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Which (n_4) configurations exist ?

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An **(n_k) configuration** is a family of n points and n (straight) lines in the Euclidean plane such that each point is on precisely k of the lines, and each line contains precisely k of the points. While the study of (n_3) configurations goes back more than a century, very little has been written about the geometric (n_k) configurations for $k \geq 4$. (See [1], [2], [3], [4], which seems to be a complete list of published references on this topic.) It is well known that there exist (n_k) configurations for each k and suitably large n. However, these arguments yield only configurations with very large values of n, and their size makes it quite pointless to represent them in the plane. The present note is devoted to the study of the question for which n do there exist (n_4) configurations, and to show that many of these can be reasonably drawn in the plane. In a later note the (n_k) configurations for $k \geq 5$ will be considered.

An (n_k) configuration is said to be **connected** if it is possible to reach every point starting from an arbitrary point and stepping to other points only if they are on one of the lines of the configuration. Equivalently, it is connected if it is not the union of two configurations with the same k but smaller n. The six configurations shown in Figure 1 of [3] are examples of connected (36_4) configurations; the (48_4) configuration shown in Figure 3 below is not connected. In the sequel, we shall freely use the terminology and notation of [3].

Theorem. Connected (n_4) configurations exist for every $n \geq 21$ except possibly if $n = 32$ or $n = p$ or $n = 2p$ or $n = p^2$ or $n = 2p^2$ or $n = p_1p_2$, where p, p_1, p_2 are odd primes and $p_1 < p_2 < 2p_1$.

For the proof we need appropriate notation. Let $m\#b_c d_e f_g \ldots h_j \;\; p_r$ denote an (n_4) configuration constructed as follows; see Figure 1 for an illustration in the particular case $18\#4_1 7_4 \;\; 8_6$. The points of the configuration are situated at the vertices of q concentric regular m-gons, where q is the number of symbols on the main level, to the right of the #-sign; thus $q = 3$ in Figure 1. The regular m-gons share all lines of mirror symmetry; in other words, they have different sizes, but for each pair, when viewed from the center, their vertices are either collinear or placed midway between each other. We call q the **number of levels** of the configuration, and we obviously have $n = qm$. The string of symbols following the #-sign encodes the structure of the configuration.

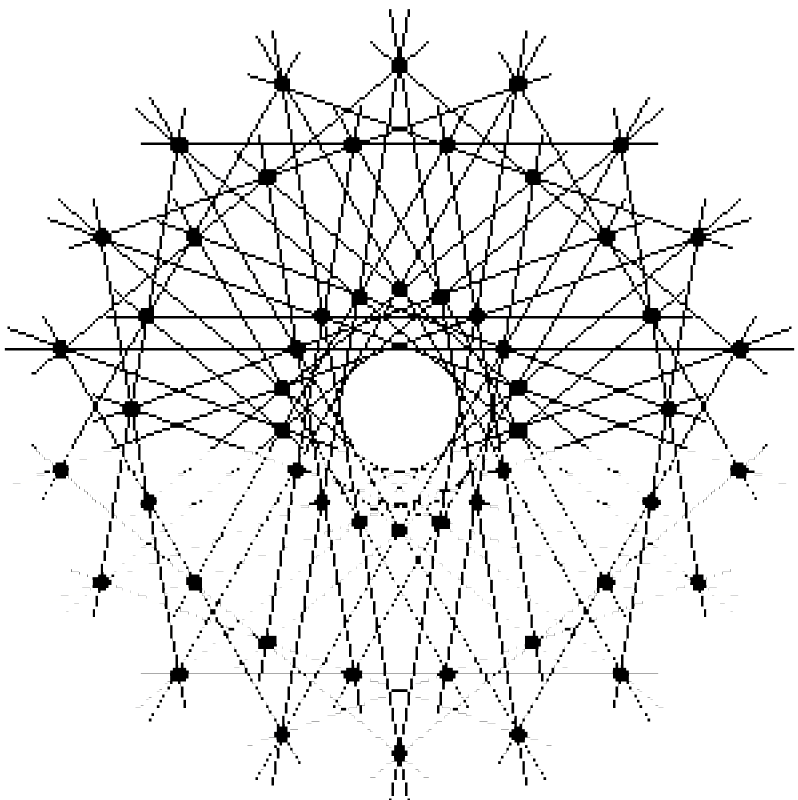

Figure 1. The (54_4) configuration $18\#4_1 7_4 \;\; 8_6$.

The first symbol, b, is the **span** of the shorter diagonals of the outer m-gon which determine m of the lines of the configuration; here span means the number of sides of the m-gon bridged by the diagonal. The second symbol, c, which is a subscript, indicates where the points of the second m-gon are located; c indicates the order, counting from the midpoint of a diagonal of spread b, of a point of the configuration, among all intersections of the diagonal of span b with other diagonals of the same span. In general, this c can be any integer from 1 to $b-1$; in Figure 1 we have $c = 1$. We continue in a similar way for the symbols d and e, but now starting from the just defined points of the second level. In Figure 1 we have $d = 7$ and $e = 4$. We continue in this way till we reach the last pair of symbols — the $(q-1)^{st}$ pair — before the break, f and g; for the last pair of symbols we perform the same construction but starting again with the outermost polygon. Naturally, arbitrary selections of the integers $m, b, c, d, e, \ldots$ will not result in an (n_4) configuration, since the points determined on the last step will in general not coincide with the points determined on the penultimate step. As is easily shown by elementary trigonometric arguments, presented in [3] for $k = 2$, an (n_4) configuration will be obtained only if

$$\frac{\cos(a\theta)}{\cos(b\theta)} \, \frac{\cos(c\theta)}{\cos(d\theta)} \cdots \frac{\cos(e\theta)}{\cos(f\theta)} = \frac{\cos(g\theta)}{\cos(h\theta)}, \qquad (*)$$

where $\theta = \pi/m$. For the example in Figure 1 relation (*) is equivalent to the validity of $\cos 10^\circ \cdot \cos 80^\circ = \cos 60^\circ \cdot \cos 70^\circ$, which is easily verified.

The main observation needed in the proof is that for every $q \geq 3$ and every $m \geq 2q+1$ there exists a configuration $m\#2_1 3_2 4_3 \ldots q_{q-1} \;\; q_1$. This follows at once from (*), since all terms cancel regardless of m, and $m \geq 2q+1$ assures that the diagonals used are all of different spans. The connectedness of such configurations is easily established, since the two outer levels are connected in a zigzag fashion. Since such a configuration contains $n = qm$ points, the validity of the theorem follows at once. □

In Figure 2 we show a configuration $11\#2_1 3_2 4_3 \;\; 4_1$, which illustrates the proof in case $m = 11$, $q = 4$ and hence $n = 44$.

Remarks.

(i) The configuration (21_4) described in [1] is the smallest one obtainable by the above construction; it results from $q = 3$, $m = 7$. It may be observed that connected $(n4)$ configurations exist for $n = 28$, 35 or 50 as well. The first two of these are described in [1], while (50_4) can be constructed from four copies of any configuration (10_3) in the following way. Start with one such configuration C, and a line L which is neither parallel nor perpendicular to any line of C. Taking L as the x-axis, construct three copies of C by stretching C away from L in

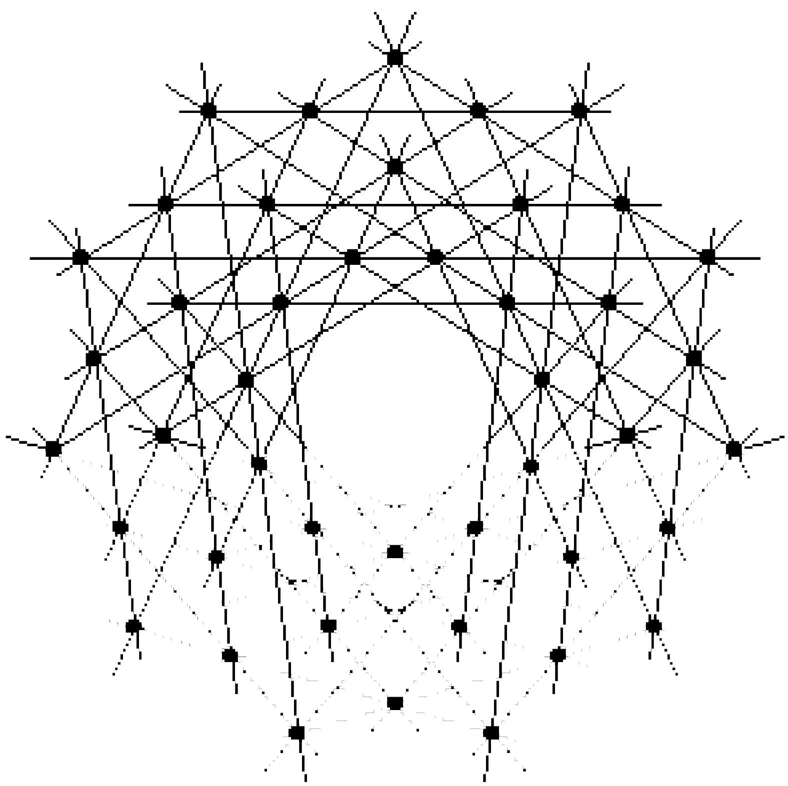

Figure 2. The (44_4) configuration $11\#2_1 3_2 4_3\ 4_1$.

the perpendicular direction, in three different ratios. The points of (50_4) are the points of the four copies of (10_3) together with their ten intersection points with L; the lines of (50_4) are the lines of the four copies of (10_3) together with the ten lines perpendicular to L passing through the points of the copies of (10_3).

(ii) Since the condition (*) is invariant under permutation of the first q-1 pairs, in many cases additional configurations can be obtained for the same set of parameters. However, not all permutations lead to (n_4) configurations, since in some cases additional collinearities or incidences result.

(iii) It was shown in [4] that configurations (13_4) and (14_4) (in the geometric sense understood in the present note) do not exist although they are combinatorially possible; in fact, it is well known that (13_4) is the finite projective plane of order 3. Combinatorial (n_4) configurations exist for every $n \geq 13$.

(iv) The following is a list of the 40 values of $n \leq 100$ for which the question of existence of a connected (n_4) configuration is undecided at this time: 15, 16, 17, 18, 19, 20, 22, 23, 25, 26, 29, 31, 32, 34, 37, 38, 41, 43, 46, 47, 49, 53, 58, 59, 61, 62, 67, 71, 73, 74, 77, 79, 82, 83, 86, 89, 91, 94, 97, 98.

(v) Any configurations (n_4) and (r_4) placed in the same plane so that no additional incidences occur yield a configuration $((n+r)_4)$. Clearly, such a configuration is not connected. If disconnected configurations are admitted, then the list of values of $n > 21$ for which no (n_4) configuration is known becomes finite; is consists of the fifteen numbers 22, 23, 25, 26, 29, 31, 32, 34, 37, 38, 41, 43, 46, 47, 53.

(vi) In [3] it was assumed that the configurations considered are connected; unfortunately, this was not stated. If disconnected configurations are admitted, the conjectured characterization of astral (n4) configurations is invalid and has to be replaced by admitting those obtained by taking two concentric copies of one of the configurations described in [3], rotated through an arbitrary angle with respect to each other.

References.

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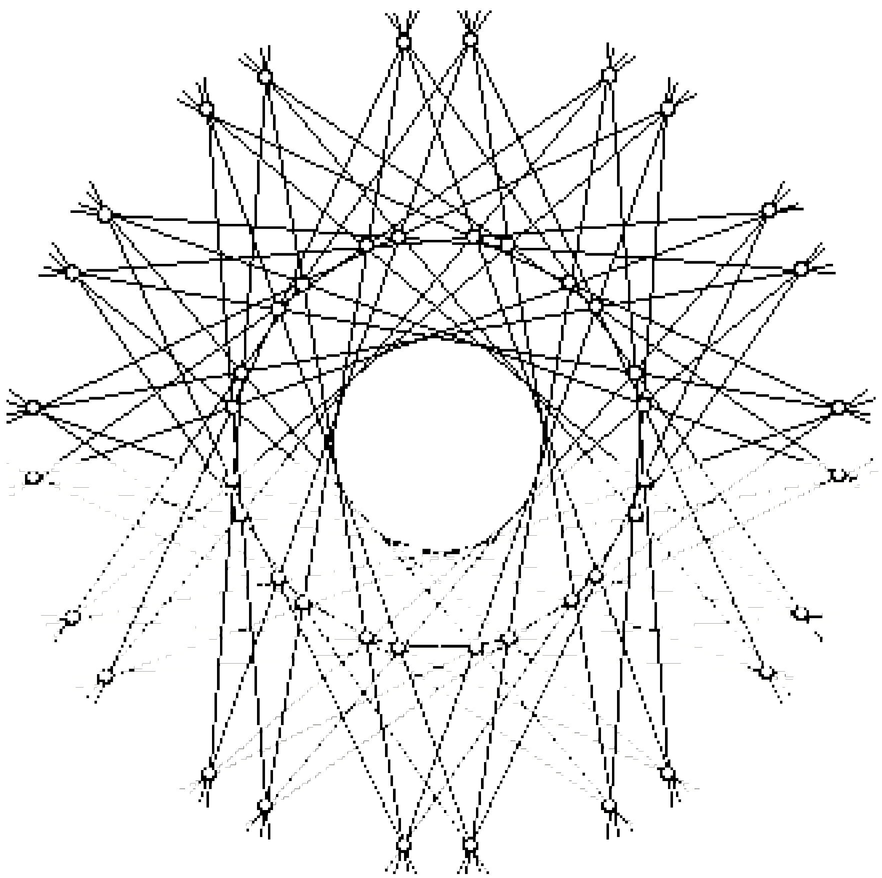

Figure 3. A (48_4) astral configuration which is not connected.