SECTION 3.3.

PROBLEM 22. The null space of a matrix $A$ is: $\mathcal{N}(A) = \{ X : AX = 0 \}$.

Here are the calculations of $AX$ for $X = a, b, c, d,$ and $e$.

\[
Aa = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 - 2 \\ 3 - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Ab = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 - 6 \\ 6 - 9 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}
\]

\[
Ac = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 + 4 \\ -6 + 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Ad = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 + 0 \\ 3 + 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\]

\[
Ae = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Therefore, the vectors $a, c, \text{ and } e$ are in $\mathcal{N}(A)$, but the vectors $b$ and $d$ are not.

PROBLEM 24. Here are the calculations of $AX$ for $X = v, w, x, y, \text{ and } z$.

\[
Av = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 + 2 + 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}, \quad Aw = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 - 1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}
\]

\[
Ax = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 + 1 - 1 \end{bmatrix} = \begin{bmatrix} -2 \end{bmatrix}, \quad Ay = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 - 2 + 2 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}
\]

\[
Az = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 + 0 + 2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}
\]

Hence, the vectors $v, w, \text{ and } z$ are in $\mathcal{N}(A)$, but the vectors $x$ and $y$ are not.

PROBLEM 34. The null space of a matrix $A$ is: $\mathcal{N}(A) = \{ X : AX = 0 \}$. We can find a simple algebraic specification for $\mathcal{N}(A)$ by reducing $A$ to its reduced echelon form:

\[
A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 1 & 0 & 7 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}
\]
The last matrix $E$ is in reduced echelon form. Since $\mathcal{N}(A) = \mathcal{N}(E)$, we have the following algebraic specification for $\mathcal{N}(A)$:

$$x_1 + 7x_3 = 0, \quad x_2 + 3x_3 = 0$$

To find an algebraic specification for $\mathcal{R}(A)$, we consider the augmented matrix $[A|b]$, where

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Row-reduction gives:

$$[A|b] = \begin{bmatrix} 1 & -2 & 1 & b_1 \\ 2 & -3 & 5 & b_2 \\ 1 & 0 & 7 & b_3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - 2b_1 \\ 0 & 2 & 6 & b_3 - b_1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - 2b_1 \\ 0 & 0 & 0 & (b_3 - b_1) - 2(b_2 - 2b_1) \end{bmatrix}$$

The last matrix is in echelon form. The matrix equation $AX = b$ can be solved if and only if $(b_3 - b_1) - 2(b_2 - 2b_1) = 0$. This simplifies to

$$3b_1 - 2b_2 + b_3 = 0$$

This equation is the algebraic specification for $\mathcal{R}(A)$.

PROBLEM 36. The null space of a matrix $A$ is: $\mathcal{N}(A) = \{X : AX = 0\}$. We can find a simple algebraic specification for $\mathcal{N}(A)$ by reducing $A$ to its reduced echelon form:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we see that $A$ has rank 3 and so is a nonsingular matrix. The matrix equation $AX = 0$ has only one solution, namely $X = 0$. Hence

$$\mathcal{N}(A) = \{0\}$$

Also, since $A$ is nonsingular, it follows that for every vector $b$ in $\mathbb{R}^3$, the matrix equation $AX = b$ has at least one solution (in fact, exactly one solution). Therefore,

$$\mathcal{R}(A) = \mathbb{R}^3$$
SECTIOIN 3.4.

PROBLEM 2. We solve this system of equations by Gauss-Jordan elimination. Subtracting
the 2nd equation from the 1st equation gives the following equivalent system of equations:

\[
\begin{align*}
    x_1 + x_3 + 2x_4 &= 0 \\
    x_2 - 2x_3 - x_4 &= 0
\end{align*}
\]

Thus, the leading variables are \( x_1 \) and \( x_2 \). The free variables are \( x_3 \) and \( x_4 \). The solutions to the given system of equations are given in vector form by:

\[
X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - 2x_4 \\ 2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\]

where \( x_3 \) and \( x_4 \) are arbitrary numbers. In this question, \( W \) is a subspace of \( \mathbb{R}^4 \) and has the following basis:

\[
\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

PROBLEM 4. We just have a system of 1 equation in the 4 unknowns \( x_1, x_2, x_3 \) and \( x_4 \). Note that \( x_1 \) is the leading variable and \( x_2, x_3 \) and \( x_4 \) are the free variables. Here is a description of the solutions in vector form:

\[
X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

In this question, \( W \) is a subspace of \( \mathbb{R}^4 \) and has the following basis:

\[
\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]
PROBLEM 10b. The vector

\[ \mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix} \]

is in \( W \) because \( x_1 = 0, x_2 = 3, x_3 = 2, x_4 = -1 \) is a solution to the homogeneous system of equations given in exercise 2. That is,

\[
\begin{align*}
0 + 3 - 2 + (-1) &= 0 \\
3 - 2(2) - (-1) &= 0
\end{align*}
\]

In order to express \( \mathbf{x} \) as a linear combination of the basis vectors found in exercise 2, we just consider the vector equation

\[
a \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}
\]

We get the solution \( \mathbf{x} \) by letting \( a = x_3 = 2 \) and \( b = x_4 = -1 \). Thus,

\[
2 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}
\]

PROBLEM 10c. The vector

\[ \mathbf{x} = \begin{bmatrix} 7 \\ 8 \\ 3 \\ 2 \end{bmatrix} \]

is not in \( W \) because \( x_1 = 7, x_2 = 8, x_3 = 3, x_4 = 2 \) is not a solution to the system of equations given in exercise 2. The first equation in the system is not satisfied because

\[ 7 + 8 - 3 + 2 = 14 \neq 0 \]

PROBLEM 11b. We can find a basis for the null space of \( A \) by finding the reduced echelon form for \( A \). We do the row-reduction as follows:

\[
A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 5 & 8 & -2 \\ 1 & 1 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 1 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 2 & 3 & -1 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad E = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The last matrix \( E \) is the reduced echelon form for \( A \). The solutions to \( AX = 0 \) are the same as the solutions to \( EX = 0 \). Denoting the unknowns by \( x_1, x_2, x_3, x_4 \), the solutions to \( EX = 0 \) are described by the equations \( x_1 = -x_3 - x_4 \), \( x_2 = -x_3 + x_4 \) where \( x_3 \) and \( x_4 \) are arbitrary numbers. In vector form, the solutions are given by

\[
X = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-x_3 - x_4 \\
-x_3 + x_4 \\
x_3 \\
x_4
\end{bmatrix} = x_3 \begin{bmatrix}
-1 \\
1 \\
1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
\]

In this question, \( W = \mathcal{N}(A) \) is a subspace of \( \mathbb{R}^4 \) and has the following basis:

\[
\left\{ \begin{bmatrix}
-1 \\
1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix} \right\}
\]

**PROBLEM 12b.** We can find a basis for the null space of \( A \) by finding the reduced echelon form for \( A \). We do the row-reduction as follows:

\[
A = \begin{bmatrix}
1 & 1 & 2 \\
1 & 1 & 2 \\
2 & 3 & 5
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 2 \\
0 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix}, \quad E = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

The last matrix \( E \) is the reduced echelon form for \( A \). The solutions to \( AX = 0 \) are the same as the solutions to \( EX = 0 \). Denoting the unknowns by \( x_1, x_2, x_3 \), the solutions to \( EX = 0 \) are described by the equations \( x_1 = -x_3 \), \( x_2 = -x_3 \) where \( x_3 \) is an arbitrary number. In vector form, the solutions are given by

\[
X = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-x_3 \\
-x_3 \\
x_3
\end{bmatrix} = x_3 \begin{bmatrix}
-1 \\
-1 \\
1
\end{bmatrix}
\]

In this question, \( W = \mathcal{N}(A) \) is a subspace of \( \mathbb{R}^3 \) and has the following basis:

\[
\left\{ \begin{bmatrix}
-1 \\
-1 \\
1
\end{bmatrix} \right\}
\]
PROBLEM 32. We first determine if the matrix $A$ which has the three given vectors as its columns is singular or nonsingular:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -3 \\ -2 & 2 & -3 \end{bmatrix} , \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 4 & 1 \end{bmatrix} , \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

The last matrix is in echelon form and so the rank of $A$ is 3. This means that $A$ is nonsingular. Thus, the set $S$ is a linearly independent set consisting of three vectors and therefore $S$ is a basis of $\mathbb{R}^3$.

PROBLEM 33. As in problem 33, we first determine if the matrix $A$ which has the three given vectors as its columns is singular or nonsingular:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix} , \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 6 & 4 \end{bmatrix} , \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in echelon form. The rank of $A$ is 2. Hence $A$ is a singular matrix. The set $S$ is a linearly dependent set. Hence $S$ is not a basis for $\mathbb{R}^3$.

PROBLEM 34. The matrix $A$ which has the four vectors in $S$ as its columns is a $3 \times 4$ matrix. Hence, $\text{rank}(A) \leq 3$ and hence the rank of $A$ cannot be four. The set $S$ is a linearly dependent set and so $S$ isn’t a basis for $\mathbb{R}^3$.

PROBLEM 35. The set $S$ consists of two vectors in $\mathbb{R}^3$. Those two vectors determine a plane $\pi$ in $\mathbb{R}^3$ through the origin. It is clear that $\text{Sp}(S)$ is the set of vectors which lie on that plane $\pi$. Thus, $\text{Sp}(S) \neq \mathbb{R}^3$. Therefore $S$ is not a spanning set for $\mathbb{R}^3$. Therefore $S$ cannot be a basis for $\mathbb{R}^3$.

SECTION 3.5.

PROBLEM 10. Since neither vector in the set $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ is equal to scalar multiple of the other, the set $S$ is a linearly independent set of two vectors in $\mathbb{R}^2$. Using part 3 of theorem 9, it follows that $S$ is a basis for $\mathbb{R}^2$. 
PROBLEM 11. Since neither vector in the set \( S = \{\mathbf{u}_2, \mathbf{u}_3\} \) is equal to scalar multiple of the other, the set \( S \) is a linearly independent set of two vectors in \( \mathbb{R}^2 \). Hence \( S \) is a basis for \( \mathbb{R}^2 \).

PROBLEM 12. The set \( S \) contains three vectors in \( \mathbb{R}^3 \). This set \( S \) will be a basis for \( \mathbb{R}^3 \) if and only if the \( S \) is a linearly independent set. We test this by determining whether the matrix \( A \) which has the vectors in \( S \) as its columns is singular or nonsingular:

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
-1 & 1 & -1 \\
1 & 2 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 2 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Thus, \( A \) has rank 3 and so \( A \) is nonsingular. Therefore, \( S \) is a basis for \( \mathbb{R}^3 \).

PROBLEM 13. We are given a set \( S \) of three vectors in \( \mathbb{R}^3 \). We will consider the matrix \( A \) which has the three vectors in \( S \) as its columns. We determine the rank of \( A \) as follows:

\[
A = \begin{bmatrix}
1 & 0 & -1 \\
-1 & 1 & 3 \\
1 & 2 & 3
\end{bmatrix}, \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 2 \\
1 & 2 & 3
\end{bmatrix}, \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 2 \\
0 & 2 & 4
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]

The last matrix is in echelon form and so \( A \) has rank 2. This implies that \( S \) is a linearly dependent set. Therefore, \( S \) is not a basis for \( \mathbb{R}^3 \).

PROBLEM 22. First, we find the reduced echelon form for \( A \):

\[
A = \begin{bmatrix}
-1 & 2 & 0 \\
2 & -5 & 1
\end{bmatrix}, \begin{bmatrix}
1 & -2 & 0 \\
2 & -5 & 1
\end{bmatrix}, \begin{bmatrix}
1 & -2 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
1 & 0 & -2 \\
0 & 1 & -1
\end{bmatrix}
\]

The matrix \( E \) is the reduced echelon form for \( A \). The solutions to the matrix equation \( AX = 0 \) are the same as the solutions to the matrix equation \( EX = 0 \). That is, \( \mathcal{N}(A) = \mathcal{N}(E) \). If we denote the unknowns by \( x_1, x_2, x_3 \), then \( x_1 \) and \( x_2 \) are the leading variables and \( x_3 \) is a free variable. The solutions to \( AX = 0 \) are described by \( x_1 - 2x_3 = 0, x_2 - x_3 = 0 \). In vector form, the solutions are described by:

\[
X = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
2x_3 \\
x_3 \\
x_3
\end{bmatrix} = x_3 \begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}
\]
A basis for $\mathcal{N}(A)$ is:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Hence $\dim(\mathcal{N}(A)) = 1$ and so the nullity of $A$ is 1. The rank of $A$ is 2 because the reduced echelon form $E$ for $A$ has 2 nonzero rows.

PROBLEM 24. The null space of $A$ is a subspace of $\mathbb{R}^4$. To find a basis for $\mathcal{N}(A)$, we first find the reduced echelon form for $A$:

$$A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

Denoting the unknowns by $x_1, x_2, x_3, x_4$, then $x_1, x_2$ and $x_4$ are the leading variables and $x_3$ is the free variables. The matrix equation $EX = 0$ is equivalent to the system of equations:

$$x_1 - 2x_3 = 0$$
$$x_2 + x_3 = 0$$
$$x_4 = 0$$

Thus, the solutions to $EX = 0$ can be described in vector form as follows:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Thus, a basis for $\mathcal{N}(A)$ is:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The nullity of $A$ is 1. The rank of $A$ is 3.
PROBLEM 26. To find a basis for $\mathcal{R}(A)$, we use the method of casting out vectors. To do this, we first find the reduced echelon form for $A$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 4 \\ 2 & 1 & 5 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 0 & -1 & 1 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the leading 1’s in $E$ occur in columns 1 and 2, we obtain a basis for $\mathcal{R}(A)$ by choosing those two columns from $A$. Here is a basis for $\mathcal{R}(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix} \right\}$$

Thus, $\dim(\mathcal{R}(A)) = 2$. Since $\dim(\mathcal{R}(A)) = \text{rank}(A)$, we have $\text{rank}(A) = 2$. Therefore, the nullity of $A$ is $\dim(\mathcal{N}(A)) = 4 - \text{rank}(A) = 4 - 2 = 2$.

PROBLEM A: This is question 1 from the Autumn, 2007 sample exam.

(a) We first reduce $A$ to its reduced echelon form $E$:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The null space of $A$ is defined to be $\mathcal{N}(A) = \{ X : AX = 0_{2} \}$. Here we are using the notation $0_{2}$ for the zero-vector in $\mathbb{R}^{2}$. The matrix equation $AX = 0_{2}$ has the same solutions as the matrix equation $EX = 0_{2}$. We find these solutions by re-introducing the unknowns to get the equations

$$x_{1} + x_{2} + 0x_{3} - x_{4} = 0, \quad x_{3} + 2x_{4} = 0$$

Note that $x_{1}$ and $x_{3}$ are the leading variables, $x_{2}$ and $x_{4}$ are the free variables. We can describe the solutions in vector form:

$$X = \begin{bmatrix} x_{1} \\ x_{2} \\ x_{3} \\ x_{4} \end{bmatrix} = \begin{bmatrix} -1x_{2} + x_{4} \\ x_{2} \\ -2x_{4} \\ x_{4} \end{bmatrix} = x_{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$
where $x_2$ and $x_4$ are arbitrary. As explained in class, we can then write down the following basis for $\mathcal{N}(A)$:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(b) The dimension of $\mathcal{N}(A)$ is the number of vectors in a basis for $\mathcal{N}(A)$. This number is 2. Hence the dimension of $\mathcal{N}(A)$ is 2.

(c) To give an example of a spanning set for $\mathcal{N}(A)$ which is not a basis for $\mathcal{N}(A)$, we need to find a set of vectors in $\mathcal{N}(A)$ which contains a basis for $\mathcal{N}(A)$, but which fails to be linearly independent. We need at least three vectors since $\mathcal{N}(A)$ has dimension 2. We can simply modify the answer to part (a) by including one extra vector from $\mathcal{N}(A)$. We include a third vector, namely the sum of the two specified vectors in the answer to (a). Here is that example:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

That set is a spanning set for $\mathcal{N}(A)$, but not a basis for $\mathcal{N}(A)$.

(d) We are assuming that $\mathcal{R}(C) = \mathcal{N}(A)$. Now $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^4$ and hence $\mathcal{R}(C)$ will also be a subspace of $\mathbb{R}^4$. Recall that the columns of a matrix $C$ form a spanning set for $\mathcal{R}(C)$. Hence each column of $C$ is contained in $\mathcal{R}(C)$. Since $\mathcal{R}(C)$ is a subspace of $\mathbb{R}^4$, as just stated, it follows that each column of $C$ is contained in $\mathbb{R}^4$. Therefore, $C$ has 4 rows. Since $C$ is $n \times n$, we must have $n = 4$.

PROBLEM B. Let $A$ denote the $2 \times 6$ zero matrix. The null space for $A$ is a subspace of $\mathbb{R}^6$. By definition, $\mathcal{N}(A) = \{ X : AX = 0 \}$. Notice that if $X$ is any vector in $\mathbb{R}^6$, then

$$AX = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, every vector $X$ in $\mathbb{R}^6$ satisfies that equation $AX = 0$. Therefore,

$$\mathcal{N}(A) = \mathbb{R}^6$$

The null space of $A$ is $\mathbb{R}^6$. 