SOLVABLE GROUPS

Definition. Suppose that G is a finite group. We say that G is a *"solvable group"* if the following statements are true: If $|G| > 1$, then G has a proper, normal subgroup H_1 such that the quotient group G/H_1 is abelian, if $|H_1| > 1$, then H_1 has a proper, normal subgroup H_2 such that the quotient group H_1/H_2 is abelian, if $|H_2| > 1$, then H_2 has a proper normal subgroup H_3 such that the quotient group H_2/H_3 is abelian,..., etc. If $|G| = 1$, then G is also considered to be a solvable group.

To state this definition another way, let us denote G by H_0. We say that G is a solvable group if there exists a sequence of subgroups $H_1, ..., H_k$ of G such that, for each j, $1 \leq j \leq k$, H_j is a normal subgroup of H_{j-1}, the quotient group H_{j-1}/H_j is abelian, and $H_k = \{id_G\}$.

Examples: If G is abelian, then G is a solvable group. The groups S_3 and S_4 are both solvable groups. For S_4, one can take

$$H_0 = S_4, \qquad H_1 = A_4, \qquad H_2 = \{\ id,\ (12)(34),\ (13)(24),\ (14)(23)\ \}, \qquad H_3 = \{id\}$$

Note that H_0/H_1 is a group of order 2, that H_1/H_2 is a group of order 3, and that $H_2/H_3 \cong H_2$ is a group of order 4, and all of these quotient groups are abelian.

All of the dihedral groups D_{2n} are solvable groups. If G is a power of a prime p, then G is a solvable group.

It can be proved that if G is a solvable group, then every subgroup of G is a solvable group and every quotient group of G is also a solvable group. Suppose that G is a group and that N is a normal subgroup of G. Then it can be proved that G is a solvable group if and only if both G/N and N are solvable groups.

There is a very deep theorem in finite group theory which is known as the Feit-Thompson theorem. It states that if $|G|$ is odd, then G is a solvable group. The paper giving this proof, entitled "Solvability of Groups of Odd Order," was published in 1963 and is more than 250 pages long.

It is an important fact that S_5 is not a solvable group. A normal subgroup of S_5 must be a union of conjugacy classes. The conjugacy classes in S_5 consists of permutations having the same cycle-decomposition type. There are seven types. Here are the types and the cardinalities of the corresponding conjugacy classes:

The identity element   Cardinality $= 1$

Type (ab) Cardinality $= 10$

Type (abc) Cardinality $= 20$

Type $(abcd)$ Cardinality $= 30$

Type $(abcde)$ Cardinality $= 24$

Type $(ab)(cd)$ Cardinality $= 15$

Type $(abc)(de)$ Cardinality $= 20$

Of course, the total of these cardinalities is $|S_5| = 120$. Any subgroup of S_5 must contain the identity element and must have order dividing 120. A normal subgroup must also be a union of distinct conjugacy classes. The conjugacy classes are disjoint. Checking all the possibilities, subject to the above constraints, it follows that the only possible normal subgroups of S_5 will be of orders 1, 40, 60, or 120. For order 60, one finds two sets of choices of the conjugacy classes, but only one subgroup of that order, namely A_5 which is the union of the conjugacy classes of type (abc), $(ab)(cd)$, and $(abcde)$ in addition to the identity element. For order 40, there is one choice of conjugacy classes with total cardinality equal to 40, but the union is not a subgroup.

Hence, for $G = S_5$, the only possible choice of the proper, normal subgroup H_1 is $H_1 = A_5$ since, if $H_1 = \{id\}$, then G/H_1 is nonabelian. To find the normal subgroups of A_5, we study the conjugacy classes of A_5 and their cardinalities. There are 5 conjugacy classes, namely

The identity element   Cardinality $= 1$

Type (abc) Cardinality $= 20$

Type $(abcde)$ Two conjugacy classes, each of cardinality $= 12$

Type $(ab)(cd)$ Cardinality $= 15$

Considering all possible unions of these sets (always including the identity element), the total cardinality cannot divide 60 except for the two obvious possibilities: $\{id\}$ and A_5 itself. Hence there is no possible choice of a proper, normal subgroup H_2 of $H_1 = A_5$ if we require that H_1/H_2 be abelian.

Therefore, S_5 is not a solvable group. The group A_5 is also not a solvable group.