SOLUTIONS FOR THE SAMPLE FIRST EXAM FROM FALL, 2001

QUESTION 1.
(a) To find $A^{-1}$, we reduce the augmented matrix $[A|I_3]$ to its reduced echelon form:

$$[A|I_3] = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

Therefore, we have

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

(b) The specified vectors $V_1, V_2, V_3$ are just the columns of the matrix $A$. Hence the vector equation given in this problem is equivalent to the matrix equation:

$$AX = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

where

$$X = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

The solution is found by multiplying by $A^{-1}$, putting the $A^{-1}$ on the left. We have

$$X = A^{-1}(AX) = A^{-1} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 + 3 - 2 \\ 10 - 3 - 2 \\ -5 + 0 + 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -3 \end{bmatrix}$$

The answer is $c_1 = 6, \quad c_2 = 5, \quad c_3 = -3$.

(c) We want to find a $2 \times 3$ matrix $B$ satisfying

$$BA = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$
Multiply both sides of this equation by $A^{-1}$, putting the $A^{-1}$ on the right:

$$(BA)A^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \end{bmatrix} A^{-1}$$

Now $(BA)A^{-1} = B(AA^{-1}) = BI_3 = B$ and so we get

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ 5 & -3 & 2 \end{bmatrix}$$

(d) The equation is $A^{-1}CA = I_3$. We use matrix algebra as follows:

$A^{-1}CA = I_3, \quad A(A^{-1}CA) = AI_3, \quad (AA^{-1})CA = A, \quad I_3CA = A, \quad CA = A,$

$(CA)A^{-1} = AA^{-1}, \quad C(AA^{-1}) = I_3, \quad CI_3 = I_3, \quad C = I_3$

The only possible $C$ is $C = I_3$.

(e) The matrix $A$ is invertible and hence nonsingular. Therefore $A$ is row-equivalent to the identity matrix $I_3$. But $A^{-1}$ is also invertible (because it has the matrix $A$ as inverse). Therefore $A^{-1}$ is also nonsingular and hence row-equivalent to the identity matrix $I_3$. Since elementary row-operations can be reversed, one can first transform $A$ to $I_3$ and then $I_3$ to $A^{-1}$ by a sequence of elementary row-operations. Thus, it is true that $A$ is row-equivalent to $A^{-1}$.

**QUESTION 2.**

(a) The matrix equation in this question is equivalent to a system of two equations in four unknowns $x_1, x_2, x_3, x_4$. The augmented matrix for this system is

$$\begin{bmatrix} 0 & 2 & 0 & 5 & 7 \\ 0 & 4 & 1 & 8 & 3 \end{bmatrix}$$

This matrix reduces to

$$\begin{bmatrix} 0 & 1 & 0 & 5/2 & 7/2 \\ 0 & 4 & 1 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 5/2 & 7/2 \\ 0 & 0 & 1 & -2 & -11 \end{bmatrix}$$

The last matrix is in reduced echelon form. The leading 1’s occur in columns 2 and 3. Thus, the leading variables are $x_2, x_3$ and the free variables are $x_1, x_4$. The solutions are given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 7/2 - 5/2x_4 \\ -11 + 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 + x_1 + x_4 \\ 7/2 + 0x_1 + (-5/2)x_4 \\ -11 + 0x_1 + 2x_4 \\ 0 + 0x_1 + 1x_4 \end{bmatrix}$$
where $x_1$ and $x_4$ are arbitrary. The solutions to the matrix equation in this question can be described in vector form as:

$$X = \begin{bmatrix} 0 \\ 7/2 \\ -11 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -5/2 \\ 2 \\ 1 \end{bmatrix}$$

where $x_1$ and $x_4$ are arbitrary.

(b) Write the $4 \times 3$ matrix $B$ as $B = [X \quad Y \quad Z]$, where $X, Y,$ and $Z$ are $4 \times 1$ matrices. Then $AB = [AX \quad AY \quad AZ]$. Thus, we want to consider the following three matrix equations:

$$AX = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad AY = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \quad AZ = \begin{bmatrix} 21 \\ 9 \end{bmatrix}$$

The first matrix equation has infinitely many solutions, but we will just take the trivial solution $X$ = the 4-dimensional zero vector. The second matrix equation has been solved in part (a) of this problem, except that $X$ has been changed to $Y$. We will just choose $Y$ to be the solution where $x_1 = x_4 = 0$. For the third matrix equation, notice that

$$AY = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \quad \Longrightarrow \quad A(3Y) = 3(AY) = 3 \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ 9 \end{bmatrix}$$

Thus, we can choose $Z$ to be $3Y$, where $Y$ is chosen as above. This gives one possible choice of $B$, namely

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 7/2 & 21/2 \\ 0 & -11 & -33 \\ 0 & 0 & 0 \end{bmatrix}$$

There are other possible choices of $B$ since each of the columns $X, Y,$ and $Z$ could have been chosen in infinitely many ways.

**QUESTION 3.**

(a) Suppose that $b$ is any vector in $\mathbb{R}^3$. The matrix equation $AX = b$ is equivalent to a system of 3 equations in 4 unknowns. Since the coefficient matrix for this system of equations is $A$ and the rank of $A$ is 3, which is the number of equations in the system, the matrix equation $AX = b$ must be consistent. In particular, this is true for

$$b = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$$
Since the rank of the coefficient matrix $A$ is 3, which is less than the number of unknowns (which is 4), it follows that the system of equations (already known to be consistent) must have an infinite number of solutions. Therefore, the matrix equation in this question has an infinite number of solutions.

(b) Based on the given information, notice that the matrix equation

$$BX = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

has at least two solutions, namely

$$X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad X = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}$$

It follows that the $3 \times 3$ matrix $B$ cannot have rank 3. We must have rank($B$) < 3. Furthermore, notice that

$$B \left( \begin{bmatrix} 1 \\ 2 \\ 1 \\ & 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right) = B \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + B \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

Therefore, the matrix equation

$$BX = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

has at least one solution, namely

$$X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 7 \end{bmatrix}$$

The matrix equation

$$BX = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

is equivalent to a consistent system of linear equations with 3 unknowns. Since rank($B$) < 3, the number of solutions must be infinite.