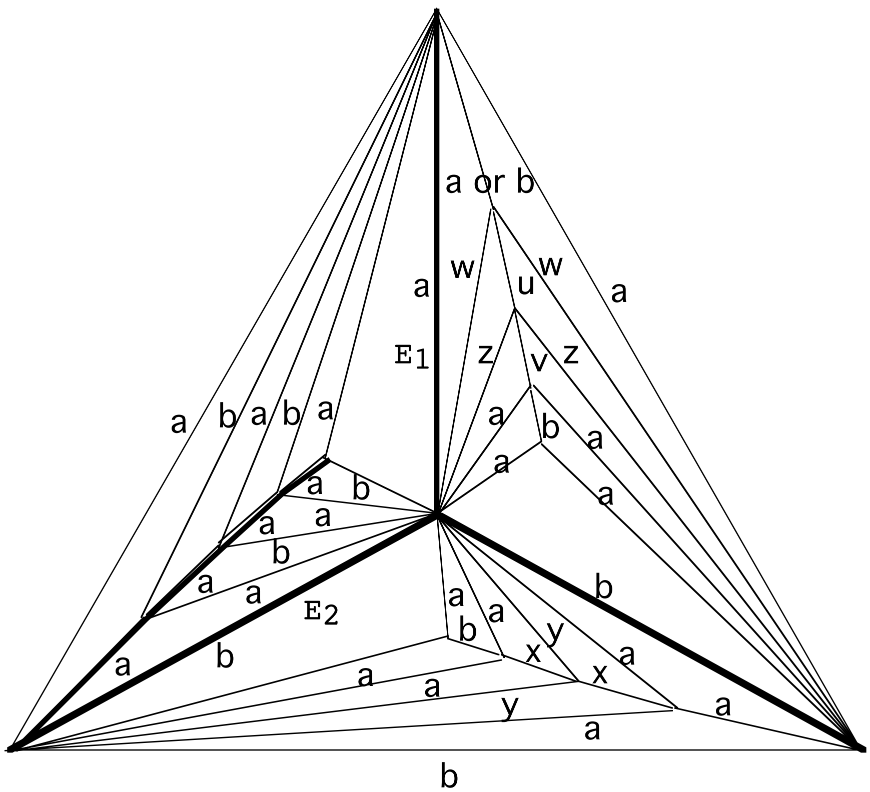


Figure 2. The labeling scheme of a polyhedron Q_1.

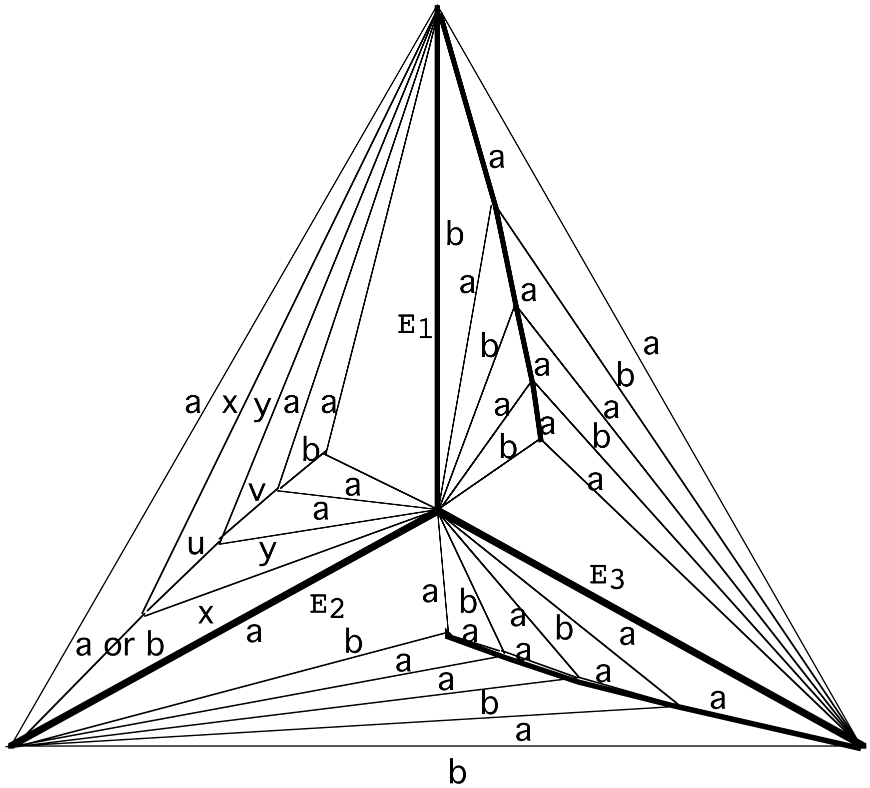


Figure 3.The labeling scheme of a polyhedron Q_2.

formed at its a and b edges respectively, and by $\square_a^*$ and $\square_b^*$ the dihedral angles along the a and b edges of T_2. (It should be noted that T_2 is *not* monohedral.) With a little trigonometry, and denoting by h is the altitude of the protoface, so that $h^2 = a^2 - p^2$, we find that

$$\cos \square_a = p^2/(a^2 - p^2) = p^2/h^2,$$

$$\cos \square_b = (a^2 - 3p^2)/(a^2 - p^2) = (h^2 - 2p^2)/h^2.$$

Similarly,

$$\cos \square_a^* = (a^2 - 2p^2)/2(a^2 - p^2) = (h^2 - p^2)/2h^2,$$

$$\cos \square_b^* = p/\sqrt{3(a^2 - p^2)} = p/\sqrt{3}\, h.$$

Agreeing to standardize the size of the protofaces so that $h = 1$, these expressions simplify to

$$\cos \square_a = p^2 \qquad \cos \square_b = 1 - 2p^2$$

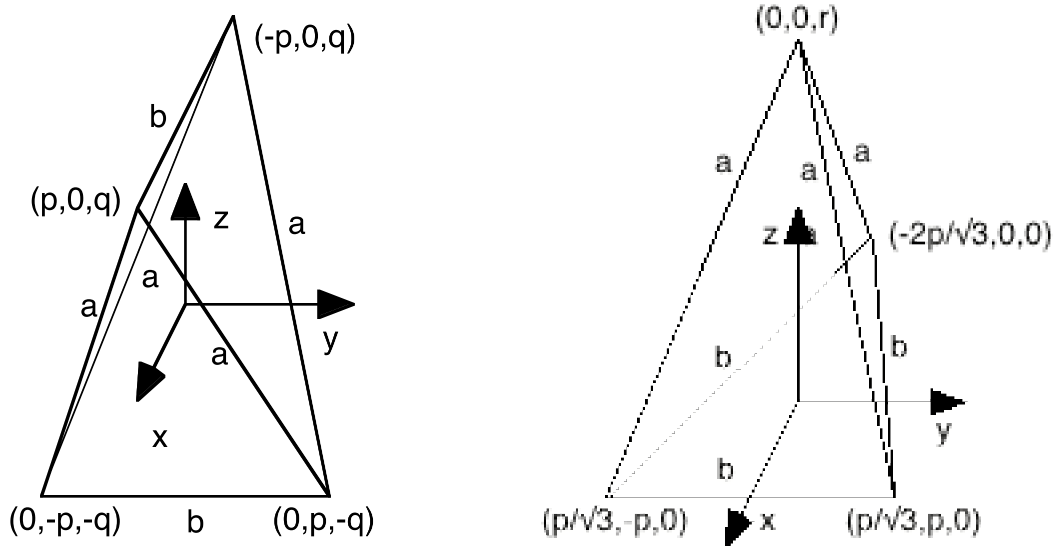
$$\cos \square_a^* = (1 - p^2)/2 \qquad \cos \square_b^* = p/\sqrt{3}\,,$$

where p can be any number such that $0 < p < 1$.

Now we are ready to give estimates of the sums of dihedral angles for any candidate Q.

We first note that the sum of the dihedral angles at edge E_1 of Q_1 is at least $\square_1 = 4\square_a + \square_a^* + \min\{\square_a, \square_a^*\}$ and at edge E_2 it is at least $\square_2 = 2\square_b + 2\square_b^* + 2 \min\{\square_b, \square_b^*\}$. As is visible from the plot of the function $\square = \max\{\square_1, \square_2\}$ for $0 < p < 1$ in Figure 5(a), the minimum of $\square$ is well over 360°. Therefore Q_1 cannot be acoptic.

Similarly, we see that the sum of the dihedral angles at edge E_1 of Q_2 is at least $\square_1^* = 4\square_b + 2 \min\{\square_b, \square_b^*\}$ and at edge E_3 it is at least $\square_2^* = 6\square_a$. As the plot of the function $\square^* = \max\{\square_1^*, \square_2^*\}$ for



T_1 T_2

Figure 4. The two tetrahedra that appear in the proof of the Theorem.

$0 < p < 1$ in Figure 5(b) shows, the minimum of $\square^*$ is also well over 360°. Therefore Q_2 cannot be acoptic, and hence the Theorem is completely proved.

3. Remarks and problems.

(i) It is somewhat surprising that the proof of the Theorem requires such lengthy and detailed calculations. It would appear very likely that the great majority of polyhedra with only one kind of faces (be they triangles, or quadrangles, or pentagons) are not isomorphic to monohedral acoptic polyhedra. But no good tools seem to be available to establish this for any extended family of polyhedra. Naturally, stacked polyhedra and other similarly constructed triangle-faced polyhedra can be shown as nonrealizable by calculations similar to the ones given above.

(ii) The main topic of the present paper (and of [2]) can be generalized to arbitrary convex polyhedra in the following form: For which given convex polyhedron P, does there exist an acoptic (or a

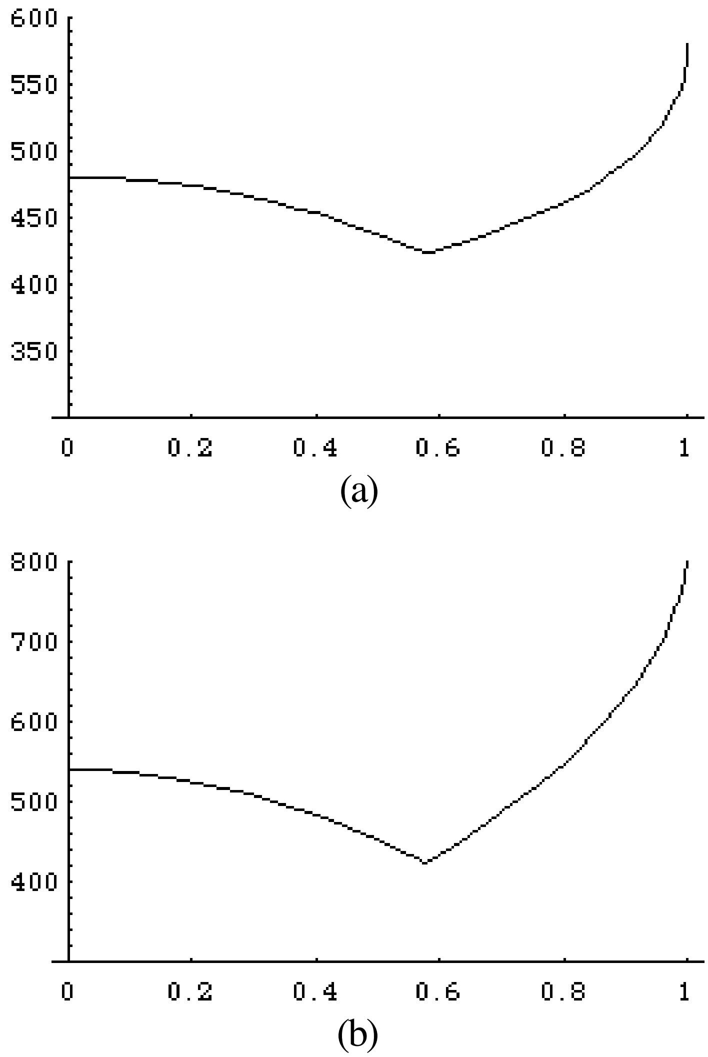


Figure 6. The plots of the functions (a) □(p) and (b) □*(p) obtained using Mathematica™ software.

convex) polyhedron Q which is isomorphic to P and is such that for each k = 3, 4, ..., all k-sided faces of Q are congruent ? For example, if P is a polyhedron obtained from the cube by "stacking" on each face a number of "prisms" as indicated for one face in Figure 6, can one show that such a polyhedron does not admit an isomorphic acoptic polyhedron Q with the property that all the quadrangles of Q are congruent, as are all its triangles ?

(iii) A different direction of investigation may treat the topic of this paper with a more positive attitude: What is the least $t = t(f)$ such that every triangle-faced convex polyhedron with f faces is isomorphic to an acoptic polyhedron with at most t different kinds of triangles as faces ? It may be conjectured that there is a constant $c > 0$ such that $t(f) \geq c f$ for sufficiently large f. Similar questions can be asked in the more general context mentioned in (ii) above, where also the distinction may be made according to whether non-convex faces are admitted or not admitted.

References.

[1] B. Grünbaum, Convex Polytopes. Wiley, London 1967.

[2] J. Malkevitch, Convex isosceles triangle polyhedra. Geombinatorics 10(2001), 122 - 132.

[3] G. Ziegler, Lectures on Polytopes. Springer, New York 1995.

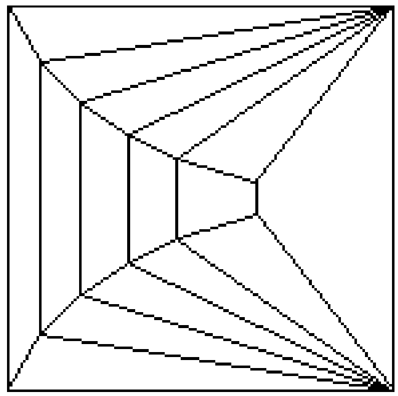


Figure 6. A proposed modification, to be applied to each face of a cube, as discussed in Remark (ii).