SOLUTIONS FOR THE SECOND EXAM - AUTUMN, 2005

QUESTION 1.

(a) We first reduce $A$ to its reduced echelon form $E$:

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 7 \\ 2 & 4 & 2 & 7 & 18 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 2 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The null space of $A$ is defined to be $\mathcal{N}(A) = \{X : AX = 0_2 \}$. Here we are using the notation $0_2$ for the zero-vector in $\mathbb{R}^2$. The matrix equation $AX = 0_2$ has the same solutions as the matrix equation $EX = 0_2$. We find these solutions by re-introducing the unknowns to get the equations

$$1x_1 + 2x_2 + 1x_3 + 0x_4 - 5x_5 = 0, \quad 1x_4 + 4x_5 = 0$$

Note that $x_1$ and $x_4$ are the leading variables, $x_2$, $x_3$, and $x_5$ are the free variables. We can describe the solutions in vector form:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 1x_3 + 5x_5 \\ x_2 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $x_2$, $x_3$, and $x_5$ are arbitrary. As explained in class, we can then write down the following basis for $\mathcal{N}(A)$:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) The statement that $\mathcal{N}(B) = \mathcal{N}(A)$ is true. We can explain this as follows. By definition, $\mathcal{N}(A) = \{X : AX = 0_2 \}$ and $\mathcal{N}(B) = \{X : BX = 0_2 \}$. The matrix equation $AX = 0_2$ is equivalent to a system of two linear equations in five unknowns. We refer to that system of equations as system$_A$. The corresponding augmented matrix is $[A \mid 0_2]$. The matrix equation $BX = 0_2$ is equivalent to a system of two linear equations in five unknowns. We refer to that system of equations as system$_B$. The corresponding augmented matrix is $[B \mid 0_2]$. Since $A$ is row-equivalent to $B$, there is a sequence of elementary row operations which transforms $A$
to $B$. It is clear that the same sequence of elementary row-operations will transform $[A \mid 0_2]$ to $[B \mid 0_2]$. Thus, the systems of equations system$_A$ and system$_B$ are equivalent; they have exactly the same solutions. Hence the matrix equations $AX = 0_2$ and $BX = 0_2$ also have exactly the same solutions. This means that $N(A) = N(B)$, as stated above.

(c) It is true that $\mathcal{R}(B) = \mathcal{R}(A)$. Here is the explanation. In this problem, the matrix $A$ is a $2 \times 5$ matrix. Thus, $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^2$. However, since $\text{rank}(A) = 2$, if $b$ is any vector in $\mathbb{R}^2$, the matrix equation $AX = b$ will have at least one solution. It follows that $\mathcal{R}(A) = \mathbb{R}^2$. Now, if $B$ is another $2 \times 5$ matrix which is row-equivalent to $A$, then $B$ and $A$ will have the same reduced echelon form, namely the matrix $E$, and so $\text{rank}(B) = 2$. It again follows that $\mathcal{R}(B) = \mathbb{R}^2$. Hence $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are both equal to $\mathbb{R}^2$. Therefore, we indeed have $\mathcal{R}(A) = \mathcal{R}(B)$ as stated above.

Remark: In general, if $A$ and $B$ are $m \times n$ matrices which are row-equivalent, one cannot assert that $\mathcal{R}(A) = \mathcal{R}(B)$. This is often false. One can say that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are both subspaces of $\mathbb{R}^m$ and that $\text{dim}(\mathcal{R}(A)) = \text{dim}(\mathcal{R}(B))$. Two subspaces of $\mathbb{R}^m$ could have the same dimension and still be different subspaces. For example, $\mathbb{R}^2$ contains many different subspaces of dimension 1. The relevant fact in part (c) of this question is that $\mathbb{R}^2$ has only one subspace of dimension 2, namely $\mathbb{R}^2$ itself.

**QUESTION 2.**

(a) The simplest example of a basis for $\mathbb{R}^3$ is the so-called standard basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) To give an example of a spanning set for $\mathbb{R}^3$ which is not a basis for $\mathbb{R}^3$, we need to find a set of vectors which contains three linearly independent vectors, but is not a linearly independent set itself. Thus, we need at least four vectors. Here is such an example:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(c) Here is a linearly independent subset of $\mathbb{R}^4$ which is not a spanning set for $\mathbb{R}^4$ and hence not a basis for $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$
**QUESTION 3.** In this question, suppose that $C$ is an unspecified $3 \times 3$ matrix with the following properties:

$CX = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$ and $CX = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ are both solvable. Also $C \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

First notice that $C$ must be a singular matrix. This is so because the matrix equation $CX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a nontrivial solution, namely $X = \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix}$. It follows that $\text{rank}(C) < 3$.

A second thing to notice is that $\mathcal{R}(C)$ contains the vectors $\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$. Since the set \( \left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \right\} \) is a linearly independent set, it follows that $\text{dim}(\mathcal{R}(C)) \geq 2$. Since $\text{rank}(C) = \text{dim}(\mathcal{R}(C))$, it then follows that $\text{rank}(C) \geq 2$. Hence, we have $2 \leq \text{rank}(C) < 3$. Therefore, $\text{rank}(C) = 2$. Therefore, $\text{dim}(\mathcal{R}(C)) = 2$. Thus, $\mathcal{R}(C)$ is a 2-dimensional subspace of $\mathbb{R}^3$. It is the set of vectors which lie on a certain plane through the origin. We can find a basis for this subspace by just choosing any two linearly independent vectors in $\mathcal{R}(C)$. We can choose the vectors $\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \right\}$. This set of vectors is a spanning set for $\mathcal{R}(C)$. Hence we can describe $\mathcal{R}(C)$ as follows:

$$\mathcal{R}(C) = \text{Sp} \left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \right\}$$

(b) By the dimension theorem, $\text{dim}(\mathcal{N}(C)) + \text{dim}(\mathcal{R}(C)) = 3$. Since, $\text{dim}(\mathcal{R}(C)) = 2$, it follows that $\text{dim}(\mathcal{N}(C)) = 1$. Thus, we can find a basis for $\mathcal{N}(C)$ by just choosing any nonzero vector in $\mathcal{N}(C)$. One such vector is $\begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix}$. Thus, we can describe $\mathcal{N}(C)$ as follows:

$$\mathcal{N}(C) = \text{Sp} \left\{ \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix} \right\}$$
(c) The matrix $C$ is singular. Hence the matrix equation $CX = \begin{bmatrix} 8 \\ 11 \\ 15 \end{bmatrix}$ will have either no solutions or infinitely many solutions. We should determine whether or not the vector $\begin{bmatrix} 8 \\ 11 \\ 15 \end{bmatrix}$ is in the range of $C$. We use the description of $\mathcal{R}(C)$ given in part (a). Thus we must decide whether the vector equation

$$x \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 15 \end{bmatrix}$$

has any solutions. This vector equation is equivalent to a system of three equations in two unknowns with augmented matrix $\begin{bmatrix} 1 & 2 & 8 \\ 4 & 1 & 11 \\ 0 & 5 & 15 \end{bmatrix}$. This matrix is row-equivalent to

$$\begin{bmatrix} 1 & 2 & 8 \\ 0 & -7 & -21 \\ 0 & 5 & 15 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 5 & 15 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

We indeed find the solution $x = 2, y = 3$. Therefore, the vector $\begin{bmatrix} 8 \\ 11 \\ 15 \end{bmatrix}$ is in $\mathcal{R}(C)$. Therefore, the matrix equation $CX = \begin{bmatrix} 8 \\ 11 \\ 15 \end{bmatrix}$ has at least one solution. Since $C$ is singular, this matrix equation must have infinitely many solutions.