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Parallelogram-faced isohedra with edges in mirror-planes

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Dedicated to Ludwig Danzer, on his 70th birthday

1. Introduction

Regular (or Platonic) polyhedra have been studied since antiquity, and many other kinds of polyhedra with various symmetry properties have been investigated since then. These include the traditional Archimedean polyhedra (regular-faced with congruent vertex figures), isohedra (polyhedra with faces all equivalent under symmetries of the polyhedron), isogonal polyhedra (all vertices of which are equivalent), uniform polyhedra (regular-faced isogonal polyhedra), Kepler–Poinsot regular polyhedra, and others. The study of rhombohedra (isohedra with rhombic faces) started with Kepler, and continued in more recent times. However, there has been no complete enumeration, or classification, of such polyhedra. In the present paper we shall make a beginning of such a systematic investigation. For reasons of time and space we restrict the consideration to polyhedra with the octahedral symmetry group, but admit any parallelograms as faces; the analogous but much more numerous polyhedra with icosahedral symmetry will be discussed elsewhere.

Another restriction imposed in the present investigation concerns the kinds of objects we shall accept as 'polyhedra'. As is quite generally accepted, we treat polyhedra as collections of planar polygons (parallelograms in the present case), such that edges are shared by two faces, faces incident with any one vertex form a single circuit, and the set of faces is strongly connected. We do not insist that distinct faces determine distinct planes, and we do not object to overlaps among faces. We also admit the possibility of several vertices being represented by the same point, as long as the circuits of faces incident with each are disjoint. (We shall comment on this permissiveness in the concluding section.) It seems that this class of polyhedra is wide enough to encompass all the traditional families mentioned above, without leading to some of the 'strange' possibilities described in [7,11] for polyhedra of more general kinds.

A final restriction on the polyhedra considered here is that every edge of the polyhedron is to be contained in a mirror, that is, a plane of reflective symmetry of the

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polyhedron. The effect of this restriction will be discussed in the last section; at the moment, it should suffice to note that all rhombohedra (except some affine images of the cube) considered in the literature have this 'edges-are-mirrors' property.

The result of our enumeration is:

Theorem. *There are precisely nine parallelogram-faced isohedra with octahedral symmetry, all edges of which are contained in mirrors.*

More precisely, two of the polyhedra have square faces, three have nonsquare rectangles, two have nonsquare rhombic faces and two have nonrhombic, nonrectangular parallelograms as face. All these polyhedra are described below, and graphic representations are shown in Fig. 1. While it is clear that the polyhedra we describe satisfy the imposed conditions, the more interesting part of the proof of the Theorem is the argument showing the completeness of the enumeration. The method appears to be applicable in a variety of other cases as well.

2. Proof of the Theorem

Möbius nets have been invented for the study of symmetries of polyhedra, and have been used, among other things, to describe all uniform polyhedra, as well as in the investigation of more general isogonal polyhedra. Somewhat unexpectedly, they seem not to have been applied so far to the investigation of isohedra. As we show here, they are very appropriate for this purpose as well.

The Möbius net of the octahedral symmetry group consists of the 48 triangles into which the planes of mirror symmetry of the regular octahedron divide a sphere centered at the center of the octahedron. Clearly, the central projection onto a sphere centered at the center of the polyhedron will map every face of a parallelogram-faced isohedron all edges of which are mirrors onto a union of Möbius triangles. Moreover, this union will be a spherically convex quadrangle consisting of two or more Möbius triangles. It is easy to obtain a list of such quadrangles; as it turns out, there are precisely the nine possibilities which are shown in Fig. 2.

The crucial next step is the following simple observation: If Q is a spherically convex quadrangle, and if C is the cone generated by Q, with apex at the center of the sphere, then there is a plane H which intersects C in a parallelogram; moreover, all such planes are parallel, and hence the parallelograms are mutually homothetic so that the parallelogram is unique up to size. The proof of this assertion reduces at once to showing that for every point inside an angle (pointed cone) in the plane there is a unique chord that has the point as its midpoint. Clearly, the center of the parallelogram will be on the line of intersection of the two planes determined by the diagonals of the spherical quadrangle. The precise shape of the parallelogram can be easily determined, either trigonometrically or using numerical calculations.

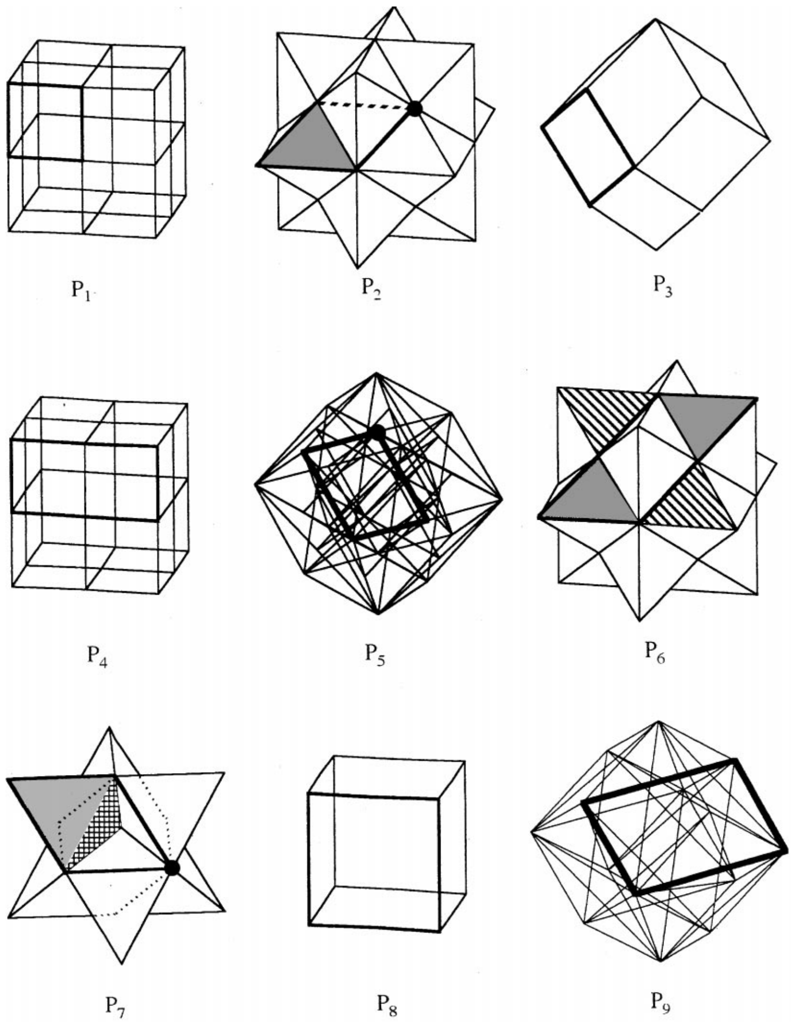

Fig. 1. The nine parallelogram-faced isohedra with octahedral symmetry and edges in planes of mirror symmetry. For clarity, one face of each polyhedron is emphasized by heavy edges. For polyhedra P_2, P_5 and P_7 points marked by solid dots (and their images under symmetries of the polyhedron) represent two distinct vertices each.

This observation implies that for every one of the nine quadrangles q_j in Fig. 2 there is a parallelogram whose central projection is the quadrangle in question. Reflections of this parallelogram in the planes of symmetry of the octahedron generate a finite family of parallelograms, a polyhedron P_j, which is one of the polyhedra we are enumerating. We now briefly describe each of them, and supply the following data: the symbol in square brackets, which is the valence cycle of each face; the numbers f, e, and v of faces, edges and vertices; the density d and the genus g of the polyhedron; and details about the shape of the faces.

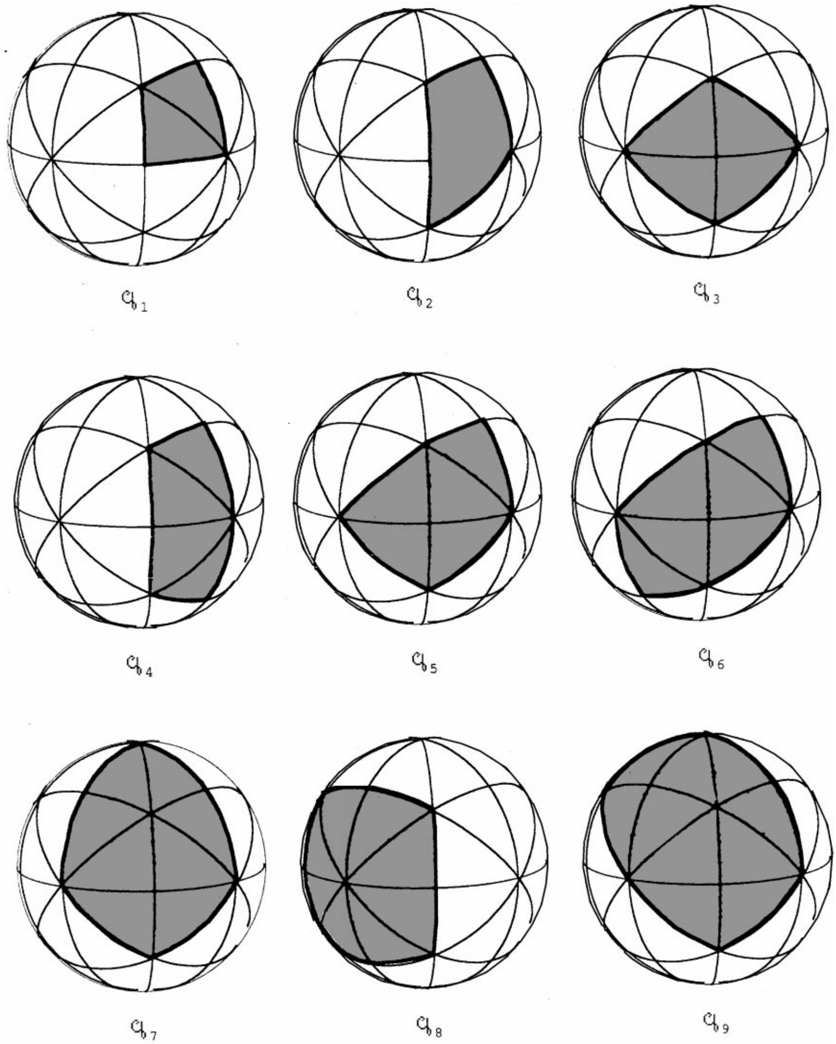

Fig. 2. The nine convex quadrangles which are unions of triangles in the octahedral Möbius net.

q_1: The polyhedron P_1 is obtained by subdividing each face of a cube into four squares. Symbol: $[3,4,4,4]$; $f=24$, $e=48$, $v=6+8+12=26$; $d=1$, $g=0$. The faces are squares.

q_2: The polyhedron P_2 has eight points each of which represents two distinct vertices. As a consequence, certain segments represent two edges of the polyhedron each. Symbol: $[4,\frac{6}{2},6,\frac{8}{3}]$; $f=48$; $e=96$, $v=6+8+8+12=34$; $d=3$, $g=8$. The faces are parallelograms with sides 2 and $\sqrt{3}$, and diagonals 2 and $\sqrt{11}$; the acute angles are 54.735610°.

q_3: This polyhedron P_3 is the rhombic dodecahedron. Symbol: $[3,4,3,4]$; $f=12$, $e=24$, $v=6+8=14$; $d=1$, $g=0$. Faces are rhombi, with ratio of diagonals $\sqrt{2}$ and acute angles 70.528780°.

q_4: The faces of P_4 are 2-by-1 rectangles, obtained by dividing into two each face of a cube, in the two possible ways. Hence quadruplets of faces are coplanar. It should be noted that the vertices at midpoints of the edges of the cube are *incident* with just four faces, although two more faces *contain* these points. Symbol: $[4,4,\frac{6}{2},\frac{6}{2}]$; $f=24$, $e=48$, $v=8+12=20$; $d=2$, $g=3$.

q_5: The faces of P_5 are rectangles, coplanar in pairs. Three vertices of the polyhedron project onto each of the octahedral vertices of the Möbius net; two of these vertices are represented by the same point. Symbol: $[4,4,\frac{6}{2},\frac{8}{3}]$; $f=48$, $e=96$, $v=6+6+6+8+12=38$; $d=5$, $g=6$. Faces are rectangles, with sides in ratio $\sqrt{3}/\sqrt{2}$.

q_6: The faces of P_6 are parallelograms which overlap in their central parts. Symbol: $[4,\frac{8}{3},4,\frac{8}{3}]$; $f=24$, $e=48, v=6+12=18$; $d=3$, $g=4$. Faces are parallelograms with sides 1 and $\sqrt{3}$, and diagonals $\sqrt{2}$ and $\sqrt{6}$; the acute angles are 54.735610°.

q_7: Polyhedron P_7 has rhombic faces, triplets of which are coplanar. Pairs of vertices are represented by the same point. Symbol: $[3,\frac{8}{3},4,\frac{8}{3}]$; $f=24$, $e=48$, $v=6+6+8=20$; $d=4$, $g=3$. Faces are rhombi, with ratio of diagonals $\sqrt{3}$ and acute angles 60°.

q_8: Polyhedron P_8 is the cube. Symbol: $[3,3,3,3]$; $f=6$, $e=12$, $v=8$; $d=1$, $g=0$.

q_9: The faces of P_9 are rectangles. Symbol: $[\frac{6}{2},\frac{6}{2},\frac{8}{3},\frac{8}{3}]$; $f=24$, $e=48$, $v=6+8=14$; $d=5$, $g=6$. The edges of the rectangles are in ratio $\sqrt{8}/\sqrt{3}$.

This completes the proof of the Theorem. □

3. Comments

(a) Three of the polyhedra listed above (P_2, P_5, P_7) have two or three vertices represented by the same point; this may be found objectionable by some people. In defense of nondiscrimination against such polyhedra one may note that polyhedra with coplanar faces have been accepted for a long time; see for example, Hess [12, p. 34], Brückner [1, p. 215; 2, pp. 154, 310], Coxeter et al. [5, Figs. 41, 80, 91, 92], or the corresponding Figs. 110, 115, 118, 119 in Wenninger [14]. Since the polar (reciprocal) polyhedra have coinciding vertices, it would be natural to expect that they have been as frequently accepted. However, this happened only on rare occasions (Hess [12] and Brückner [1,2] actually admit two vertices of the same face to be at the same point), and such polyhedra were never consistently investigated. In any case, it seems to us that disallowing the possibility of distinct vertices being represented by the same point is a needless and arbitrary restriction, without any intrinsic justification.

(b) The rectangle-faced polyhedra P_4, P_5 and P_9 seem to have been first described in Coxeter and Grünbaum [4]; P_1 appeared earlier, see for example, Unkelbach [13]. The absence of any mention of the rhombohedron P_7 in the literature is rather strange; the only explanation that comes to mind is that the representation of

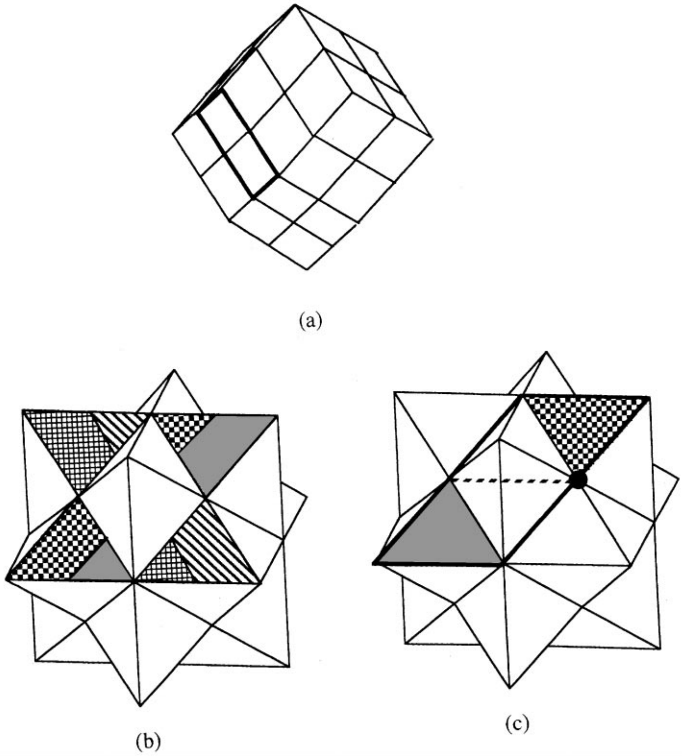

Fig. 3. (a) The 48-faced isohedron with parallelogrammatic faces, obtained by halving two copies of each face of the rhombic dodecahedron P_3. (b), (c) The two isohedra with 48 parallelogrammatic faces, obtained by dividing into two each face of the polyhedron P_6.

distinct vertices by the same point was responsible for the omission. The fact that P_2 and P_6 are apparently new is less surprising, since it seems that nobody ever looked for polyhedra with parallelogrammatic faces that are not rhombi. However, P_6 is a polyhedron even under more restrictive definitions than the ones used here.

(c) There are many other parallelogram-faced isohedra besides the ones listed above. However, their symmetry groups are not generated by the reflections in their edges. For example, in analogy to the derivation of P_4 from the cube P_8 by cutting in half two copies of each face, the rhombic dodecahedron P_3 yields an isohedron with 48 parallelograms as faces, as Fig. 3(a). Analogously, P_6 leads to the two distinct parallelogram-faced isohedra with 48 faces shown in Fig. 3(b) and (c). In all three cases, the two parallelograms that resulted from the splitting of one face of the original polyhedron are related by halfturns, and not by mirror symmetry. It may be conjectured that there are no additional parallelogram-faced isohedra in which each edge is either a mirror or else its midpoint is a center of a halfturn. It may also be noted that the faces of the polyhedron in Fig. 3(c) coincide as sets with the faces of P_2; however, their adjacencies are different, hence this is a new polyhedron.

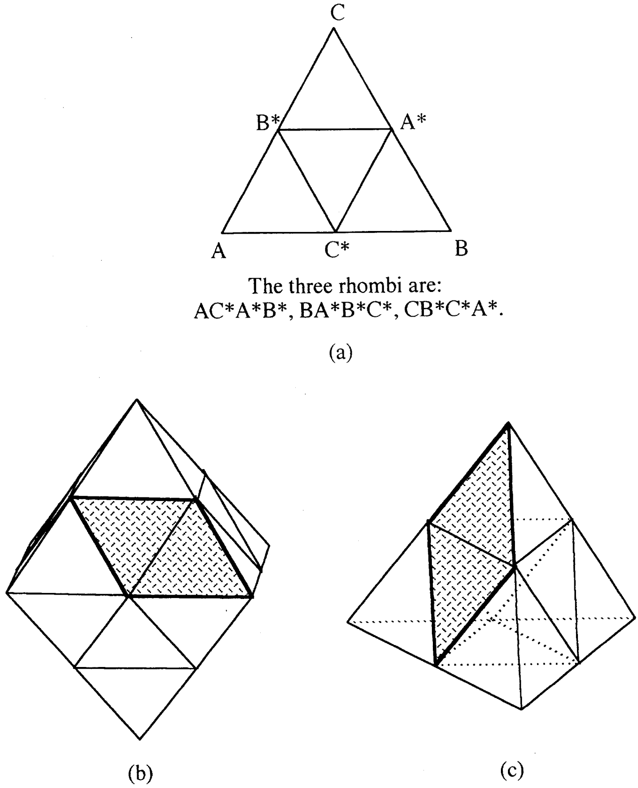

Fig. 4. (a) Three rhombi whose union is an equilateral triangle. (b), (c) Using the pattern from (a), rhombic isohedra with 24 or 12 faces can be generated from the regular octahedron and tetrahedron.

(d) There exist parallelogram-faced isohedra of still other kinds. In Fig. 4(a) is shown a set of three congruent rhombi which are related in such a way that their union covers an equilateral triangle. If every face of the regular octahedron is replaced by such a triplet of rhombi, the resulting nonorientable rhombic isohedron has 24 faces; see Fig. 4(b). Naturally, analogous replacements can be made starting from the regular tetrahedron and icosahedron. The former case is particularly interesting since it leads to a new type of isohedral rhombic dodecahedron, see Fig. 4(c). There are various other constructions of analogous kinds that lead to parallelogram-faced isohedra. A detailed account is in preparation.

(e) If we apply the splitting operation of (c) above to the polyhedron P_4, the resulting collection of 48 squares is another polyhedron, in which the eight vertices of the original cube are 6-valent (with a vertex figure of rotation number 2) while the other 36 vertices are 4-valent. The polyhedron looks exactly like P_1, but all faces and edges are now coinciding in pairs, as are all vertices except the ones of the original cube. We did not include this polyhedron, or other polyhedra constructed in a similar way, in the enumeration of the theorem because the concept of symmetry has to be modified for such polyhedra. It is not enough to note that the reflection in a plane is a symmetry; to specify its effect on the polyhedron it is necessary to declare which of the coinciding faces is mapped onto which face. For example, one may agree that one of each pair of coinciding faces is red and the other green, and that reflections in mirrors along edges of the cube reverse colors while the other reflections do not change colors. Again, there are many possible extensions of this construction, which would lead us beyond the scope of this article.

(f) The rectangle-faced polyhedron P_9 is of the isohedral type denoted (21) in the listing described by Brückner [1, p. 191], and an example of an isohedron of this type is shown in his Fig. 2 of Plate 10. In fact, although Brückner describes its faces as trapezoids (noting specifically that it has a pair of parallel sides, of which one connects two vertices of a cube, the other two vertices of an octahedron), and although the faces of the model seem to be rectangles, he does not seem to have noticed this feature and the particular character of this polyhedron.

(g) Until recently, the only parallelogram-faced isohedra with icosahedral symmetry described in the literature were the three rhombic triacontahedra (see [3, p. 25, 103]; Cundy and Rollett [6, p. 121, 125, 126]), and a rhombic hexecontahedron described by Unkelbach [13]. Recently, three additional rhombic hexecontahedra have been described [8–10]. The method of the present paper allows the complete determination of all the parallelogram-faced isohedra with edges contained in mirrors; this will be described in detail elsewhere. Among these are three rectangle-faced isohedra. However, there is large number of other rhombic isohedra with icosahedral symmetry; these have not been completely determined so far.

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