VECTOR SPACES.

**DEFINITION:** Suppose that $F$ is a field. A vector space $V$ over $F$ is a nonempty set with two operations, “addition” and “scalar multiplication” satisfying certain requirements.

Addition is a map $V \times V \rightarrow V$: $(v_1, v_2) \mapsto v_1 + v_2$.

Scalar multiplication is a map $F \times V \rightarrow V$: $(f, v) \mapsto f v$.

The requirements are:

(i) $V$ is an abelian group under the addition operation $+$.

(ii) $f(v_1 + v_2) = f v_1 + f v_2$ for all $f \in F$ and $v_1, v_2 \in V$.

(iii) $(f_1 + f_2)v = f_1 v + f_2 v$ for all $f_1, f_2 \in F$ and $v \in V$.

(iv) $f_1(f_2 v) = (f_1 f_2)v$ for all $f_1, f_2 \in F$ and $v \in V$.

(v) $1_F v = v$ for all $v \in V$.

**Easy results:**

1. $f 0_V = 0_V$ for all $f \in F$.
2. $0_F v = 0_V$ for all $v \in V$.
3. $(-f) v = -(fv)$ for all $f \in F$ and $v \in V$.
4. Assume that $f \in F$ and $v \in V$. The $fv = 0_V \implies f = 0_F$ or $v = 0_V$.

**DEFINITION:** Suppose that $V$ is a vector space over a field $F$ and that $W$ is a subset of $V$. We say that $W$ is a “subspace of $V$” if

1. $W$ contains $0_V$,
2. $W$ is closed under addition, and
3. $W$ is closed under scalar multiplication, i.e., $fw \in W$ for all $f \in F$ and $w \in W$.

**DEFINITION:** Suppose that $V$ is a vector space over a field $F$ and that $S = \{v_1, \ldots, v_n\}$ is a finite sequence of elements of $V$. We say that “$S$ is a generating set for $V$ over $F$” if, for every element $v \in V$, there exist elements $f_1, \ldots, f_n \in F$ such that $v = f_1 v_1 + \ldots + f_n v_n$. If such a finite sequence $S$ exists, then we say that “$V$ is a finitely generated vector space over $F$.” We then say that “$S$ generates $V$ over $F$.” One might also say that “$S$ spans $V$ over $F$” or that “$S$ is a spanning set for $V$ over $F$.”

**DEFINITION:** Suppose that $V$ is a vector space over a field $F$ and that $S = \{v_1, \ldots, v_n\}$ is a finite sequence of elements of $V$. We say that “$S$ is linearly dependent over $F$” if there
exist elements $f_1, \ldots, f_n \in F$, not all equal to $0_F$, such that $f_1v_1 + \ldots + f_nv_n = 0_V$. We say that “$S$ is linearly independent over $F$” if, for $f_1, \ldots f_n \in F$,

$$f_1v_1 + \ldots + f_nv_n = 0_V \implies f_1 = \ldots = f_n = 0_F.$$

**DEFINITION:** Suppose that $v_1, \ldots, v_n \in V$. Let $S = \{v_1, \ldots, v_n\}$. We say that $S$ is a basis for $V$” if $S$ is a generating set for $V$ over $F$ and $S$ is also a linearly independent set over $F$.

**DEFINITION:** Suppose that $V$ is a finitely generated vector space over a field $F$. Suppose that $V$ has a generating set over $F$ of cardinality $d$, but does not have a generating set over $F$ of cardinality $d - 1$. We then say that “$V$ has dimension $d$ over $F$.” We will write $d = \dim F(V)$.

**IMPORTANT PROPOSITIONS.**

We assume that $V$ is a finitely generated vector space over a field $F$.

**Proposition 1.** Suppose that $S$ is a finite generating set for $V$ over $F$ and that $T$ is a subset of $V$ which is linearly independent over $F$. Then $|T| \leq |S|$.

**Proposition 2.** Let $d = \dim F(V)$. Every generating set for $V$ over $F$ has cardinality at least $d$. Every linearly independent subset of $V$ over $F$ has cardinality at most $d$.

**Proposition 3.** If $S$ is a basis for $V$ over $F$, then $S$ has cardinality equal to $d$.

**Proposition 4.** Let $v_1, \ldots, v_n \in V$ and let $S = \{v_1, \ldots, v_n\}$. Consider the following three statements:

(a) $S$ is linearly independent over $F$.

(b) $S$ is a generating set for $V$ over $F$.

(c) $n = d$.

Any two of these three statements imply the third and that $S$ is a basis for $V$ over $F$.

**Proposition 5.** Suppose that $W$ is a subspace of $V$. Then $W$ is also finitely generated over $F$. Furthermore, $\dim F(W) \leq \dim F(V)$ and equality holds if and only if $W = V$. 