VECTOR SPACES.

DEFINITION: Suppose that F is a field. A vector space V over F is a nonempty set with two operations, *"addition"* and *"scalar multiplication"* satisfying certain requirements.
Addition is a map $V \times V \longrightarrow V$: $(v_1, v_2) \longrightarrow v_1 + v_2$.
Scalar multiplication is a map $F \times V \longrightarrow V$: $(f, v) \longrightarrow fv$.

The requirements are:

(i) V is an abelian group under the addition operation $+$.

(ii) $f(v_1 + v_2) = fv_1 + fv_2$ for all $f \in F$ and $v_1,\ v_2 \in V$.

(iii) $(f_1 + f_2)v = f_1v + f_2v$ for all $f_1,\ f_2 \in F$ and $v \in V$.

(iv) $f_1(f_2v) = (f_1f_2)v$ for all $f_1,\ f_2 \in F$ and $v \in V$.

(v) $1_F v = v$ for all $v \in V$.

Easy results:

(1) $f0_V = 0_V$ for all $f \in F$.
(2) $0_F v = 0_V$ for all $v \in V$.
(3) $(-f)v = -(fv)$ for all $f \in F$ and $v \in V$.
(4) Assume that $f \in F$ and $v \in V$. The $fv = 0_V \Longrightarrow f = 0_F$ or $v = 0_V$.

DEFINITION: Suppose that V is a vector space over a field F and that W is a subset of V. We say that W is a *"subspace of V"* if (1) W contains 0_V, (2) W is closed under addition, and (3) W is closed under scalar multiplication, i.e., $fw \in W$ for all $f \in F$ and $w \in W$.

DEFINITION: Suppose that V is a vector space over a field F and that $S = \{v_1, ..., v_n\}$ is a finite sequence of elements of V. We say that *"S is a generating set for V over F"* if, for every element $v \in V$, there exist elements $f_1, ..., f_n \in F$ such that $v = f_1v_1 + ... + f_nv_n$. If such a finite sequence S exists, then we say that *"V is a finitely generated vector space over F."* We then say that *"S generates V over F."* One might also say that *"S spans V over F"* or that *"S is a spanning set for V over F."*

DEFINITION: Suppose that V is a vector space over a field F and that $S = \{v_1, ..., v_n\}$ is a finite sequence of elements of V. We say that *"S is linearly dependent over F"* if there exist elements $f_1, ..., f_n \in F$, not all equal to 0_F, such that $f_1 v_n + ... + f_n v_n = 0_V$. We say that *"S is linearly independent over F"* if, for $f_1, ... f_n \in F$,

$$f_1 v_1 + ... + f_n v_n = 0_V \implies f_1 = ... = f_n = 0_F \quad .$$

DEFINITION: Suppose that $v_1, ..., v_n \in V$. Let $S = \{v_1, ..., v_n\}$. We say that S *is a basis for V"* if S is a generating set for V over F and S is also a linearly independent set over F.

DEFINITION: Suppose that V is a finitely generated vector space over a field F. Suppose that V has a generating set over F of cardinality d, but does not have a generating set over F of cardinality $d-1$. We then say that *"V has dimension d over F."* We will write $d = dim_F(V)$.

IMPORTANT PROPOSITIONS.

We assume that V is a finitely generated vector space over a field F.

Proposition 1. *Suppose that S is a finite generating set for V over F and that T is a subset of V which is linearly independent over F. Then $|T| \leq |S|$.*

Proposition 2. *Let $d = dim_F(V)$. Every generating set for V over F has cardinality at least d. Every linearly independent subset of V over F has cardinality at most d.*

Proposition 3. *If S is a basis for V over F, then S has cardinality equal to d.*

Proposition 4. *Let $v_1, ..., v_n \in V$ and let $S = \{v_1, ..., v_n\}$. Consider the following three statements:*

(a) S is linearly independent over F.

(b) S is a generating set for V over F.

(c) $n = d$.

Any two of these three statements imply the third and that S is a basis for V over F.

Proposition 5. *Suppose that W is a subspace of V. Then W is also finitely generated over F. Furthermore, $dim_F(W) \leq dim_F(V)$ and equality holds if and only if $W = V$.*