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ASTRAL (n_4) CONFIGURATIONS.

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A family of n points and n (straight) lines in the Euclidean plane is said to be an (n_4) **configuration** provided each point is on four of the lines and each line contains four of the points. A configuration may have various symmetries, that is, there may exist isometric mappings of the plane onto itself that map the configuration onto itself; all the symmetries of a configuration form its group of symmetries. It is obvious that no more than two points of a line can be in the same transitivity class with respect to the group of symmetries, and no more than two lines passing through one point can be in the same transitivity class unless all lines pass through that point. Hence, under its symmetry group each (n_4) configuration must have at least two transitivity classes of points, and at least two equivalence classes of lines. An (n_4) configuration is called **astral** if its points, as well as its lines, form precisely two transitivity classes. While it is not completely trivial that any astral (n_4) configurations exist, there is, in fact, a large number of possibilities which will be precisely described below. A few examples of astral (n_4) configurations were given in an earlier paper [2], and additional ones are shown in Figure 1. The aim of the present paper is to give this description, and to explain how it was found and established.

In order to proceed, we need some notation. It is easily verified that the points of any astral (n_4) configuration must lie at the vertices of two concentric regular m-gons, where $m = n/2$, and that the lines of the configuration must be determined by common diagonals of these m-gons. The **size** b of a diagonal D of a regular m-gon is the number of sides of P bridged by D; hence the angle subtended by D at the center of P is $2\pi b/n = \pi b/m$, and we need consider only the range $2 \le b \le m-1$. We

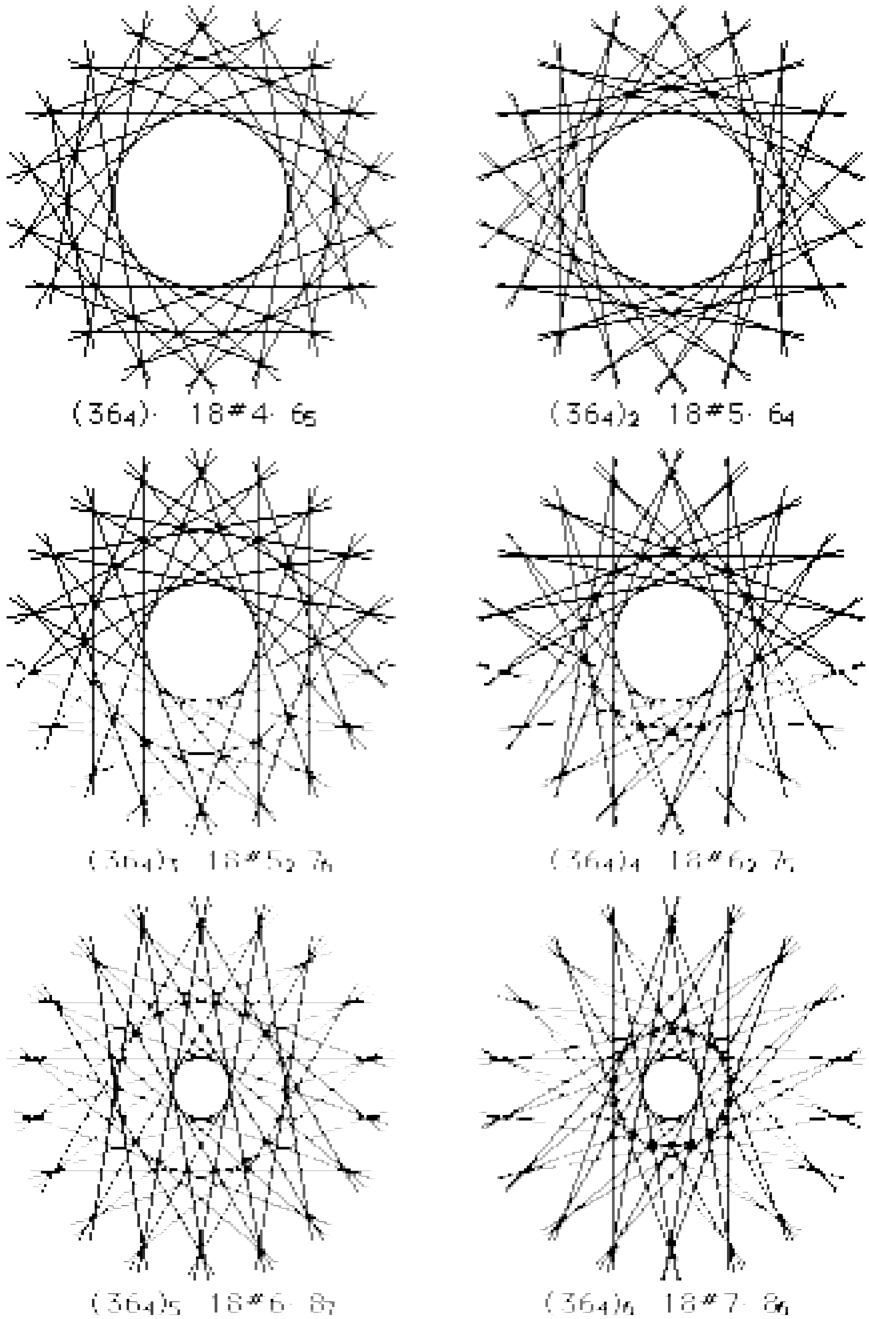

Figure 1. The six astral (36_4) configurations and their symbols.

use the symbol b_c to denote a point on a b-diagonal D which is the c^{th} among the intersection-points of D with the other b-diagonals, counting from the midpoint of D. This is illustrated in Figure 2. A point which is simultaneously b_c and d_e has symbol b_cd_e. Finally, an astral (n_4) configuration which has as points the $m = n/2$ vertices b_bd_d of a regular m-gon and m points b_cd_e, will be designated by the symbol $m\#b_cd_e$.

With this notation, we have the following result:

Theorem 1. There exist two infinite families of astral (n_4) configurations. One consists of the configurations $(6k)\#(3k-j)_{3k-2j}(2k)_j$ where $k \geq 2$, $1 \leq j \leq 2k-1$ but $j \neq k$ and, if k is even, $j \neq 3k/2$ as well. The other family consists of configurations $(6k)\#(3k-2j)_j(3k-j)_{2k}$ where $k \geq 2$ and $1 \leq j \leq k-1$.

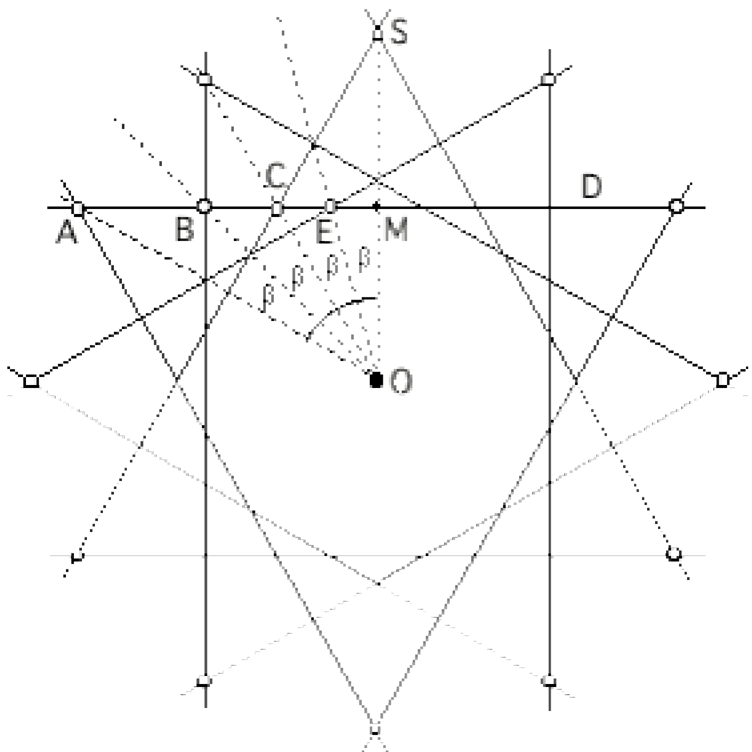

Figure 2. In the notation explained in the text, the points A, B, C, E have the symbols $4_4, 4_3, 4_2, 4_1$, respectively.

It may be noted that if k is even, the configurations of the two families coincide for $j = k/2$.

To prove the theorem, we first note that the midpoint M of a selected b-diagonal D, together with the intersection points of the other b-diagonals with D, determine angles which are multiples of $□ = \pi/m$. Therefore, in the notation of Figure 2 and with distance OA equal 1, the distance from O to b_c is $\frac{\cos(b\pi/m)}{\cos(c\pi/m)}$, where $1 \le c < b \le [\frac{m-1}{2}]$. Putting $\frac{\pi}{m} = \frac{\pi}{6k} = □$, and noting that the existence of a point $b_c d_e$ implies $\frac{\cos(b\pi/m)}{\cos(c\pi/m)} = \frac{\cos(d\pi/m)}{\cos(e\pi/m)}$, the existence of a configuration of type $(6k)\#(3k-j)_{3k-2j}(2k)_j$ is easily seen to be equivalent to

$$\cos j□ \cdot \cos (3k - j)□ = \cos 2k□ \cdot \cos (3k - 2j)□.$$

Since $\cos 2k□ = \cos \frac{\pi}{3} = 1/2$ and since

$$2 \cos a \cdot \cos b = \cos (a + b) + \cos (a - b),$$

this is equivalent to

$$\cos 3k□ + \cos (3k - 2j)□ = \cos (3k - 2j)□ .$$

This relation holds since $\cos 3k□ = \cos \pi/2 = 0$. A similar argument establishes the existence of the configurations in the second family.

It is easily seen that for every positive integer s, given any astral configuration $m\#b_c d_e$ there is an astral configuration $m^*\#b^*_{c^*}d^*_{e^*}$, where $m^* = sm$, $b^* = sb$, $c^* = sc$, $d^* = sd$, and $e^* = se$. We shall call each such configuraion a **multiple** of $m\#b_c d_e$. The multiple is obtained by symmetrically placing s copies of $m\#b_c d_e$. For the configurations of the families listed in Theorem 1 all multiples are configuration of the same family.

The configurations listed in Theorem 1 are by no means the only astral (n_4) configurations. There exist also a number of "sporadic" configurations, specified in Theorem 2.

Theorem 2. The following sporadic configurations exist, together with their multiples:

$30\#4_1\,7_6$,	$30\#6_1\,7_4$,	$30\#6_2\,8_6$,
$30\#6_1\,11_{10}$,	$30\#7_2\,12_{11}$,	$30\#8_1\,13_2$,
$30\#10_1\,11_6$,	$30\#10_6\,12_{10}$,	$30\#10_7\,13_{12}$,
$30\#11_2\,12_7$,	$30\#11_6\,14_{13}$,	$30\#12_1\,13_8$,
$30\#12_4\,14_{12}$,	$30\#12_7\,13_{10}$,	$30\#13_6\,14_{11}$
$42\#6_1\,13_{12}$,	$42\#11_6\,18_{17}$,	$42\#12_1\,13_6$,
$42\#12_5\,19_{18}$,	$42\#17_6\,18_{11}$,	$42\ \#\ 17_2\,19_{14}$,
$60\#9_2\,22_{21}$,	$60\#12_5\,25_{24}$,	$60\#14_3\,27_{26}$,
$60\#21_2\,22_9$,	$60\#24_5\,25_{12}$,	$60\#26_3\,27_{14}$.

The proof of Theorem 2 proceeds along lines similar to the proof of Theorem 1, but utilizing specific information about the values involved. For $m = 30$ we have $\alpha = \frac{\pi}{30} = 6^\circ$. The existence of $30\#6_2\,8_6$ is equivalent to:

$$\cos^2 36^\circ = \cos 48^\circ \cdot \cos 12^\circ,$$

which is the same as

$$1 + \cos 72^\circ = \cos 60^\circ + \cos 36^\circ,$$

or

$$1 + 2\cos 72^\circ + 2\cos 144^\circ = 0.$$

This relation is valid, as can be seen without computations, since it expresses the fact that the origin coincides with the centroid of a regular pentagon inscribed in the unit circle, with one vertex at (1,0). Completely analogous reasoning establishes the existence of configurations with symbols $30\#6_1\,11_{10}$ and $30\#12_4\,14_{12}$.

For the other sporadic cases with $m = 30$, explicit values of the cosines can be used. With

$$\cos 36^\circ = \frac{1 + \sqrt{5}}{4}$$

and

$$\cos 12° = \frac{-1 + \sqrt{5} + \sqrt{30 + 6\sqrt{5}}}{8}$$

it is easy to verify the existence of the remaining irreducible sporadic astral (60_4) configurations.

Similar calculations establish the existence of the sporadic (120_4) configurations.

The existence of the sporadic (84_4) configurations can be established by reducing all the coincidence conditions to the relation

$$1 + 2\cos 2\alpha + 2\cos 4\alpha + 2\cos 6\alpha = 0,$$

where $\alpha = \frac{\pi}{7}$. The validity of this equation can again be seen without calculations by noting that it expresses the coincidence of the center of a circle with the centroid of a regular heptagon inscribed in the circle. Alternatively, for direct proofs one can use the fact that $\cos\frac{\pi}{7}$ is the largest positive root of the equation $8y^3 - 4y^2 - 4y + 1 = 0$, which is explicitly given by $\cos\frac{\pi}{7} = \frac{1}{6} + z + \frac{7}{36z}$, where

$$z = \sqrt[3]{\frac{-7}{432} + \frac{7i}{48\sqrt{3}}} \quad ;$$

numerically, $\cos\frac{\pi}{7} = 0.90096886790\ldots$.

This completes the proof of Theorem 2.

The following statement should be considered as having a status between well-supported conjecture and fully proved theorem. The reasons will be explained below.

The only astral configurations (n_4) *are the ones given by Theorems 1 and 2.*

The history of my involvement with the astral (n_4) configurations started more than fifteen years ago, when I empirically found several, with small n. However, beyond the smallest ones I was not be sure of their existence, either by drawing them by hand or using MacDraw® software.

In particular, I found no pattern for the parameters of the ones I believed existed. Somewhat later I was initiated (by Stan Wagon, to whom I am greatly indebted for this) to Mathematica® software. High precision calculations easily established that, for $n \leq 60$, astral (n_4) configurations exist only if $n = 2m = 12k$, for some integer k with $k \geq 2$. However, the pattern of possible configurations remained puzzling. This is possibly best understood by considering Figure 3. In it a configuration $m\#b_c d_e$, where $b > d$, is represented by solid a solid square, centered at (b, d). A square with diagonals drawn indicates the existence of two distinct configurations with the same values of b and d.

The solution of this puzzle was suggested by additional numerical calculations, which led to results such as the ones shown in Figure 4. The pattern revealed is that one should distinguish between the two kinds of configurations, the ones that exist for every $m = 6k$ with $k \geq 2$, and those that exist only for certain multiples of 6. This is reflected in the above Theorems 1 and 2. Once the pattern was recognized, it was easy to establish that it holds for values beyond the ones for which the evidence was obtained through numerical calculations.

In order to turn the italicized statement into a theorem we need to show that there are no other astral configurations $m\#b_c d_e$. In principle, this should be possible with the information available in the literature. Bol

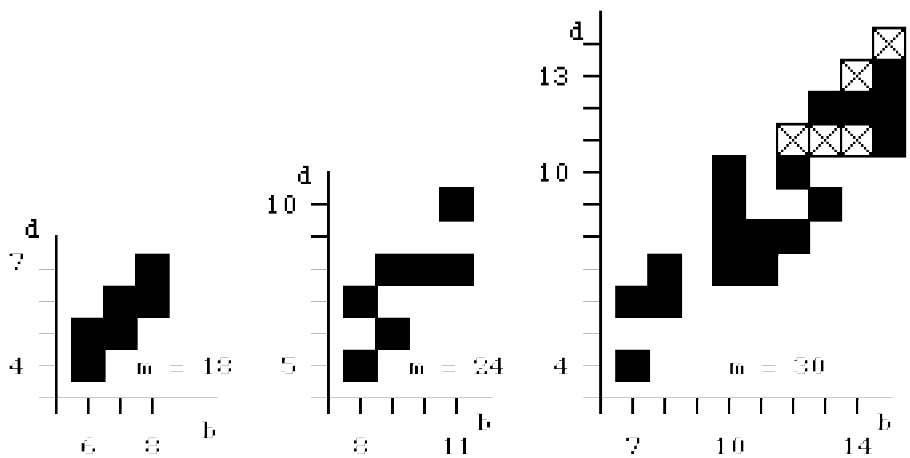

Figure 3. The values of b and $d < b$ for which there exist configurations $m\#b_c d_e$.

[1] was the first to determine all multiple intersection points of the diagonals of regular polygons. Other determinations of these points were given by Rigby [4] and by Poonen and Rubinstein [3]; the latter gives references to the many related papers. The problem is surprisingly complicated, and the results cannot be expressed in simple terms. The astral (n_4) configurations obviously arise from the intersection points of four diagonals, two of each of two spans. But the practical difficulty is to extract the descriptions of such intesection points from the general results given in the papers listed above. I have no doubt in the validity of the italicized statement, but have so far not derived a formal proof.

References

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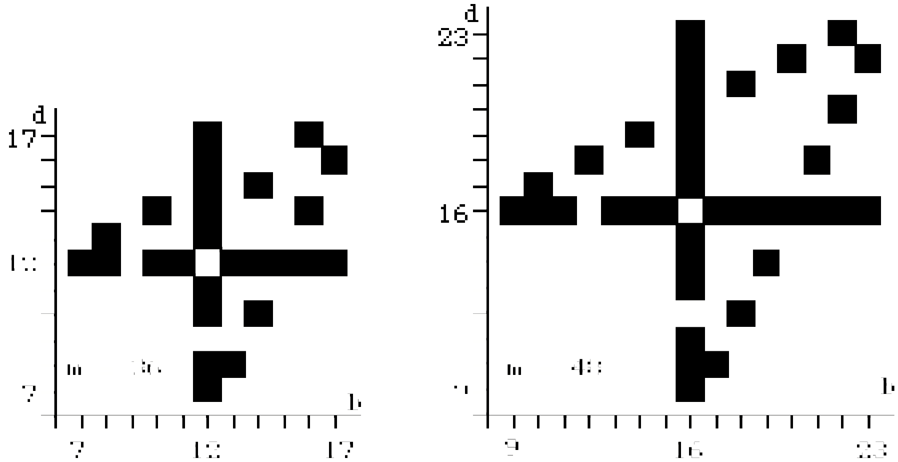

Figure 4. An illustration of the existing configurations m#$b_c d_e$ for additional values of m. The configurations correspond to the ones specified by Theorem 1.