A. (Groups of order 8.)

(a) Which of the five groups $G$ have the following property: $G$ has a normal subgroup $N$ such that

$$N \cong \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad G/N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$  

Answer: Among the abelian groups, if $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or if $G \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $G$ has the stated property, as is easily verified. However, if $G$ is cyclic, then one of the homework problems shows that every quotient group of $G$ is also cyclic. Therefore $G$ does not have the stated property.

If $G$ is nonabelian and has order 8, then $G$ has the stated property. Thus, both $Q$ and $D_8$ have the stated property. We will justify this statement. One helpful observation is that if $G$ is nonabelian, then $G/Z(G)$ cannot be cyclic. This is because of the following lemma proved in class.

**Lemma.** If $G$ is any group and $G/Z(G)$ is cyclic, then $G$ is abelian.

It follows that the order of $G/Z(G)$ must be at least 4 if $G$ is nonabelian. We also proved that if $G$ is a group of order $p^n$, where $p$ is a prime and $n \geq 1$, then the center $Z(G)$ has order at least $p$. Thus, if $G$ is nonabelian and has order 8, then $|Z(G)| = 2$. One can actually verify this fact directly for the groups $Q$ and $D_8$. Thus, $G/Z(G)$ has order 4 and is not cyclic. Hence $G/Z(G) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Combining these observations, if we let $N = Z(G)$, then the above stated property holds.

(b) Which of the five groups $G$ have the following property: $G$ has a normal subgroup $N$ such that

$$N \cong \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad G/N \cong \mathbb{Z}/4\mathbb{Z}.$$  

Answer: Only two of the five groups have this property. We must have $N \subseteq Z(G)$. (See sample midterm question 4(i).) Also, $G/Z(G)$ will be cyclic. Hence, the lemma stated above shows that $G$ must be abelian. Thus, $G$ must be isomorphic to one of the following two groups (up to isomorphism)

$$\mathbb{Z}/8\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$
since $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ clearly doesn’t have the stated property. The above two groups do have the stated property, as is easily verified.

(c) Which of the five groups $G$ have the following property: $G$ has a normal subgroup $N$ such that

$$N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad G/N \cong \mathbb{Z}/2\mathbb{Z}$$

Answer: It suffices to have a subgroup $N$ of $G$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Such a subgroup would have index 2 and hence would automatically be a normal subgroup of $G$. The corresponding quotient group would have order 2 and would then be isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Groups of order 8 having such a subgroup are isomorphic to

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad \text{or} \quad D_8.$$ 

As shown in the solutions for the sample final exam, $Q$ doesn’t contain any subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Also, $\mathbb{Z}/8\mathbb{Z}$ is cyclic and hence does not contain a subgroup isomorphic to the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ because that group is not cyclic.

(d) Determine all the conjugacy classes in each of the five groups of order 8.

Answer: For the three abelian groups, the conjugacy classes are just the singletons consisting of the individual elements.

For the quaternionic group $Q$, the conjugacy classes are:

$$\{1\}, \quad \{-1\}, \quad \{i, -i\}, \quad \{j, -j\}, \quad \{k, -k\}.$$ 

For $D_8$, the conjugacy classes are:

$$\{i\}, \quad \{\beta^2\}, \quad \{\beta, \beta^3\}, \quad \{\alpha, \alpha\beta^2\}, \quad \{\alpha\beta, \alpha\beta^3\}$$

where $\beta = (1234)$ (which corresponds to a 90° rotation of a square with vertices labeled $1, 2, 3, 4$) and $\alpha = (12)(34)$ (which corresponds to a reflection of that square).
B. (Questions about $S_4$.)

(a) Suppose that $H$ is a subgroup of $S_4$ of order 8. Show that $H \cap A_4$ is the Klein 4-group. Then show that $S_4$ has exactly three distinct subgroups of order 8.

Using a homework exercise, we know that the intersection of two subgroups of a group is also a subgroup. Hence, $H \cap A_4$ is a subgroup of $S_4$. It is also a subgroup of $H$. Hence the order of $H \cap A_4$ must divide 8. However, we proved in class that every subgroup $H$ of $S_4$ of order 8 must contain the Klein 4-group $K$. Also, recall that the Klein 4-group $K$ is a subgroup of $A_4$. Therefore, we have

$$K \subseteq H \cap A_4 \subseteq H.$$ 

Hence the order of $H \cap A_4$ is divisible by 4. As already pointed out, the order of $H \cap A_4$ must divide 8.

For the above reasons, we have just two possibilities to consider:

1. $|H \cap A_4| = 4$, in which case we have $H \cap A_4 = K$,

or

2. $|H \cap A_4| = 8$, in which case we have $H \cap A_4 = H$.

However, (2) implies that $H \subset A_4$. Every element of $H$ has order dividing 8. Thus, $A_4$ would then contain at least 8 elements whose order divides 8. But, as pointed out in class, $A_4$ has 8 elements of order 3, namely all the 3-cycles in $S_4$. Those elements cannot be in $H$. Since, $|A_4| = 12$, possibility (2) cannot actually occur. Hence, only possibility (1) can occur, which implies that we indeed have $H \cap A_4 = K$.

The second part of this question was explained in class by using the correspondence theorem and the fact that $S_4/K \cong S_3$.

(b) Suppose that $K$ is a subgroup of a group $G$ and that $H$ is a subgroup of $K$. Then it is rather obvious that $H$ is a subgroup of $G$. In general, it is not necessarily true that if $H$ is a normal subgroup of $K$ and $K$ is a normal subgroup of $G$, then $H$ must be a normal subgroup of $G$. Give a counterexample where $G = S_4$.

Solution: Take $K$ to be the Klein 4-group, a normal subgroup of $S_4$. Let $H = \{i, (12)(34)\}$, a normal subgroup of $K$ because $K$ is abelian. However, $H$ is not a normal subgroup of $S_4$. This is because the conjugacy class of $(12)(34)$ in $S_4$ has cardinality 3 and is not contained in $H$. 

(c) Show that $A_4$ has no subgroup of order 6. (Hence the converse of Lagrange’s theorem is false since 6 divides $|A_4| = 12$.)

Solution: Suppose to the contrary that $A_4$ has a subgroup $N$ of order 6. Then $[A_4 : N] = 2$ and so $N$ would be a normal subgroup of $A_4$. Hence, the quotient group $A_4/N$ is a group of order 2. Suppose that $a \in A_4$ is an element of order 3. Thus, $a^3 = i$, the identity element of $A_4$. It follows that $(aN)^3 = N$. Thus, $aN$ is an element of $A_4/N$ of order dividing 3. However, $A_4/N$ has no elements of order 3. Hence, $aN$ has order 1. That is, $aN = N$, the identity element in $A_4/N$. Therefore, we have $a \in N$.

We have proved that every element $a$ in $A_4$ of order 3 must be contained in $N$. However, $A_4$ contains 8 elements of order 3. They cannot all be contained in $N$ because $|N| = 6$. This is a contradiction. Hence we conclude that $A_4$ cannot have a subgroup of order 6.

(d) Show that $A_4$ can be generated by two elements. Can $S_4$ be generated by two elements?

Solution: $A_4$ is generated by $S = \{(123), (124)\}$. To verify this, note first that those elements are in $A_4$. Let $H$ be the subgroup of $A_4$ generated by $S$. By Lagrange’s theorem, $|H|$ divides 12. The element (123) generates a subgroup of order 3 which doesn’t contain (124). Thus, $|H|$ is divisible by 3, but greater than 3. By question 3, it follows that $|H| \neq 6$. Hence, we have $|H| = 12$ and so $H = A_4$.

The answer to the second question is yes. We will show that $S_4$ is generated by the set $T = \{(34), (123)\}$. To see this, note that the subgroup generated by $T$ contains

$$(34)(123)(34)^{-1} = (124).$$

(Use the Conjugacy Principle to verify this immediately.) Let $H$ be the subgroup of $S_4$ generated by $T$. Thus, $H$ contains both (123) and (124). Hence $H$ contains the subgroup generated by those two elements, namely $A_4$. Therefore,

$$A_4 \subseteq H \subseteq S_4$$

Hence, by Lagrange’s theorem, $|H|$ is divisible by $|A_4| = 12$ and divides $|S_4| = 24$. But $H$ contains (34), an element of $S_4$ not contained in $A_4$. Hence $H$ is strictly bigger than $A_4$ and has order $> 12$. Thus, $|H| = 24$. This implies that $H = S_4$. 