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A RELATIVE OF "NAPOLEON'S THEOREM"

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1. Introduction. A remarkable theorem of elementary geometry is often called "Napoleon's Theorem". It is not clear whether it is appropriate to assign the credit to the French general and emperor, but the name seems to be widely accepted; see, for example, [1, Section 3.3], [11, pp. 38, 93], [5, Section 3.3], [4, pp. 60, 112, 180]. Additional comments, and references, will be given in Section 3 below.

Here is the theorem, illustrated in Figure 1.

Napoleon's theorem. Given an arbitrary triangle ABC. Equilateral triangles AC'B, BA'C and CB'A are constructed, not overlapping ABC, and their centroids denoted by C*, A*, B*. Then

(i) A*B*C* is an equilateral triangle with the same orientation as ABC.

Similarly, let the equilateral triangles AC"B, BA"C and CB"A be constructed so that each overlaps ABC, and let now their centroids be denoted C**, A**, B**. Then

(ii) A**B**C** is an equilateral triangle with orientation opposite to that of ABC.

Moreover,

(iii) the area of ABC equals the algebraic sum of the areas of A*B*C* and A**B**C**.

The triangles A*B*C* and A**B**C** are often called the outer and the inner Napoleon triangles. The purpose of this short note is to present a result related in spirit to Napoleon's theorem, but more complex. Even so, it is a surprising aspect of the new result that — as far as I could ascertain — it has not been noticed earlier. The new construction will also lead to two triangles analogous to the Napoleon triangles but having side only half as long; these triangles are obtained from triplets of intermediate triangles which are also equilateral, and each involves one of the original vertices A, B, C.

The formulation of our result is somewhat long, but the illustration in Figure 2 should make the steps obvious.

Theorem. Given an arbitrary triangle ABC. Equilateral triangles AC'B, BA'C and CB'A are constructed, not overlapping ABC. The midpoints of A'B', B'C', C'A' are denoted by C_1, A_1, B_1, respectively. Then

(i) A_1B_1C, B_1C_1A and C_1A_1B are equilateral triangles with the same orientation as ABC;

(ii) the centroids A^*, B^*, C^* of these triangles are vertices of an equilateral triangle with orientation opposite that of ABC.

Similarly, let the equilateral triangles AC"B, BA"C and CB"A be constructed so that each overlaps ABC. We denote the midpoints of A"B", B"C", C"A" by C_2, A_2, B_2. Then

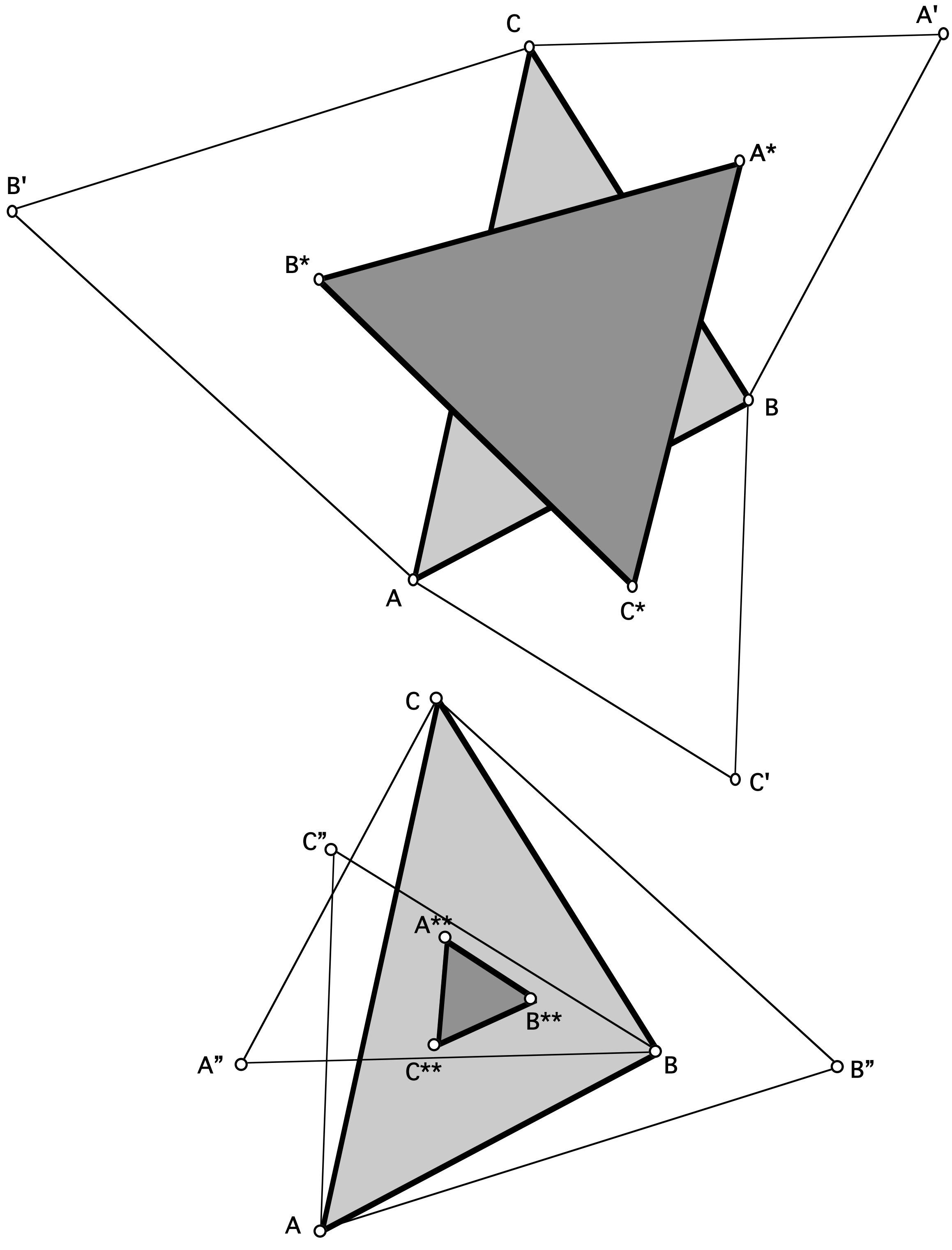

Figure 1. Illustration of parts (i) and (ii) of Napoleon's theorem.

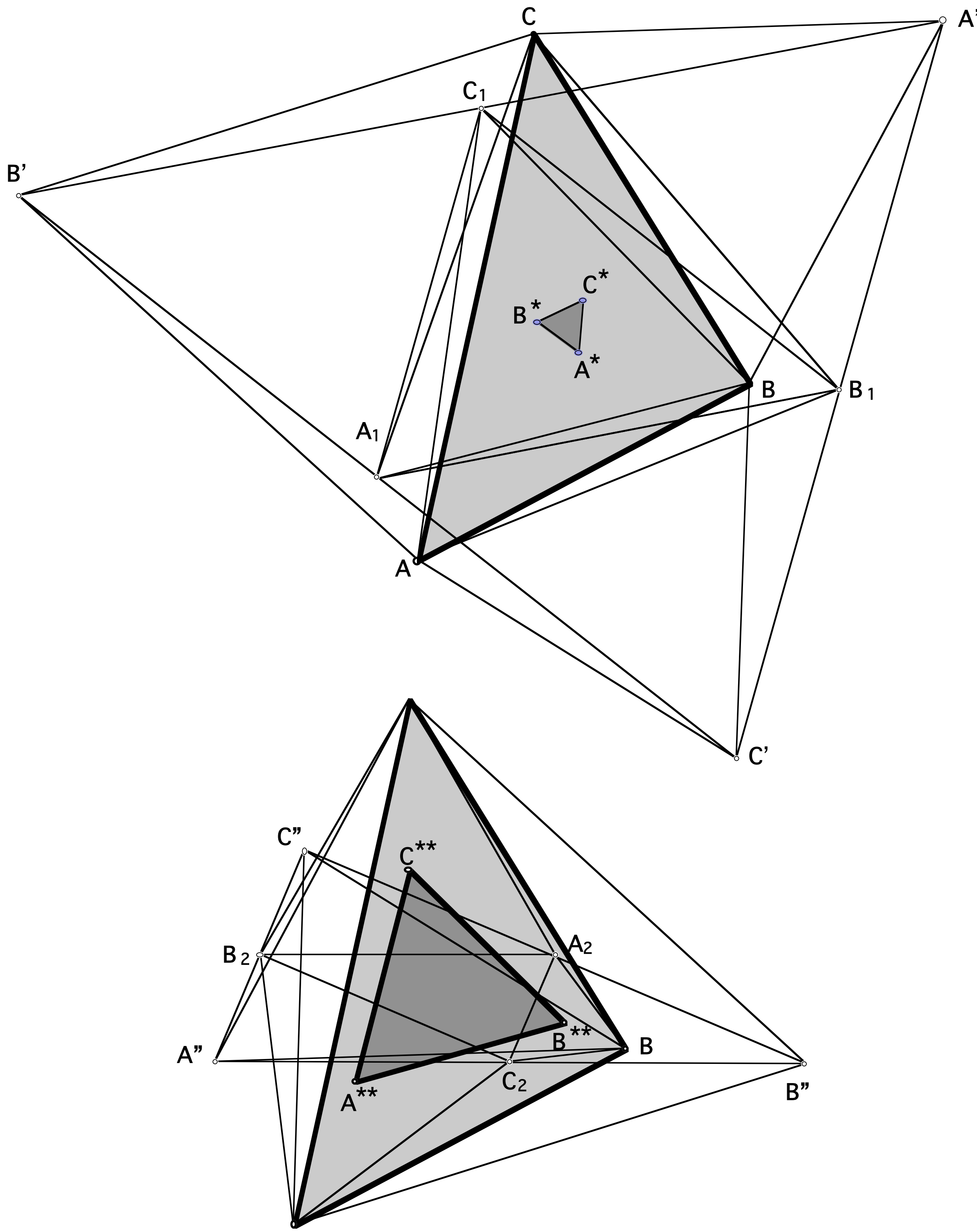

Figure 2. An illustration of the first four parts of the Theorem.

(iii) A_2B_2C, B_2C_2A and C_2A_2B are equilateral triangles with orientation opposite to that of ABC;

(iv) the centroids A**, B**, C** of these triangles are vertices of an equilateral triangle with the same orientation as ABC.

(v) Moreover, the area of ABC equals four times the algebraic sum of the areas of A*B*C* and A**B**C**.

2. Proofs. Napoleon's theorem has been given many different proofs. Some use simple Euclidean geometry -- so, for example, [4, p. 112], [9]; others use trigonometry, see [1, p. 64]. Still others use the coordinatization of the Euclidean plane in which each point is represented by a complex number; this representation goes back two centuries, to C. F. Gauss and others. It was applied to the solution of geometric problems in a somewhat esoteric way by Giusto Bellavitis, but turned into a powerful tool by C.-A. Laisant [7]. In many ways, this last method is the most appropriate one. It easily leads to proofs of many far-reaching generalizations of Napoleons theorem, some of which will be mentioned in Section 3. In order to make the presentation of this paper accessible to readers not used to complex numbers, we shall present the proofs using the customary coordinate system in the plane. It is essentially equivalent to the use of complex numbers, although requiring expressions that are somewhat longer.

For the proofs of the Napoleon theorem and the new result, let the vertices be given by their coordinates $A = (a_1,a_2)$, $B = (b_1,b_2)$ and $C = (c_1,c_2)$. Then it is easily verified that

$A' = \big((b_1+c_1+\sqrt{3}(-b_2+c_2))/2,\ (b_2+c_2+\sqrt{3}(b_1-c_1))/2\big)$,

$A'' = \big((b_1+c_1+\sqrt{3}(b_2-c_2))/2,\ (b_2+c_2+\sqrt{3}(-b_1+c_1))/2\big)$,

and similarly for B', B", C', C". Hence

$A^* = \big((3(b_1+c_1)+\sqrt{3}(-b_2+c_2))/6,\ (3(b_2+c_2)+\sqrt{3}(b_1-c_1))/6\big)$,

$A^{**} = \big((3(b_1+c_1)+\sqrt{3}(b_2-c_2))/6,\ (3(b_2+c_2)+\sqrt{3}(-b_1+c_1))/6\big)$, etc.,

and therefore, by easy but somewhat tedious calculations, it follows that both Napoleon triangles are equilateral, and that

$\text{areaA*B*C*} = \frac{1}{2}\ \text{areaABC} + \frac{1}{6}(\text{areaAC'B} + \text{areaBA'C} + \text{areaCB'A})$

$\text{areaA**B**C**} = \frac{1}{2}\ \text{areaABC} + \frac{1}{6}(\text{areaAC''B} + \text{areaBA''C} + \text{areaCB''A})$

,

which proves that areaABC = areaA*B*C* + areaA**B**C**. ◊

Analogous computations establish the new result. Here

$A_1 = \big((2a_1+b_1+c_1+\sqrt{3}(b_2-c_2))/4,\ (2a_2+b_2+c_2+\sqrt{3}(-b_1+c_1))/4\big)$,

$A_2 = \big((2a_1+b_1+c_1+\sqrt{3}(-b_2+c_2))/4,\ (2a_2+b_2+c_2+\sqrt{3}(b_1-c_1))/4\big)$,

$A^* = \big((6a_1+3b_1+3c_1+\sqrt{3}(-b_2+c_2))/12,\ (6a_2+3b_2+3c_2+\sqrt{3}(b_1-c_1))/12\big)$,

$A_2 = \big((6a_1+3b_1+3c_1+\sqrt{3}(b_2-c_2))/12,\ (6a_2+3b_2+3c_2+\sqrt{3}(-b_1+c_1))/12\big)$,

etc., from which the assertions about equilaterality of A_1B_1C and the other triangles in parts (i) and (iii) of the Theorem can be verified, as well as the fact that areaABC = 4(areaA*B*C* + areaA**B**C**). ◊

The illustrations in Figures 1 and 2 show (and calculations confirm) that the triangle A*B*C* in Figure 2 is homothetic in ratio

–1/2 to the inner Napoleon triangle, while the triangle A**B**C** is homothetic in the same ratio to the outer Napoleon triangle.

In further analogy to the situation regarding the "Napoleon configuration" (see, e.g., [10]), it is easy to show that the lines AA*, BB* and CC* are concurrent, as are the lines AA**, BB**, CC**.

3. Remarks and problems.

One may consider the new result as just an exercise in manipulating algebraic expressions involving coordinates of points; a similar observation applies to Napoleon's theorem. The challenge in both cases is not the proof, but the discovery of the result. While the source of Napoleon's theorem seems to be lost, I can easily divulge how I found the new result. It was through somewhat systematic experimenation using Geometers Sketchpad™ software on a Macintosh™ computer. After one is convinced in its validity, the formal proof becomes straightforward.

The main reason for hope that there may be greater significance in the new result is the question whether it can be generalized in some of the different directions that have been developed for Napoleon's theorem. Among these, I would like to mention two specific instances.

The simpler generalization is what is known as the Napoleon–Barlotti theorem, see [2], [8]. It states that given an affine-regular n-gon (that is, an affine image of a regular n-gon), if regular n-gons of the same kind are constructed on its sides all outwards or else all inwards, their centroids form another regular n-gon. Here, and in the sequel, by "regular n-gon" we understand any polygon of n-sides such that symmetries act transitively on its vertices and on its edges; thus not only convex polygons can be regular, but starshaped ones as well, and even ones in which there are coincident sets of vertices. (For more details on these concepts see, for example, [3].) Since every triangle is affine–regular, the Napoleon–Barlotti theorem is clearly a generalization of Napoleon's theorem. Is there a similar generalization of the new theorem?

The more complex generalization (see [2]) should be known as the Petr–Douglas–Neumann theorem, although many writers fail to mention some or all of these names. In particular, the 1905 publication of K. Petr is rarely quoted, the theorem being attributed to either J. Douglas or B. H. Neumann even though they published only in 1940 and 1941, respectively. This theorem is quite complicated to state, and I will describe it only in a rough way. It states that if on the sides of an arbitrary n-gon one constructs regular n-gons of a certain kind, then on the n-gon of centroids of these polygons one constructs regular n-gons of a different, kind, on the n-gon of their centroids regular n-gons of a still different kind, ... ; then, the centroids of the regular n-gons of the one-but last kind will be the vertices of a regular n-gon of the last kind. Here the accounting is that there are two kinds of regular triangles (the equilateral one taken clockwise, or else counterclockwise), three kinds of regular quadrangles (squares taken clockwise or counterclockwise, as well as the regular quadrangle that looks like a segment, but whose vertices alternate at the endpoints of the segment), four kinds of regular pentagons (the convex, and the pentagram, both taken clockwise or counterclockwise), and so on. In general, there are n–1 kinds of regular n-gons that need to be considered in the Petr–Douglas–Neumann theorem. The challenge here is, again, to determine whether there is an analogous generalization of the new theorem.

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