Discrete Geometry: In Honor of W. Kuperberg's 60th Birthday
Monographs and Textbooks in Pure and Applied Mathematics, vol. 253.
Marcel Dekker, New York, 2003. Pp. 331 – 350.
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"New" uniform polyhedra

Dedicated to Wlodek Kuperberg on his sixtieth birthday

Abstract. Definitions of polygons and polyhedra, more general than the one traditionally accepted, allow the construction of "new" uniform polyhedra. Although regular polyhedra, as well as some other classes of polyhedra, have recently been discussed from this point of view, the uniform polyhedra seem not to have been considered till now.

1. Introduction. By their nature, facts do not change. However, our interpretation of facts changes quite often, frequently due to changes in definitions. *Uniform polyhedra* (also called Archimedean by some — but we shall come back to this later), that is, polyhedra with regular polygons as faces, and with all vertices in a single orbit under symmetries of the polyhedron, have been studied for a long time. The fact that the family of *convex* uniform polyhedra consists — besides the regular polyhedra — of the infinite families of prisms and antiprisms together with thirteen individual polyhedra, has been established countless times. In contrast, the enumeration of *all* uniform polyhedra, convex and nonconvex, has been carried out only gradually, and much more recently. Only in 1953 was the complete list published [1], without a claim of completeness. In fact, that enumeration was proved to be complete in [15] and [14]; a different approach to the enumeration and a proof of completeness is reported to be contained in [16] — unfortunately, I have not had the opportunity to see this work, and probably would not have been able to overcome the language barrier in any case. Illustrations and data can be found in [1], [10] and [17]. However, these "facts" should be replaced by new ones as soon as more inclusive definitions are accepted for regular polygons, for polyhedra and for their symmetries.

There are many reasons for generalizing the traditional definitions. The main motivations are the wish to avoid needless restrictions, and to introduce consistency among the concepts and their applications — which is lacking in the traditional literature. Detailed critiques of the traditions can be found in [6], [7], [8] and [9], among others. Here I shall not repeat these arguments; instead, I shall only briefly describe what I believe are much more appropriate concepts. After that I shall present an account, as complete as I could make it within bounds for this article and with available time and energy, of the "new" uniform polyhedra.

2. Polygons and polyhedra. An n-*gon,* for some $n \geq 3$, is a cyclically ordered sequence of arbitrary points labeled $V_1, V_2, \ldots, V_n$, (called *vertices*), together with the segments E_i determined by pairs of vertices V_i, V_{i+1} adjacent in the cyclic order (the *edges*); each edge is *incident* with, and only with, the two vertices that determine it. If the value of n (the *size* of the n-gon) is not relevant, we speak of a *polygon.* Here we restrict attention to planar polygons, that is polygons all vertices of which are coplanar. A polygon is *regular* if each of its flags can be mapped onto any other flag by a *symmetry*. (A *flag* is a pair consisting of a vertex and an edge incident with it.)

All this should sound familiar — countless publications give these definitions or equivalent ones. However, most of them — in particular, almost all those devoted to the study of polyhedra — interpret (tacitly or explicitly) the definition of polygons as including the qualifier "distinct" when referring to "arbitrary points". Despite appearances, this is not a minor difference. To begin with, the unrestricted definition includes the possibility of two or more vertices, adjacent of not, to be situated at the same point of the plane. These vertices are still *distinguished* by their labels, and each is *incident* with just the edges specified in the definition. Also, edges can be of zero length, collinear, overlapping, or coinciding in pairs or larger sets, passing through vertices not incident with them, and intersecting at triple or multiple points. In contrast, each (labeled) vertex of a polygon appears only once in the cyclic sequence describing the polygon; in other words, the polygon does not "revisit" any *vertex*, although it may return to the *point* representing a vertex. These possibilities require that "symmetry" be defined in a way that meaningfully accounts for them. To achieve this, by *symmetry* we understand a *pair* consisting of a *permutation* of the vertices that preserves incidences and adjacencies, and a *compatible* isometric map of the polygon onto itself. An illustration is given in Figure 1.

For *regular* polygons this understanding of symmetries means that for every n and d with $0 \leq d < n$ there is a regular polygon that can be denoted $\{n/d\}$ and obtained by the following construction. Start with a fixed circle, and a point chosen as vertex V_1. Locate vertex V_2 on the circle, at arc distance $2\pi d/n$ from V_1 in the positive orientation of the circle, and continue analogously for n steps. Clearly, the n vertices will be represented by distinct points if and only if n and d are relatively prime; otherwise some points will represent several distinct vertices. It is also clear that $\{n/d\}$ and $\{n/e\}$ with $d + e = n$ differ only by the orientation; since this is not important in the present context, we may restrict attention to $0 \leq d \leq n/2$. In fact, since $\{n/0\}$ is the *trivial* regular polygon, with all vertices at the same point, and since $\{n/d\}$ with $d = n/2$ is a polygon with two sets of n/2 vertices, each set represented by the one point, hence not suitable for the production of polyhedra interesting in the present context, we may assume that $0 < d < n/2$.

A *polyhedron* is best described as being a geometric realization of an underlying combinatorial object which we call an "abstract polyhedron". An *abstract polyhedron* is a family of objects called *vertices*, *edges*, and *faces*, some pairs of which are *incident*, subject to the conditions which we state here informally (formal statements appear in [9]).

Each edge is incident with two distinct vertices and two distinct edges. If two edges are incident with the same two vertices [faces], then these edges are incident with four distinct faces [vertices]. For each *flag* (triplet of mutually incident vertex, edge, face) there is precisely one other flag with the same vertex and face. Each face is a cyclically ordered sequence of vertices and edges, and analogously the faces and edges incident with any vertex form a cyclically ordered sequence (the *vertex star* of that vertex). Finally, any two faces [vertices] are connected by a chain of mutually incident faces [vertices] and edges.

A (geometric) *polyhedron* is obtained by a mapping of an abstract polyhedron into 3-space in such a way that vertices are mapped to points, edge to segments (possibly of zero length) and faces to polygons. Two polyhedra are *combinatorially equivalent*, or of the same *combinatorial type*, if they have the same underlying abstract polyhedron. As in the case of polygons, a *symmetry* of a polyhedron is a pair consisting of an incidence-preserving automorphism of the underlying abstract polyhedron together with a compatible isometric map of the polyhedron onto itself. With this understanding of symmetries, the definition of *uniform polyhedra* given above remains valid. An illustration of these concepts is provided by the two combinatorially equivalent uniform polyhedra in Figure 2; one is a traditional prism, the other is "new".

It is customary to designate uniform polyhedra by a symbol of type (p . q . r . ...), which specifies the cyclic sequence of the sizes of faces surrounding one (hence every) vertex of the polyhedron. Among the possible choices, the lexicographically first is usually selected.

A refinement of this notation takes into account the orientation of the faces incident to a vertex, with respect to the centroid O of the polyhedron. All faces of such a cycle have one side selected as the "outer" one, the selection being such that the "outer" sides of adjacent faces agree. If the "outer" side of a face is not visible from O its size receives a + sign in the symbol (but this sign is usually omitted); if the "outer" side is visible from O, its size receives a sign – ; if the plane of the face contains O, the sign is ±.

3. Generating "new" uniform polyhedra. We turn now to a description of the various methods for the construction of "new" uniform polyhedra that have been found so far.

1. *Vertex-doubling* replaces each vertex by two, one red and one green. For each face, we follow around it, but connect by edges only vertices that differ in color. Hence vertex-doubling doubles the number of vertices and edges; it also doubles the size of odd-sized faces, and doubles the number of even-sized faces without change in their size. As is easily seen, a new polyhedron results if and only if at least one face of the starting polyhedron has odd size; if the starting polyhedron is uniform, so is the one obtained by vertex-doubling. For example, the polyhedron (6/2.4.4) in Figure 2(b) arises by vertex-doubling of the prism (3.4.4). Similarly, (3.6.6) leads to (6/2.6.6), as illustrated in Figure 3. If all faces of a polyhedron P have even sizes then vertex-doubling produces two separate polyhedra, each congruent with P.

2. *Face-doubling* replaces each face by one red and one green face, with edges joining only faces of different colors; hence the number of edges is also doubled. Face-doubling doubles the valence of odd-valent vertices, replaces even-valent vertices by two vertices each. Face-doubling results in a polyhedron if and only if the starting polyhedron has at least one odd-valent vertex. Since all vertices of a uniform polyhedron have the same valence, face-doubling is applicable only to uniform polyhedra of odd valence; in such cases, it produces another uniform polyhedron with the same number of vertices. For example, the 3-valent (3 . 6 . 6) leads to the 6-valent (3 . ***6*** . 6 . ***3*** . 6 . ***6***); here the different fonts are substituting for different colors. Notice that in the latter polyhedron the cycle of faces incident with a vertex winds *twice* around the vertex. Hence it is reasonable to say that its *vertex rotation* is 2. Similar explanations hold for the vertex rotation data for other polyhedra.

It should be observed that if a uniform polyhedron is odd-valent and has some faces of odd size, then both vertex-doubling and face-doubling are applicable; the result of carrying out both constructions is independent of order, and produces an even-valent uniform polyhedron with all faces even-sized. In the example of (3 . 6 . 6), the resulting polyhedron is (6/2 . ***6*** . 6 . ***6/2*** . 6 . ***6***), with 24 six-valent vertices, 72 edges, eight faces of type {6/2} and 16 faces of type {6}.

3. *Deleting one transitivity class of faces* of a uniform polyhedron, accompanied by the replacement of each remaining face by one red and one green copies, leads in many cases to one or two new uniform polyhedra. To achieve this, we join the different-colored faces at the edges that were incident to the deleted faces, and we join the faces at their other edges

while insisting that all the joining edges either be incident with faces of the same color, or else that all be incident with faces of different colors. It may happen that these two procedures yield congruent polyhedra — for example, if the triangles are deleted from (3 . 4 . 4 . 4) — but for (3 . 8 . 8) with the triangles deleted the two resulting uniform polyhedra (8 . ***8*** . 8 . ***8***) and (8 . 8 . ***8*** . ***8***) are distinct. The precise conditions under which the two choices result in isomorphic polyhedra remain to be determined.

4. *Slitting along one transitivity class of edges* (that is, deleting some or all edges of this class), accompanied by duplication of all faces, with no new vertices. Call one face of each pair red, the other green. In place of each edge that was slit, two edges are introduced. Each is incident with one red and one green face, but the incidences can happen in one of two ways: the faces either overlap, or else they are on opposite sides of the slit. This construction is applicable whenever there is a perfect matching in the graph of vertices and edges of the starting uniform polyhedron, such that the subgroup of the symmetry group which maps the matching onto itself leaves all vertices in one transitivity class.

For example (see Figure 5), on a cube make two parallel slits on the front face, and two parallel slits on the back face. One possibility is that all four slits are parallel (Figure 5a), another that not all are parallel (Figure 5b). Both variants of incidences at slits work in each case, for a total of four distinct polyhedra.

5. *Double covers of nonorientable uniform polyhedra.* The orientable double cover of each nonorientable uniform polyhedron is itself a uniform polyhedron. An easy way to visualize this is to begin with the old procedure for deciding that a surface is nonorientable: start painting and continue till all reachable parts of the surface have been painted; if both sides of the surface are covered, the surface is nonorientable. Now, assume you use latex or oil paint, let it dry, and then make the original surface disappear. The double cover consisting of the paint is left over. This construction clearly doubles the numbers of faces, edges and vertices of the polyhedron, but leaves its (unsigned) symbol unchanged. The simplest example is the nonorientable heptahedron (3 . ±4 . -3 . ±4), also known [17] as the tetrahemihexahedron. Its orientable double cover is combinatorially equivalent to the cuboctahedron (3 . 4 . 3 . 4).

6. *Including additional polygons as faces.* The vertices of many traditional uniform polyhedra determine regular polygons that are not faces of the polyhedron. In some instances, one can include such polygons, doubled up if necessary, and redefine adjacencies so as to obtain new uniform polyhedra. For example, introducing in the regular octahedron

(3 . 3 . 3 . 3) pairs of equatorial squares leads to a uniform polyhedron (3 . ±4 . 3 . ±4 . 3 . ±4 . 3 . ±4). Similarly, introducing the twelve regular pentagons determined by the neighbors of each vertex of the regular icosahedron (3 . 3 . 3 . 3 . 3), we obtain two uniform polyhedra, one with symbol (3 . 5 . 3 . 5 . 3 . 5 . 3 . 5 . 3 . 5) and the other with symbol (3 . -5 . 3 . -5 . 3 . -5 . 3 . -5 . 3 . -5). Additional examples of this kind are given in Table 1.

7. *Special constructions*. There are a few additional uniform polyhedra I have found, which appear not to fit any systematic construction method. Naturally, it may well be that I just do not see the possibly larger scope involved. In any case, here they are.

(i) The sixty diagonals of the faces of the regular dodecahedron can be organized into thirty squares, and also into twelve regular pentagons, see Figure 6. Double-up the pentagons, and color one of each pair red, the other green. One pair of opposite sides of each square, parallel to the edge of the dodecahedron which is "between" them, is colored red, the other pair green. Pentagons are adjacent only to squares, along edges of squares of their own color. This yields a uniform polyhedron with symbol (4 . 5 . 4 . -5 . 4 . 5 . 4 . -5 . 4 . 5 . 4 . -5). It is orientable, with "density" 5 and is topologically of genus 24.

(ii) Starting with the regular Keplerian polyhedron {5/2, 5}, we truncate its "points". The results of different levels of truncation are illustrated in parts (a) and (b) of Figure 7. If the truncation proceeds to the end, so that only the dodecahedral "core" is left (diagram (c) in Figure 7), a uniform polyhedron (5 . 10/2 . 10/2) is obtained. Notice that it has sixty vertices, three of which are at each vertex of the dodecahedron. Each pentagon of the dodecahedron represents both a pentagonal face of this uniform polyhedron and a decagonal face of type {10/2}. Similarly, from the other Keplerian polyhedron {5/2, 3}, we may obtain another uniform polyhedron with sixty vertices and with symbol (3 . 10/2 . 10/2).

(iii) Figure 8 shows an analogous approximation to a uniform polyhedron (4 . 10 . 10/2), obtained by replacing the pentagons in (3 . 4 . 5 . 4) by decagons of type {10/2}, deleting the triangles, and introducing the decagons that surround the pentagons in (3 . 4 . 5 . 4). Another "modification and replacement" produces the uniform polyhedron (6/2 . 10/2 . –10) from the convex uniform polyhedron (3 . 5 . 3 . 5). There probably exist other such possibilities.

(iv) Skilling [14] described a "polyhedron" that is uniform according to the criteria imposed in [1] and accepted in his paper, with one exception: some edges belong to four faces. However, as Skilling points out on p. 123, this object *is* a polyhedron if the exceptional edges are interpreted as two distinct edges which happen to be represented by the same segment although they are determined by different pairs of faces; in other words, it is a polyhedron in the sense adopted here. The vertex figure — as shown in [14] — appears in Figure 9; the caption gives the resolution of the "double edges" as given by Skilling, together with five other possible resolutions.

4. The main table. In Table 2 we give a detailed survey of the traditional uniform polyhedra, slightly modifying the presentation in [1] and [10], as well as a summary of results of the methods of construction of new polyhedra, following their general exposition given in Section 3.

From the constructions in Section 3, and from the table, it is clear that there are hundreds of "new" uniform polyhedra (besides the several infinite families). It would be very nice if it were possible to determine all such polyhedra, and to explicitly describe the parameters for each of them. I do not have the time, nor the energy, to undertake such an effort. Moreover, finding a venue for the publication of a detailed accounting of all uniform polyhedra may be quite challenging. In any case, the data presented here are the best I could do. While acknowledging its shortcomings (and even that a few errors may have crept in), I do hope that the presentation here will provoke some investigators to devote part of their energies to the study of "new" uniform polyhedra, and of other special classes of general polyhedra.

Here are some explanations of the entries in the Table 2:

± in the symbol indicates that the plane of the face passes through the center.

WS is the Wythoff symbol, which describes a method of generating the polyhedron; see Coxeter *et al.* [C].

VW stands for "vertex winding"; it is the winding number of the vertex figure.

d is the "density" as given by Coxeter et al., that is, the winding number (of the surface) with respect to the center. NO indicates that the polyhedron is nonorientable.

χ is the Euler characteristic, and g is the genus of the (orientable) map which is isomorphic to the polyhedron.

V, E, F denote the numbers of vertices, edges and faces, respectively.

T = equilateral triangle; S = square; P = regular pentagon; H = regular hexagon; O = regular octagon; D = regular decagon; Pg = pentagram {5/2}; Og = octagram {8/3}; Dg = decagram {10/3}.

C# is the number in the Coxeter et al. paper [1]; W# is the number in Wenninger's book [17]. B# is the number of the in-text figure, or, if preceded by a Roman numeral, in the appropriate plate, in [B]. Note that some polyhedra shown in [B] are only isogonal, not uniform. H# is the number in [10].

The "Notes" in the last column refer to the traditional uniform polyhedra. CP stands for "coplanar" and means that there are pairs of coplanar faces. NEP stands for "no edge pairs" and means that there are pairs of vertices incident with pairs of faces but not defining an edge. OR means that the polyhedron is orientable even though there is no density defined. O? means that I do not know whether the polyhedron is orientable or not.

N1 to N5 denote the various methods for obtaining "new" uniform polyhedra, as described in Section 3. The numbers shown in these columns indicates how many distinct uniform polyhedra are possible in each method of construction. 1* means that there are either one or more possibilities, depending on the parity of n; an exponent + means that there may be additional possibilities which I did not investigate, while ? means that I do not know the answer.

5. Remarks.

(i) It seems wasteful (and inappropriate) to use the terms "uniform" and "Archimedean" interchangeably. While Archimedes may well have discovered (at least some of) the various regular-faced polyhedra in which all vertices are surrounded by the same cycle of polygons — these *Archimedean polyhedra* are both conceptually and effectively not the same as the *uniform polyhedra*. In the latter, symmetries have to act transitively on the vertices — and this is *not* a consequence of the characteristic property of Archimedean polyhedra, nor is it reasonable to assume that Archimedes had any group-theoretic ideas. The distinction between "uniform" and "Archimedean" is clearly demonstrated by the "pseudorhombicuboctahedron" of J. C. P. Miller and V. G. Ashkinuze. An analogous distinction is necessary between the uniform *quasi-rhombicuboctahedron* [17, p. 132] and the nonuniform *pseudo-quasirhombicub-octahedron* discovered by Jones [11] in 1994. The existence of two concepts and two words makes it desirable and possible to pair them in a logical way.

(ii) A few additional uniform polyhedra are described in [9]. Their descriptions have not been repeated here in order to avoid excessive length.

(iii) Two other kinds of generalization are possible (see [5]). First, one may admit infinite polyhedra, provided they are discrete (that is, every compact set meets only a finite number of vertices, edges and faces). Such polyhedra have been studied by various authors, but except in very restricted situations the results are fragmentary. The most detailed account for a special class is that of Jones [12], but even this is incomplete. Second, the definition of "polygon" may be extended to include nonplanar ones, which may then be used as building blocks of polyhedra. However, the relevant literature is restricted to regular polyhedra of this kind (see [2], [3], [13]), with no study of the possibilities of uniform polyhedra with such "skew" faces. The work of Farris [4] was a start in this direction, but the intended development seems not to have been published.

(iv) Any traditional uniform polyhedron with all faces *congruent* regular polygons is a regular polyhedron. This is not the case for the "new" uniform polyhedra. For example, let a "slitting" operation be applied to the regular icosahedron along the matching indicated in Figure 10. Let the faces be doubled-up, to yield one red and one green face. Across the edges of the matching, let faces of different colors be adjacent, while along the other edges faces of the same color are adjacent. It is clear that this determines a uniform polyhedron of type (3 . 3 . 3 . 3 . 3 . 3 . 3 . 3 . 3 . 3) with all faces congruent equilateral triangles — but the twelve faces adjacent to edges of the matching are not in the same orbit under symmetries of the polyhedron as the other eight faces.

(v) From the constructions described above, one may get the impression that all "new" uniform polyhedra (except the regular ones) have pairs of coinciding faces. However, this is not always the case. For example, the polyhedron (4 . 10 . 10/2) described above and in Figure 8 has no such faces; other examples can be found as well.

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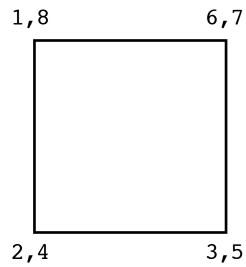
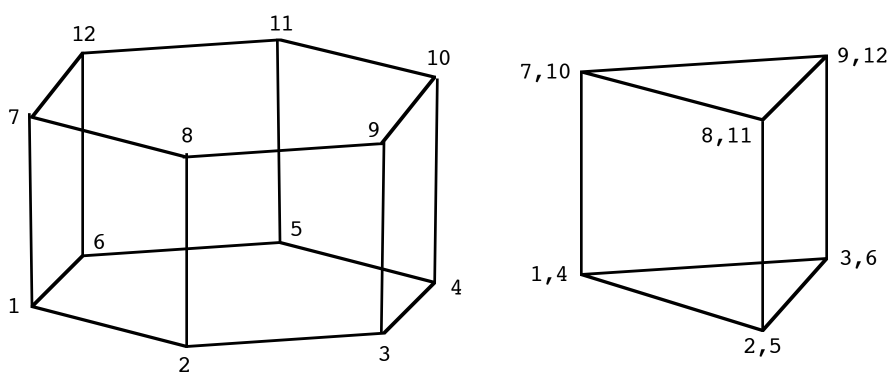
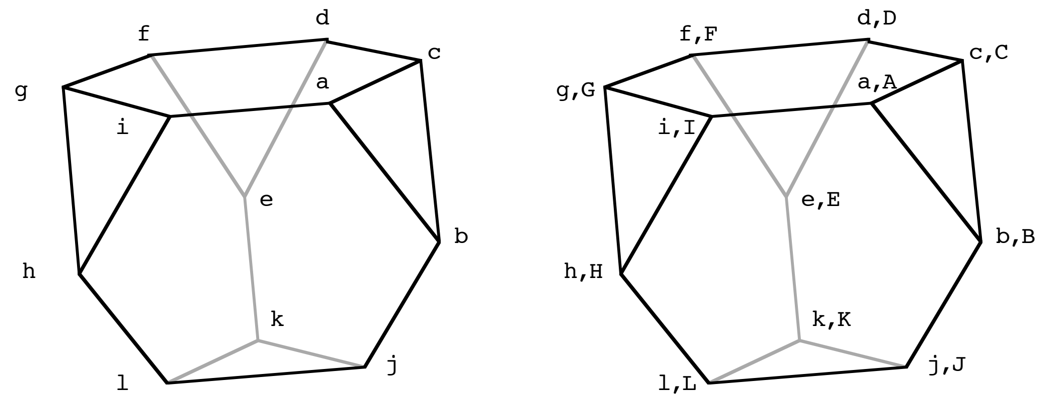


Figure 1. An octagon, with two edges of zero length, and three coinciding edges. The only symmetry (besides the identity) is the reflection in a vertical mirror, paired with the permutation (16)(25)(34)(78) of the vertices. No proper rotation is a symmetry of this octagon. Here, and in all diagrams, to avoid clutter vertices are labeled only by numerals.



(a) (b)

Figure 2. Two combinatorially equivalent uniform polyhedra. Both are prisms; the prisms in (a) has basis {6/1} while the one in (b) has basis {6/2}. The prism in (b) can be interpreted as arising by vertex-doubling (see Section 3) of the prism with triangular basis.



(a) (b)

Figure 3. (a) The uniform truncated tetrahedron (3 . 6 . 6), with the obvious faces such as [a b c a], [a i h l j a], ... Here and in the sequel, when listing the vertices of a face we repeat the first vertex in order to stress the cyclic nature of the symbol. (b) The uniform polyhedron (6/2 . 6 . 6), obtained from (3 . 6 . 6) by vertex-doubling. Instead of using colors, we distinguish doubled-up vertices by upper and lower case characters. Note that all the faces of the polyhedron in (b) are hexagons — four of type {6/2}, and eight of type {6} = {6/1}. The faces incident with the vertex A are: [A b C a B c A], [A c D f G i A] and [A i H l J b A].

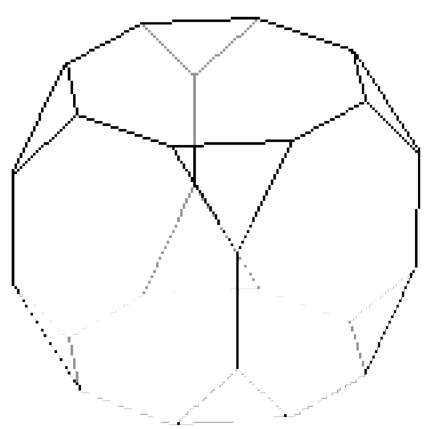
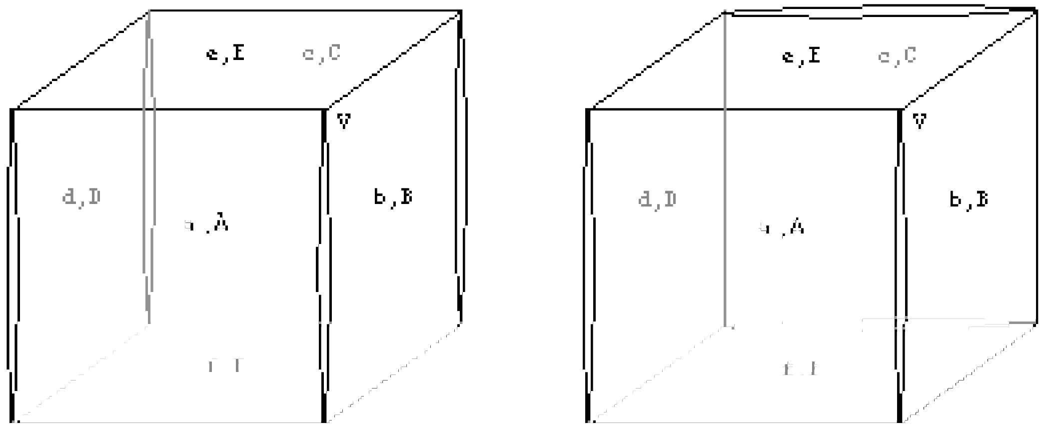


Figure 4. Deleting the triangles of the uniform truncated cube (3.8.8), and replacing each octagon by a pair of differently colored octagons, leads to two distinct uniform polyhedra, as explained in the text. Both have symbol (8.8.8.8), but are not of the same combinatorial type: in one there are two cycles of three faces around each "hole", while in the other there is only one cycle of length six.



(a) (b)

Figure 5. The two kinds of slits possible on a cube. The two edges at each slit are shown slightly curved, to simplify their visualization. The pairs of faces obtained by duplication are distinguished by lower and upper case characters. If we indicate each edge by the labels of the faces incident with it and put in parentheses, one of the "new" uniform polyhedra obtained in (a) can be described by the cycle of faces incident with vertex V as (a(aA)A(Ae)e(eB)B(Bb)b(bE)E(Ea)a), the other as (a(aB)B(Be)e(eA)A(Ab)b(bE)E(Ea)a). According to our conventions, the latter has the symbol (4 . 4 . 4 . 4 . 4 . 4) and vertex rotation 2, while the former has symbol (4 . -4 . -4 . -4 . 4 . 4) and vertex rotation 0. The alternatives in case (b) are the same, and so the resulting uniform polyhedra have the same symbols and the same vertex rotations. However, the polyhedra resulting in case (b) are distinct from the ones in case (a), since — among other distinctions — their symmetry groups are different.

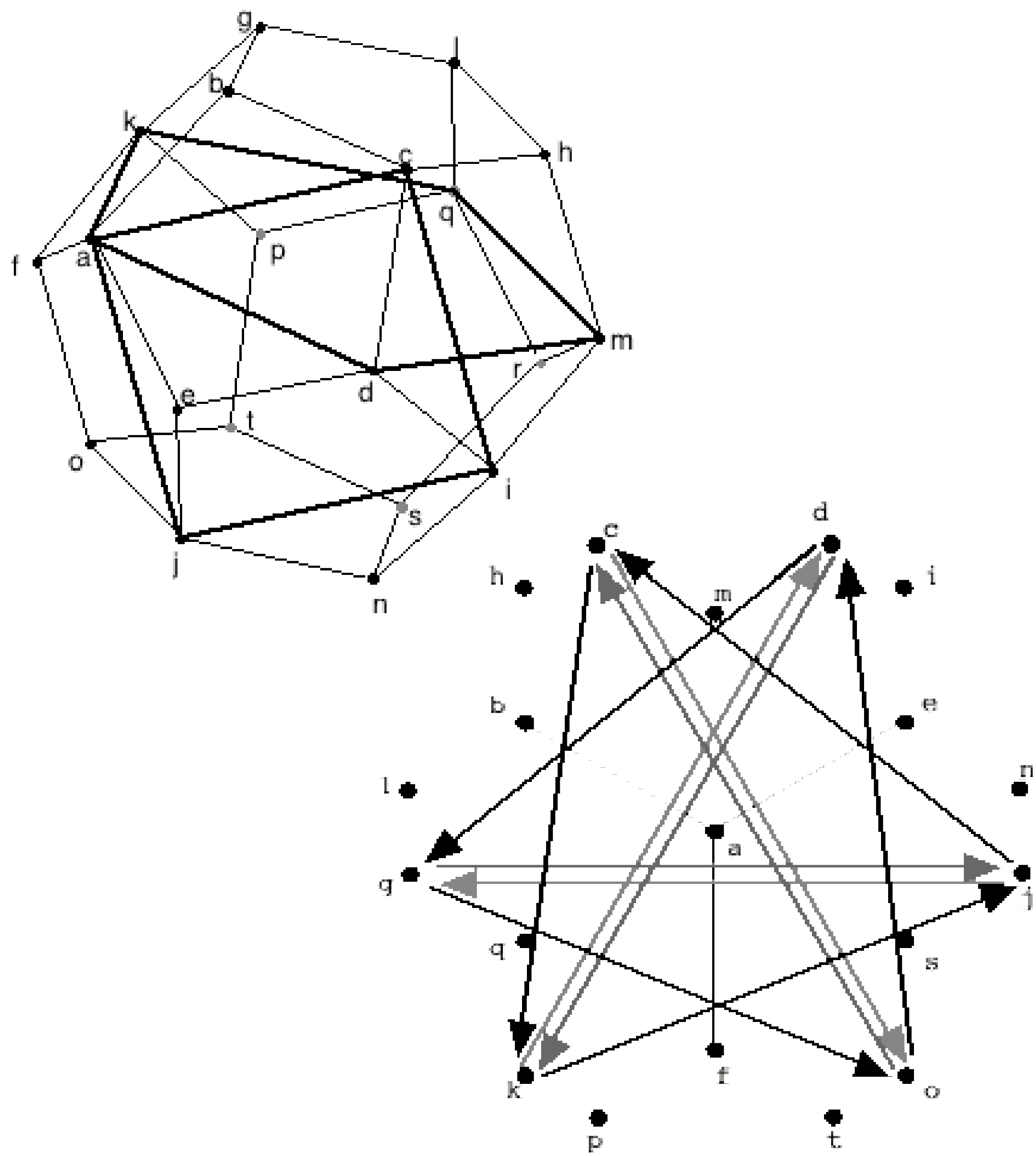
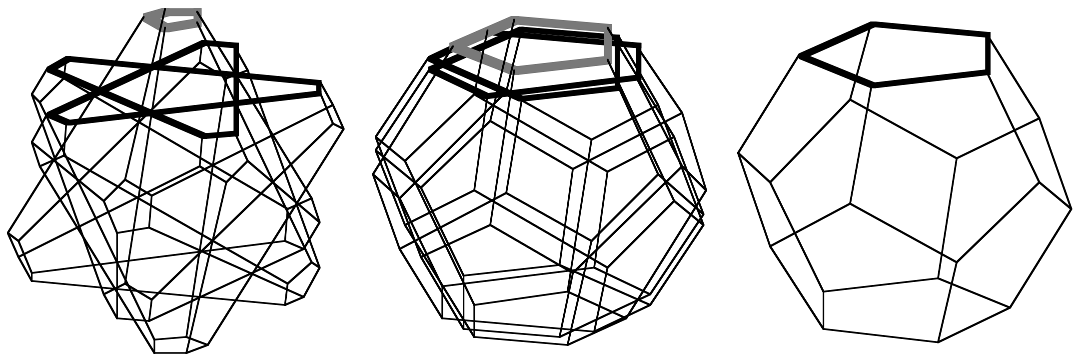


Figure 6. The construction of the polyhedron (4 . 5 . 4 . -5 . 4 . 5 . 4 . -5 . 4 . 5 . 4 . -5). On top is shown the regular dodecahedron that serves as the scaffold for the construction, together with one of the pentagonal faces of the new uniform polyhedron, and one of its square faces. The pentagon represents a pair of coinciding faces, one red the other green. The square has red edges [ac] and [ij], and green edges [ci] and [ja]. The lower diagram shows the cycle of faces that are incident with the vertex a. In this modification of the "vertex figure" used in [1] and [17], the view is along the line from vertex a towards the center of the polyhedron, and each directed line segment represents one of the faces.



(a) (b) (c)

Figure 7. Different levels of truncation of the regular polyhedron {5/2, 5}. In (a) and (b) one of the pentagonal faces and one of the decagonal faces are emphasized. In the full truncation (shown in (c)) the emphasized pentagon represents both faces; the resulting uniform polyhedron has symbol (5 . 10/2 . 10/2).

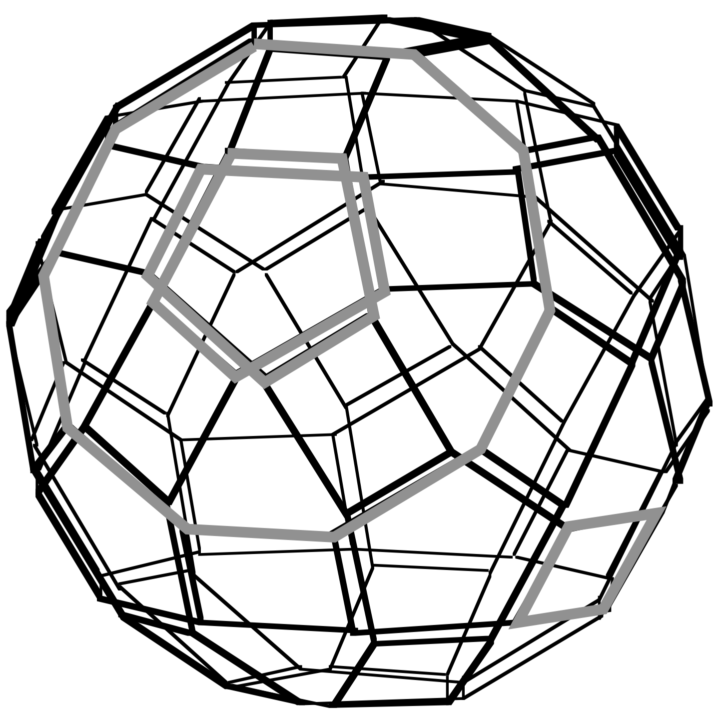


Figure 8. An approximation to the uniform polyhedron with symbol (4 . 10 . 10/2). One face of each kind is emphasized. The uniform polyhedron has vertices at the vertices of the convex uniform polyhedron (3 . 4 . 5 . 4).

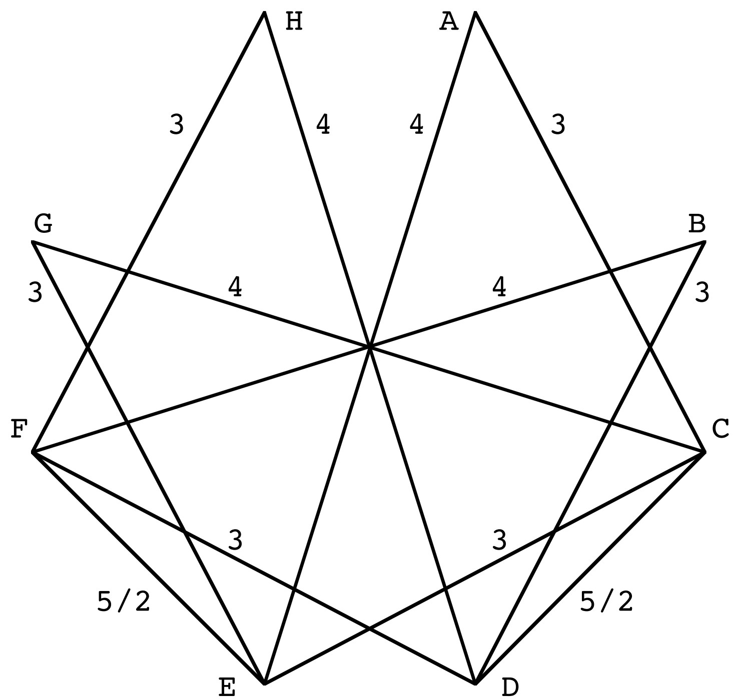


Figure 9. The vertex figure of Skilling's "great disnub dirhombidodecahedron", sketched from [14]. Since the edges leading to vertices C, D, E, F are incident with four faces, to make a uniform polyhedron each of these edges has to be split into two; this is possible, by having two pairs of faces each determining an edge. Skilling proposed to have the cycle of faces proceed as A E F B D F H D C G E C A. However, five other cycles seem as effective: A E F D B F H D C E G C A, A E C G E F H D F B D C A, A E C D B F H D F E G C A, A E G C E F H D B F D C A, and A E G C D F H D B F E C A, for a total of six uniform polyhedra. Additional uniform polyhedra can be obtained from these by applying the different constructions described in Section 3.

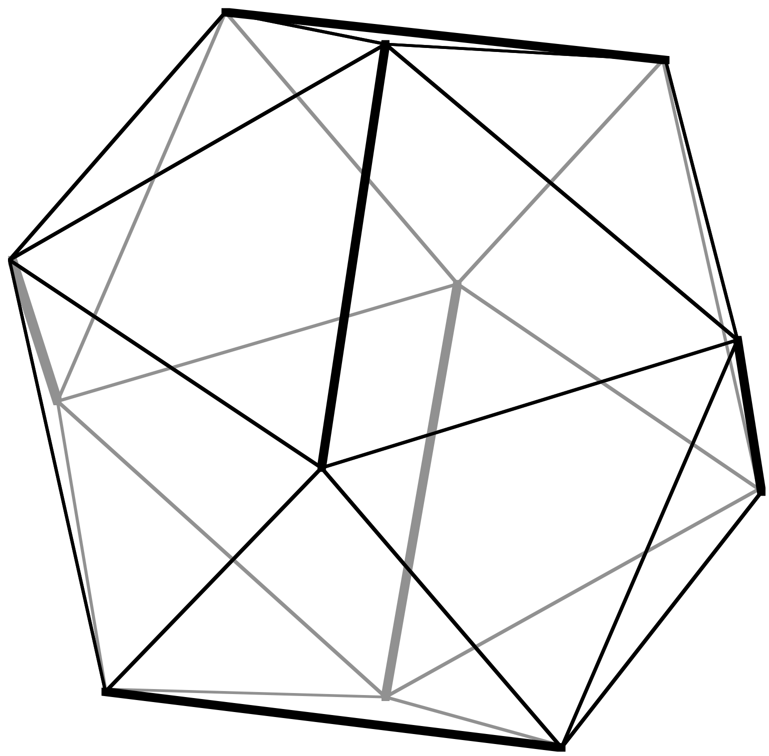


Figure 10. The matching in the graph of the regular icosahedron, used in the construction of the uniform but nonregular polyhedron having all faces congruent, described in Section 5.

Table 1. The following table lists the uniform polyhedra obtainable by introducing some of the additional regular polygons determined by the vertices and edges of certain traditional uniform polyhedra. In each case, pairs of coinciding (but distinct) polygons are introduced. Such polygons are indicated in the table by bold-faced numerals in italics. The list is certainly incomplete since it does not deal with possibilities of additional faces in "new" uniform polyhedra; probably there are omissions even with respect to traditional uniform polyhedra.

Starting uniform polyhedron	"New" uniform polyhedron
(3.4.3.4)	(3.***6***.4.***6***.3.***6***.4.***6***).
(3.±4.-3.±4)	(3.***3***.±4.***3***.-3.***3***.±4.***3***).
(3.4.4.4)	(3.***8***.4.***8***.4.***8***.4.***8***)
(-3.4.4.4)	(-3.***8/3***.4.***8/3***.4.***8/3***.4.***8/3***).
(3.4.5.4)	(3.1***10***.4.***10***.5.***10***.4.***10***)
(3.4.-5/2.4)	(3.***10/3***.4.***10/3***.-5.***10/3***.4.***10/3***).
(3.5.3.5) or (5.10.-5.-10)	(3.***10***.5.***10***.3.***10***.5.***10***)
(3.±6.-3.±6)	(3.***4***.±6.***4***.-3.***4***.±6.***4***).
(-3.6.5.6)	(-3.***10/3***.6.***10/3***.5.***10/3***.6.***10/3***).
(3.6.5/2.6)	(3.***10***.6.***10***.5/2.***10***.6.***10***)
(-3.8.4.8)	(-3.***4***.8.***4***.4.***4***.8.***4***).
(3.±10.-3.±10)	(3.***5***.±10.***5***.-3.***5***.±10.***5***).
(-3.10.5.10)	(-3.***4***.10.***4***.5.***4***.10.***4***)
(3.10.-5/2.10)	3.***6***.10.***6***.-5/2.***6***.10.***6***).
(3.5/2.3.5/2.3.5/2)	(3.***4***.5/2.***4***.3.***4***.5/2.***4***.3.***4***.5/2.***4***)
(3.8/3.4.8/3)	(3.***4***.8/3.***4***.4.***4***.8/3.***4***)
(4.5.4.5/2) or (4.6.-4.6)	(4.***6***.5.***6***.4.-***6***.5/2.-***6***)
(4.±6.-4.±6)	(4.***3***.±6.-***3***.-4.-***3***.±6.***3***)
(5.6.-5.-6)	(5.***5/2***.6.***5/2***.-5.***5/2***.-6.***5/2***)