SECTION 1.8.

PROBLEM 4. We want to find a polynomial \( p(t) = a + bt + ct^2 \) such that
\[
p(1) = 5, \quad p(3) = 11, \quad p(4) = 14
\]
Equivalently, we want to find \( a, b, \) and \( c \) such that
\[
a + b + c = 5
a + 3b + 9c = 11
a + 4b + 16c = 14
\]
The augmented matrix for this system of equations is
\[
\begin{bmatrix}
1 & 1 & 1 & 5 \\
1 & 3 & 9 & 11 \\
1 & 4 & 16 & 14
\end{bmatrix}
\]
Row-reduction transforms this as follows:
\[
\begin{bmatrix}
1 & 1 & 1 & 5 \\
0 & 2 & 8 & 6 \\
0 & 3 & 15 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 5 \\
0 & 1 & 4 & 3 \\
0 & 3 & 15 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -3 & 2 \\
0 & 1 & 4 & 3 \\
0 & 0 & 3 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]
Thus, the solution is \( a = 2, \ b = 3, \ c = 0. \) That is, the polynomial we want is
\[
p(t) = 2 + 3t.
\]
It turns out to be just a linear polynomial because the three points \((1,5), (3,11)\) and \((4,14)\) happen to lie on a line.

SECTION 3.1.

PROBLEM 16. This should be graphed as the line through the origin of the \( xy \)-plane which has slope equal to 3. That is, it is the line defined by \( y = 3x. \)

PROBLEM 22. This subset \( W \) can be described as follows:
\[
W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \ x - 2y = 1 \right\}
\]

SECTION 3.2.

PROBLEM 2. The zero vector
\[
0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
is not in $W$ because $0 - 0 \neq 2$. Hence $W$ is not a subspace of $\mathbb{R}^2$.

**PROBLEM 6.** The only solution to the equation $|x_1| + |x_2| = 0$ is $x_1 = x_2 = 0$. That is,

$$W = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

This subset $W$ of $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$. It satisfies all three of the subspace requirements.

**PROBLEM 8.** This subset $W$ of $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^2$. $W$ is not closed under addition. Here is a counterexample. Let

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then both $u$ and $v$ are in $W$ because $1 \times 0 = 0$ and $0 \times 1 = 0$. But

$$u + v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is certainly not in $W$ because $1 \times 1 \neq 0$.

**PROBLEM 10.** $W$ is a subspace of $\mathbb{R}^3$. Here is the verification.

(a) The zero vector $0$ of $\mathbb{R}^3$ is in $W$ because $0 = 0 + 0$.

(b) Suppose that $u$ and $v$ are in $W$. Then

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

where $u_2 = u_3 + u_1$ and $v_2 = v_3 + v_1$. Consider $u + v$. We have

$$u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

We have

$$u_2 + v_2 = (u_3 + u_1) + (v_3 + v_1) = (u_3 + v_3) + (u_1 + v_1)$$

Thus $u + v$ is a vector in $\mathbb{R}^3$ whose entries $x_1, x_2, x_3$ satisfy the equation $x_2 = x_3 + x_1$ defining the set $W$. Thus, $u + v$ is in $W$. This verifies that the set $W$ is closed under addition.
(c) Suppose that \( \mathbf{u} \) is in \( W \). Let \( c \) be any real number. Then

\[
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
\]

where \( u_2 = u_3 + u_1 \). Consider the vector

\[
\mathbf{c}\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}
\]

We have \( cu_2 = c(u_3 + u_1) = cu_3 + cu_1 \). Therefore, \( \mathbf{c}\mathbf{u} \) is a vector in \( \mathbb{R}^3 \) whose entries \( x_1, x_2, x_3 \) satisfy the equation \( x_2 = x_3 + x_1 \) defining the set \( W \). This verifies that the set \( W \) is closed under scalar multiplication.

We have verified that \( W \) satisfies the three subspace requirements. Hence \( W \) is a subspace of \( \mathbb{R}^3 \).

**SECTION 3.3.**

**PROBLEM 4.** We must consider the vector equation

\[
x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\]

The corresponding augmented matrix is

\[
\begin{bmatrix} 1 & 2 & b_1 \\ -1 & -3 & b_2 \end{bmatrix}
\]

This matrix is row-equivalent to

\[
\begin{bmatrix} 1 & 2 & b_1 \\ -1 & -3 & b_2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_1 + b_2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & -b_1 - b_2 \end{bmatrix}
\]

It is clear that the vector equation being considered has a solution. This is true for every choice of the numbers \( b_1, b_2 \). Hence \( \text{Sp}(S) = \mathbb{R}^2 \).

**PROBLEM 6.** We must consider the vector equation

\[
x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\]
The corresponding augmented matrix is

\[
\begin{bmatrix}
1 & -2 & b_1 \\
-1 & 2 & b_2
\end{bmatrix}
\]

This matrix is row-equivalent to

\[
\begin{bmatrix}
1 & -2 & b_1 \\
-1 & 2 & b_2
\end{bmatrix}, \quad \begin{bmatrix}
1 & -2 & b_1 \\
0 & 0 & b_1 + b_2
\end{bmatrix}
\]

It follows that the vector equation under consideration has a solution if and only if \( b_1 + b_2 = 0 \). This is the algebraic specification of \( \text{Sp}(S) \). The subspace \( \text{Sp}(S) \) of \( \mathbb{R}^2 \) is a line through the origin with slope equal to -1.

PROBLEM 10. In problem 4, we found that \( \text{Sp}\{\mathbf{a}, \mathbf{b}\} = \mathbb{R}^2 \). If \( \mathbf{e} \) is any vector whatsoever in \( \mathbb{R}^2 \), then \( \text{Sp}\{\mathbf{a}, \mathbf{b}, \mathbf{e}\} \) will be a subspace of \( \mathbb{R}^2 \). But it will also contain \( \text{Sp}\{\mathbf{a}, \mathbf{b}\} = \mathbb{R}^2 \). Therefore, \( \text{Sp}\{\mathbf{a}, \mathbf{b}, \mathbf{e}\} = \mathbb{R}^2 \).

PROBLEM 12. Since \( \mathbf{v} \) is a nonzero vector in \( \mathbb{R}^3 \), it is clear that \( \text{Sp}\{\mathbf{v}\} \) is a certain line through the origin. To find an algebraic specification, we must consider the vector equation

\[
\begin{bmatrix}
x \\
x \\
x
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
0
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]

The corresponding augmented matrix is

\[
\begin{bmatrix}
1 & b_1 \\
2 & b_2 \\
0 & b_3
\end{bmatrix}
\]

which is row-equivalent to

\[
\begin{bmatrix}
1 & b_1 \\
0 & b_2 - 2b_1 \\
0 & b_3
\end{bmatrix}
\]

Thus the algebraic specification for \( \text{Sp}\{\mathbf{v}\} \) is:

\[ b_2 - 2b_1 = 0, \quad b_3 = 0 \]

As we already stated, this describes a line through the origin.
PROBLEM 14. We must consider the vector equation

\[
x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\]

The corresponding augmented matrix is

\[
\begin{bmatrix}
1 & 0 & b_1 \\
2 & -1 & b_2 \\
0 & 1 & b_3 \\
\end{bmatrix}
\]

This is row-equivalent to

\[
\begin{bmatrix}
1 & 0 & b_1 \\
0 & 1 & 2b_1 - b_2 \\
0 & 1 & b_3 \\
\end{bmatrix} ,
\begin{bmatrix}
1 & 0 & b_1 \\
0 & 1 & 2b_1 - b_2 \\
0 & 0 & b_3 - (2b_1 - b_2) \\
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & b_1 \\
0 & 1 & 2b_1 - b_2 \\
0 & 0 & -2b_1 + b_2 + b_3 \\
\end{bmatrix}
\]

Thus the vector equation being considered has at least one solution if and only if the equation

\[-2b_1 + b_2 + b_3 = 0\]

is satisfied. This is the algebraic specification for \(\text{Sp}\{v, w\}\). Thus \(\text{Sp}\{v, w\}\) is the plane through the origin with equation \(-2x + y + z = 0\).

PROBLEM 16. We must consider the vector equation

\[x_1v + x_2w + x_3x = b\]

where

\[
b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\]

This vector equation corresponds to a system of 3 equations in the 3 unknowns \(x_1, x_2, x_3\). The augmented matrix for this system is

\[
\begin{bmatrix}
1 & 0 & 1 & b_1 \\
2 & -1 & 1 & b_2 \\
0 & 1 & -1 & b_3 \\
\end{bmatrix}
\]

This matrix is row equivalent to

\[
\begin{bmatrix}
1 & 0 & 1 & b_1 \\
0 & -1 & -1 & b_2 - 2b_1 \\
0 & 1 & -1 & b_3 \\
\end{bmatrix} ,
\begin{bmatrix}
1 & 0 & 1 & b_1 \\
0 & 1 & 1 & 2b_1 - b_2 \\
0 & 1 & -1 & b_3 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 1 & b_1 \\
0 & 1 & 1 & 2b_1 - b_2 \\
0 & 0 & -2 & b_3 - (2b_1 - b_2)
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 1 & b_1 \\
0 & 1 & 1 & 2b_1 - b_2 \\
0 & 0 & 1 & -1/2b_3 + 1/2(2b_1 - b_2)
\end{bmatrix}
\]

It is clear that the vector equation being considered has a solution. This is true for every choice of the numbers \(b_1, b_2, b_3\). Hence \(\text{Sp}(S) = \mathbb{R}^3\).

PROBLEM 18. We must consider the vector equation

\[x_1 \mathbf{v} + x_2 \mathbf{w} + x_3 \mathbf{z} = \mathbf{b}\]

where

\[
\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\]

This vector equation corresponds to a system of 3 equations in the 3 unknowns \(x_1, x_2, x_3\). The augmented matrix for this system is

\[
\begin{bmatrix}
1 & 0 & 1 & b_1 \\
2 & -1 & 0 & b_2 \\
0 & 1 & 2 & b_3
\end{bmatrix}
\]

This matrix is row equivalent to

\[
\begin{bmatrix}
1 & 0 & 1 & b_1 \\
0 & -1 & -2 & b_2 - 2b_1 \\
0 & 1 & 2 & b_3
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 1 & b_1 \\
0 & 1 & 2 & 2b_1 - b_2 \\
0 & 1 & 2 & b_3
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 1 & b_1 \\
0 & 1 & 2 & 2b_1 - b_2 \\
0 & 0 & 0 & b_3 - (2b_1 - b_2)
\end{bmatrix}
\]

The last matrix simplifies to

\[
\begin{bmatrix}
1 & 0 & 1 & b_1 \\
0 & 1 & 2 & 2b_1 - b_2 \\
0 & 0 & 0 & -2b_1 + b_2 + b_3
\end{bmatrix}
\]

It follows that the system of equations, and hence the equivalent vector equation, has a solution if and only if \(-2b_1 + b_2 + b_3 = 0\). That is the algebraic specification for \(W\). The subspace \(\text{Sp}\{\mathbf{v}, \mathbf{w}, \mathbf{z}\}\) of \(\mathbb{R}^3\) is a plane defined by the equation \(-2x + y + z = 0\).

PROBLEM 20. This problem is a continuation of problem 14. The algebraic specification for \(\text{Sp}(S)\), where \(S = \{\mathbf{v}, \mathbf{w}\}\), is \(-2b_1 + b_2 + b_3 = 0\). We can use this to test the specified vectors.

(a): \(-2 \times 1 + 1 + 1 = 0\)
(b): \(-2 \times 1 + 1 + (-1) \neq 0\)
(c): \(-2 \times 1 + 2 + 0 = 0\)
(d): \(-2 \times 2 + 3 + 1 = 0\)
(e): \(-2 \times (-1) + 2 + 4 \neq 0\)
(f): \(-2 \times 1 + 1 + 3 \neq 0\)

Thus, the vectors in (a), (c), and (d) are in \(\text{Sp}(S)\). They can be expressed as linear combinations of the vectors \(v\) and \(w\). That is, the vector equation considered in problem 14 has at least one solution. We can find a solution by considering the augmented matrix in problem 14. Assuming that \(-2b_1 + b_2 + b_3 = 0\), that matrix is row-equivalent to

\[
\begin{bmatrix}
1 & 0 & b_1 \\
0 & 1 & 2b_1 - b_2 \\
0 & 0 & 0
\end{bmatrix}
\]

Thus the solution is given by \(x = b_1, y = 2b_1 - b_2\). For the vector in (a), we have \(b_1 = b_2 = 1\) and so \(x = 1, y = 1\). For the vector in (c), \(b_1 = 1, b_2 = 2\) and so \(x = 1, y = 0\). For the vector in (d), \(b_1 = 2, b_2 = 3\) and so \(x = 2, y = 1\). Thus, for the three vectors in (a), (c), and (d), we find the following expressions for those vectors as linear combinations of \(v\) and \(w\):

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} = v + w,
\begin{bmatrix}
1 \\
2 \\
0
\end{bmatrix} = v,
\begin{bmatrix}
2 \\
3 \\
1
\end{bmatrix} = 2v + w
\]

PROBLEM A. A vector \(X\) lies on the plane of reflection if and only if \(X\) satisfies the equation \(SX = X\), This equation is equivalent to \(AX = 0\), where \(A\) is the following matrix

\[
A = S - I_3 = \begin{bmatrix}
1/3 & 2/3 & 2/3 \\
2/3 & 1/3 & -2/3 \\
2/3 & -2/3 & 1/3
\end{bmatrix} - \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
-2/3 & 2/3 & 2/3 \\
2/3 & -2/3 & -2/3 \\
2/3 & -2/3 & -2/3
\end{bmatrix}
\]

We can find the solutions to \(AX = 0\) by considering the augmented matrix \([A|0]\). This matrix is row-equivalent to

\[
\begin{bmatrix}
-2/3 & 2/3 & 2/3 & 0 \\
2/3 & -2/3 & -2/3 & 0 \\
2/3 & -2/3 & -2/3 & 0
\end{bmatrix}, \begin{bmatrix}
1 & -1 & -1 & 0 \\
2/3 & -2/3 & -2/3 & 0 \\
2/3 & -2/3 & -2/3 & 0
\end{bmatrix}, \begin{bmatrix}
1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Thus the solutions to $AX = 0$ are precisely the solutions to the equation $x - y - z = 0$. That is the equation for the plane of reflection.

PROBLEM B. We must solve the vector equation stated in this problem. The corresponding augmented matrix is:

$$
\begin{bmatrix}
3 & 1 & 3 & 150,000 \\
4 & 0 & -4 & 100,000 \\
1 & -1 & 1 & 50,000
\end{bmatrix}
$$

This matrix is row-equivalent to the following matrices:

$$
\begin{bmatrix}
1 & -1 & 1 & 50,000 \\
4 & 0 & -4 & 100,000 \\
3 & 1 & 3 & 150,000
\end{bmatrix},
\begin{bmatrix}
1 & -1 & 1 & 50,000 \\
0 & 4 & -8 & -100,000 \\
3 & 1 & 3 & 150,000
\end{bmatrix},
\begin{bmatrix}
1 & -1 & 1 & 50,000 \\
0 & 4 & -8 & -100,000 \\
0 & 4 & 0 & 0
\end{bmatrix}
$$

Thus the solution is given by $c_1 = 37,500$, $c_2 = 0$, $c_3 = 12,500$. That is, we have:

$$
\begin{bmatrix}
150,000 \\
100,000 \\
50,000
\end{bmatrix} = 37,500 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 12,500 \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}
$$