SECTION 1.5.

PROBLEM 44. The matrix is already in reduced echelon form. It corresponds to a system of 3 equations in 4 unknowns, which we denote by $x_1, x_2, x_3,$ and $x_4$. The leading 1's are in columns 1, 2, and 4. Hence, the leading variables are $x_1, x_2,$ and $x_4$. The variable $x_3$ is a free variable. The equations are

\[
\begin{align*}
    x_1 - x_3 &= -1 \\
    x_2 + 2x_3 &= 1 \\
    x_4 &= 1
\end{align*}
\]

Thus, we have:

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} =
\begin{bmatrix}
    -1 + x_3 \\
    1 - 2x_3 \\
    0 + 1x_3 \\
    1 + 0x_3
\end{bmatrix}
\]

where $x_3$ is arbitrary. We can then describe the solutions in vector form as follows:

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} =
\begin{bmatrix}
    -1 \\
    1 \\
    0 \\
    1
\end{bmatrix}
+ x_3
\begin{bmatrix}
    1 \\
    -2 \\
    1 \\
    0
\end{bmatrix}
\]

where $x_3$ is arbitrary.

PROBLEM 57. First we calculate $P \mathbf{x} =$

\[
\begin{bmatrix}
    .70 & .15 & .30 \\
    .20 & .80 & .20 \\
    .10 & .05 & .50
\end{bmatrix}
\begin{bmatrix}
    150,000 \\
    100,000 \\
    50,000
\end{bmatrix}
= \begin{bmatrix}
    105,000 + 15,000 + 15,000 \\
    30,000 + 80,000 + 10,000 \\
    15,000 + 5,000 + 25,000
\end{bmatrix}
= \begin{bmatrix}
    135,000 \\
    120,000 \\
    45,000
\end{bmatrix}
\]

This represents the state vector after one year has passed.

Next, we calculate $P^2 \mathbf{x}$. This is the same as $P(P \mathbf{x})$ by the associative law of multiplication. Using the above calculation, we obtain $P^2 \mathbf{x} = P(P \mathbf{x}) =$

\[
\begin{bmatrix}
    .70 & .15 & .30 \\
    .20 & .80 & .20 \\
    .10 & .05 & .50
\end{bmatrix}
\begin{bmatrix}
    135,000 \\
    120,000 \\
    45,000
\end{bmatrix}
= \begin{bmatrix}
    94,500 + 18,000 + 13,500 \\
    27,000 + 96,000 + 9,000 \\
    13,500 + 6,000 + 22,500
\end{bmatrix}
= \begin{bmatrix}
    126,000 \\
    132,000 \\
    42,000
\end{bmatrix}
\]
This represents the state vector after two years have passed.

**SECTION 1.6**

**PROBLEM 2.** We have

\[(FE)D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ 5 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 19 \\ 19 \end{pmatrix}\]

On the other hand, we have

\[F(ED) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 12 & 27 \\ 7 & 14 \end{pmatrix} = \begin{pmatrix} 19 \\ 19 \end{pmatrix}\]

This verifies that \((FE)D = F(ED)\).

**PROBLEM 12.** We have \((EF)v = \begin{pmatrix} 3 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -3 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\)

**PROBLEM 20.** First we compute \(Dv\), obtaining

\[\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}\]

Then \(||Dv|| = \sqrt{(-3)^2 + 9^2} = \sqrt{90} = 3\sqrt{10}||\).

**PROBLEM 26.** We can use the distributive law to obtain


This is equal to \(A^2 - B^2\) if and only if \(AB - BA\) is the \(2 \times 2\) zero matrix. Thus, the statement in question can hold if and only if \(AB = BA\). There are counterexamples to the statement. Just choose any two matrices \(A\) and \(B\) such that \(AB \neq BA\). For example, one could choose

\[A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\]
One easily checks that $AB \neq BA$ and so the statement in this question is false for this choice of $A$ and $B$.

PROBLEM 42(a). The equation is $A^T + B = C$. This implies that

$$A^T = C - B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

It follows that

$$A = (A^T)^T = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

SECTION 1.7.

PROBLEM 2. Notice that $\mathbf{v}_3 = 2\mathbf{v}_1$. Hence $2\mathbf{v}_1 + (-1)\mathbf{v}_3 = \mathbf{0}$. Therefore the set $\{\mathbf{v}_1, \mathbf{v}_3\}$ is a linearly dependent set.

PROBLEM 4. Consider the matrix $A = [\mathbf{v}_2 \quad \mathbf{v}_3]$. We reduce this matrix to a matrix in echelon form:

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The matrix $A$ has rank 2 and so the set $\{\mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

Much more simply, note that neither vector is a scalar multiple of the other vector. Since there are just two vectors in the set, it is a linearly independent set.

PROBLEM 6. The set $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ must be a linearly dependent set because the matrix $A = [\mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]$ cannot have rank 3. The matrix $A$ is a $2 \times 3$ matrix and so $\text{rank}(A) \leq 2$.

We can express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ by solving the vector equation $x\mathbf{v}_2 + y\mathbf{v}_3 = \mathbf{v}_4$

The corresponding augmented matrix is

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

and this is row-equivalent to

$$\begin{bmatrix} 1 & 1 & 1/2 \\ 3 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & -1/2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \end{bmatrix}$$
Thus, we have $x = 1, y = -1/2$ as the solution to the above vector equation. Hence
\[
v_4 = 1v_2 - 1/2v_3
\]

**PROBLEM 12.** Consider the matrix $A = \begin{bmatrix} u_1 & u_2 & u_4 \end{bmatrix}$. We will reduce this matrix to a matrix in echelon form.

\[
A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ -1 & -3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ -1 & -3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -4 \\ 0 & -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4/3 \\ 0 & -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4/3 \\ 0 & 0 & 16/3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}
\]

Thus $\text{rank}(A) = 3$. By the criterion for linear independence, the set of vectors $\{u_1, u_2, u_4\}$ is a linearly independent set.

**PROBLEM 14.** The set of vectors $\{u_0, u_2, u_3, u_4\}$ is a linearly dependent set. This is because the matrix $A$ which has these four vectors as columns is a $3 \times 4$ matrix. It is impossible for $\text{rank}(A)$ to be equal to $4$. Thus, there will be nontrivial solutions to the vector equation
\[
xu_0 + yu_2 + zu_3 + wu_4 = 0
\]

where
\[
0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

The corresponding augmented matrix is:
\[
\begin{bmatrix}
1 & 2 & -1 & 4 & 0 \\
0 & 1 & 4 & 4 & 0 \\
0 & -3 & 3 & 0 & 0
\end{bmatrix}
\]

Row-reduction gives the following sequence of matrices:
\[
\begin{bmatrix}
1 & 2 & -1 & 4 & 0 \\
0 & 1 & 4 & 4 & 0 \\
0 & -3 & 3 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & -9 & -4 & 0 \\
0 & 1 & 4 & 4 & 0 \\
0 & -3 & 3 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & -9 & -4 & 0 \\
0 & 1 & 4 & 4 & 0 \\
0 & 0 & 15 & 12 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & -9 & -4 & 0 \\
0 & 1 & 4 & 4 & 0 \\
0 & 0 & 1 & 4/5 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & 16/5 & 0 \\
0 & 1 & 4 & 4 & 0 \\
0 & 0 & 1 & 4/5 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & 16/5 & 0 \\
0 & 0 & 1 & 4/5 & 0
\end{bmatrix}
\]
The solutions to the above vector equation are given by

\[ x = -16/5w, \quad y = -4/5w, \quad z = -4/5w, \] where \( w \) is arbitrary. Taking \( w = -1 \) gives the solution \( x = 16/5, y = 4/5, z = 4/5, w = -1 \). That is,

\[ 16/5u_0 + 4/5u_2 + 4/5u_3 - 1u_4 = 0 \]

using this equation, we can express \( u_4 \) as a linear combination of the other vectors as follows:

\[ u_4 = 16/5u_0 + 4/5u_2 + 4/5u_3. \]

**PROBLEM 18.** The matrix \( C \) is \( 2 \times 2 \). We perform row-reduction on \( C \) as follows:

\[
\begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix}, \quad \begin{bmatrix}
1 & 3 \\
0 & -2
\end{bmatrix}, \quad \begin{bmatrix}
1 & 3 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

We see that \( \text{rank}(C) = 2 \) and so \( C \) is nonsingular.

**PROBLEM 22.** The matrix \( F \) is \( 3 \times 3 \). Multiplying row 2 by 1/3, we obtain:

\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 2/3 \\
0 & 0 & 1
\end{bmatrix}
\]

This matrix is in echelon form. Thus, we see that \( \text{rank}(F) = 3 \) and so \( F \) is nonsingular.

**PROBLEM 24.** The matrix \( E \) is \( 3 \times 3 \). Notice that the three columns are linearly dependent because

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} + 0 \begin{bmatrix}
1 \\
0
\end{bmatrix} + 0 \begin{bmatrix}
2 & 0 \\
3 & 1
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Since the three columns of \( E \) form a linearly dependent set, it follows that \( \text{rank}(E) \neq 3 \) and so \( E \) is a singular matrix.

Now we will solve the matrix equation \( EX = 0 \), where

\[
X = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

We reduce the augmented matrix \([E \mid 0]\) as follows:

\[
E = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 1 & 3 & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 3 & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
(Remark: Obviously, it is not really necessary to always include the 4-th column in this case. Since we are considering a homogeneous system of equations, the 4th column will consist of 0’s and those won’t change in the process of row-reduction.)

The solutions to $EX = 0$ are given by $y = z = 0$, or in vector form,

$$X = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where $c$ is arbitrary.

PROBLEM 38. The question concerns the matrix equation $FX = u_1$, where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

We will solve the matrix equation

$$FX = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

We will perform row-reduction on the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2/3 & 2/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1/3 & -1/3 \\ 0 & 1 & 2/3 & 2/3 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 2/3 & 2/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The solution to the matrix equation $FX = u_1$ is $x = -2/3, y = 4/3, z = -1$. Thus, the equivalent vector equation has the same solution. That is,

$$u_1 = -2/3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4/3 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

This expresses $u_1$ as a linear combination of the columns of the matrix $F$. 
ADDITIONAL QUESTIONS

QUESTION A. Let \( A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 0 & 3 \end{bmatrix} \).

(a) We want to find all solutions to the matrix equation \( AX = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) and to express the answer in “vector form.” That matrix equation is equivalent to a system of 2 equations in 5 unknowns. We denote the unknowns by \( x_1, x_2, x_3, x_4 \) and \( x_5 \). Then we have

\[
X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.
\]

We perform row-reduction on the augmented matrix \( \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 & 3 & 3 \end{bmatrix} \) to solve the stated matrix equation:

\[
\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 & 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\]

Reintroducing the unknowns, we have the system of equations

\[
\begin{align*}
0x_1 + 1x_2 + 1x_3 + 0x_4 + 0x_5 &= 0 \\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 1x_5 &= 1
\end{align*}
\]

The leading variables are \( x_2 \) and \( x_5 \). The free variables are \( x_1, x_3 \) and \( x_4 \). The solutions are described by

\[
x_1 = x_1, \quad x_2 = -x_3, \quad x_3 = x_3, \quad x_4 = x_4, \quad x_5 = 1,
\]

where \( x_1, x_3, \) and \( x_4 \) are arbitrary. The solutions to the matrix equation are given by

\[
X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_3 \\ x_3 \\ x_4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 1x_1 + 0x_3 + 0x_4 \\ 0 + 0x_1 - 1x_3 + 0x_4 \\ 0 + 0x_1 + 1x_3 + 0x_4 \\ 0 + 0x_1 + 0x_3 + 1x_4 \\ 1 + 0x_1 + 0x_3 + 0x_4 \end{bmatrix}
\]
where \( x_1, x_3 \) and \( x_4 \) are arbitrary. Thus, we can describe the solutions to the matrix equation 
\[
AX = \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\]
in vector form as follows:
\[
X = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
where \( x_1, x_3 \) and \( x_5 \) are arbitrary.

(b) Since \( X = U_1 \) and \( X = U_2 \) are solutions to the matrix equation considered in part (a), we know that
\[
AU_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad AU_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\]
Now \( V = 4U_1 + 6U_2 \). We can calculate \( AV \) by the distributive law. We obtain
\[
AV = A(4U_1 + 6U_2) = 4AU_1 + 6AU_2 = 4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix} + \begin{bmatrix} 6 \\ 18 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \end{bmatrix}
\]
Hence \( AV \) can be determined and we have \( AV = \begin{bmatrix} 10 \\ 30 \end{bmatrix} \).

(c) The matrix \( A \) is \( 2 \times 5 \). We want \( AB \) to be a certain \( 2 \times 3 \) matrix. Hence \( B \) must be a \( 5 \times 3 \) matrix. We will write \( B = [X \ Y \ Z] \), where \( X, Y, \) and \( Z \) are the three columns of \( B \). They are \( 5 \times 1 \) matrices. With this notation, we have \( AB = [AX \ AY \ AZ] \). We want to have
\[
AX = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad AY = \begin{bmatrix} 5 \\ 15 \end{bmatrix}, \quad AZ = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
The first matrix equation has already been solved in part (a). We just need one solution and so we will simply take \( x_1 = x_3 = x_4 = 0 \). To choose a suitable \( Y \), we can simply multiply the \( X \) just chosen by 5. For the choice of \( Z \), we can just take \( Z \) to be the 5-dimensional zero vector. Thus, one possible \( B \) is the following matrix:
\[
B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 5 & 0 \end{bmatrix}
\]
QUESTION B. According to the hint, we should first solve the matrix equation $AX = 0$, where

$$A = P - I_3 = \begin{bmatrix} .70 & .15 & .30 \\ .20 & .80 & .20 \\ .10 & .05 & .50 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.30 & .15 & .30 \\ .20 & -.20 & .20 \\ .10 & .05 & -.50 \end{bmatrix}$$

This last matrix is $A$. We will find the reduced echelon form $E$ for $A$. The solutions to the matrix equations $AX = 0$ and $EX = 0$ will be the same. This is because the corresponding augmented matrix will have $0$ as its final column. The elementary row operations will not change that final column and so there is no need to keep track of it.

$$\begin{bmatrix} -.30 & .15 & .30 \\ .20 & -.20 & .20 \\ .10 & .05 & -.50 \end{bmatrix}, \begin{bmatrix} 1 & -.5 & -1 \\ .20 & -.20 & .20 \\ .10 & .05 & -.50 \end{bmatrix}, \begin{bmatrix} 1 & -.5 & -1 \\ 0 & -.10 & .40 \\ .10 & -.50 & -.50 \end{bmatrix}, \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & -4 \\ 0 & .10 & -.40 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$$E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Reintroducing the variables $a, b, c$ gives the equations

$$a - 3c = 0$$

$$b - 4c = 0$$

$$0a + 0b + 0c = 0,$$

and the solutions are given by $a = 3c, b = 4c, c$ is arbitrary. Therefore, the solutions to $PX = X$ are given by

$$X = \begin{bmatrix} 3c \\ 4c \\ c \end{bmatrix}$$

We want $a + b + c = 300,000$. That is, $3c + 4c + c = 300,000$, which gives $c = 37,500$. Thus, the solution we want is:

$$X = \begin{bmatrix} 112,500 \\ 150,000 \\ 37,500 \end{bmatrix}.$$