SOLUTIONS FOR THE SAMPLE SECOND EXAM FROM SPRING, 2003

QUESTION 1.
(a) We are given a set \( S = \{V_1, V_2, V_3\} \) of three vectors in \( \mathbb{R}^3 \). The set \( S \) will be a basis for \( \mathbb{R}^3 \) if and only if \( S \) is a linearly independent set. We check for linear independence by determining the rank of the matrix \( [V_1 \ V_2 \ V_3] \). This is done by row-reduction:

\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & 3 & 4 \\
1 & 7 & 5
\end{bmatrix}, \quad 
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 1 \\
0 & 5 & 2
\end{bmatrix}, \quad 
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 1 \\
0 & 0 & -3
\end{bmatrix}, \quad 
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

The last matrix is in echelon form. The matrix \([V_1 \ V_2 \ V_3]\) therefore has rank 3. It follows that \( S \) is a linearly independent set and therefore a basis for \( \mathbb{R}^3 \).

(b) Linear independence doesn’t depend on the order of the vectors. Hence, using the result of part (a), the set \( \{V_2, V_3, V_1\} \) is also a linearly independent set. Therefore, the matrix \( A = [V_2 \ V_3 \ V_1] \) is a nonsingular matrix. Therefore, the matrix equation \( AX = 0 \) has just one solution, namely \( X = 0 \). Therefore,

\[
\mathcal{N}(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

(c) The set \( S \) of vectors given in part (a) is a basis for \( \mathbb{R}^3 \). If we add more vectors to the set, we will still obtain a spanning set for \( \mathbb{R}^3 \). However, the new set will no longer be a linearly independent set. As an example, consider the set

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \quad 
\begin{bmatrix}
2 \\
3 \\
7
\end{bmatrix}, \quad 
\begin{bmatrix}
3 \\
4 \\
5
\end{bmatrix}, \quad 
\begin{bmatrix}
1 \\
7 \\
8
\end{bmatrix}
\]

where the first three vectors are the vectors in the set \( S \) given in part (a). This new set consists of four vectors in \( \mathbb{R}^3 \). It must be a linearly dependent set. But it is also a spanning set for \( \mathbb{R}^3 \). This is an example of a spanning set for \( \mathbb{R}^3 \) which is not a basis for \( \mathbb{R}^3 \).

QUESTION 2.
(a) The matrix \( A = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 0 & 5 & 8 \end{bmatrix} \) is row-equivalent to:

\[
\begin{bmatrix}
1 & 2 & 0 & 1 & 1 \\
0 & 0 & 0 & 3 & 6
\end{bmatrix}, \quad 
\begin{bmatrix}
1 & 2 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 2
\end{bmatrix}, \quad 
E = \begin{bmatrix}
1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2
\end{bmatrix}
\]
The matrix $E$ is in reduced echelon form. The solutions to $AX = 0$ and $EX = 0$ are the same. Denoting the variables by $x_1, x_2, x_3, x_4, x_5$, the leading variables are $x_1$ and $x_4$, and the free variables are $x_2, x_3$ and $x_5$. The solutions to $EX = 0$ are described by the equations

\[
\begin{align*}
    x_1 + 2x_2 - x_5 &= 0 \\
    x_4 + 2x_5 &= 0
\end{align*}
\]

Thus, the solutions in vector form are given by

\[
X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_5 \\ x_2 \\ x_3 \\ -2x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}
\]

A basis for $\mathcal{N}(A)$ is:

\[
\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

(b) Since $A$ is a $2 \times 5$ matrix, the null space of $A$ is a subspace of $\mathbb{R}^5$. Thus, $t = 5$.

(c) The basis for $W = \mathcal{N}(A)$ found in part (a) of this question has three vectors in it. Hence dim($W$) = 3. If $B$ is a matrix such that $\mathcal{R}(B) = W$, then dim($\mathcal{R}(B)$) = 3. But, in general, we have dim($\mathcal{R}(B)$) = rank($B$). Hence, rank($B$) = 3.

(d) The range of a matrix $B$ is spanned by the columns of $B$. Thus, we should choose the columns of $B$ so that they span $W$. A basis for $W$ was found in part (a). That set of vectors is certainly a spanning set for $W$. Thus, here is an example of a matrix $B$ such that $\mathcal{R}(B) = W$:

\[
B = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}
\]
QUESTION 3. The information given about the $3 \times 3$ matrix $C$ leads us to some conclusions about the range of $C$. First of all, $\mathcal{R}(C)$ is a subspace of $\mathbb{R}^3$ because $C$ has 3 rows. Also, the matrix equation $CX = b$ has at least one solution when $b$ is any one of the following vectors:

$$
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \quad 
\begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix}, \quad 
\begin{bmatrix}
5 \\
3 \\
7
\end{bmatrix}
$$

It follows that $\mathcal{R}(C)$ contains each of those vectors. But $CX = b$ has no solutions when

$$
b = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}
$$

Therefore, that vector is not in $\mathcal{R}(C)$. Hence, $\mathcal{R}(C)$ is a subspace of $\mathbb{R}^3$, but not equal to $\mathbb{R}^3$. Thus, $\dim(\mathcal{R}(C)) < 3$. This implies that $\dim(\mathcal{R}(C)) = 0, 1, 2$. However, $\dim(\mathcal{R}(C))$ cannot be 0 or 1, because $\mathcal{R}(C)$ contains the following linearly independent set of vectors:

$$
S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}
$$

It follows that $\dim(\mathcal{R}(C)) = 2$ and that the set $S$ is actually a basis for $\mathcal{R}(C)$. In particular, it follows that $\mathcal{R}(C) = \text{Sp}(S)$. This description of $\mathcal{R}(C)$ allows us to determine if the vector

$$
\begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix}
$$

is contained in $\mathcal{R}(C)$. We do this by considering the vector equation:

$$
x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix}
$$

This vector equation corresponds to a system of equations with the following augmented matrix.

$$
\begin{bmatrix}
1 & 2 & 5 \\
1 & 1 & 2 \\
1 & 3 & 8
\end{bmatrix}
$$
Row-reduction gives:

\[
\begin{bmatrix}
1 & 2 & 5 \\
1 & 1 & 2 \\
1 & 3 & 8
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 2 & 5 \\
0 & -1 & -3 \\
0 & 1 & 3
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 2 & 5 \\
0 & 1 & 3 \\
0 & 1 & 3
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{bmatrix}
\]

Hence, the vector equation does have a solution, namely \( x = -1, y = 3 \). Therefore, \( \mathcal{R}(C) \) contains the vector \( \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix} \) and therefore the matrix equation \( CX = \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix} \) has at least one solution. This matrix equation corresponds to a system of 3 equations in 3 unknowns. The coefficient matrix is \( C \). This matrix equation must have infinitely many solutions because \( \text{rank}(C) = \dim(\mathcal{R}(C)) = 2 \) and so \( \text{rank}(C) < 3 \).

QUESTION 4. Since \( u \) is a vector in the null space of \( A \), we have \( Au = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). This implies that

\[
\begin{bmatrix}
1 & 2 & 0 & 1 & 1
\end{bmatrix} u = 0,
\quad
\begin{bmatrix}
2 & 4 & 0 & 5 & 8
\end{bmatrix} u = 0
\]

Notice that \( \begin{bmatrix}
2 & 4 & 0 & 5 & 8
\end{bmatrix} = v^T \). Hence, the second equation above implies that \( v^T u = 0 \). That is, the dot product of the vectors \( v \) and \( u \) is equal to 0. Therefore, the vectors \( u \) and \( v \) are indeed orthogonal to each other. The statement is true.