
QUESTION 1.

(a): The reduced echelon form for $A$ is the matrix $E$ given in the question. The leading 1’s occur in the first and fourth columns of $E$. Using the method of “casting out vectors,” we obtain a basis for $\mathcal{R}(A)$ by taking the corresponding columns of $A$. Here is a basis for $\mathcal{R}(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

(b): To find a basis for $\mathcal{N}(A)$, we must solve the matrix equation $AX = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This matrix equation has the same solutions as $EX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, where $E$ is the reduced echelon form for $A$.

The reduced echelon form for the matrix $A$ is $E = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

If we denote the unknowns by $x_1, x_2, x_3, x_4$ and $x_5$, then the following equations describe $\mathcal{N}(A) = \mathcal{N}(E)$:

$$\begin{align*}
1x_1 + 2x_3 + 1x_5 &= 0 \\
1x_4 - 1x_5 &= 0
\end{align*}$$

We see that $x_1$ and $x_4$ are the leading variables and that $x_2, x_3$ and $x_5$ are the free variables. The solutions to these equations can be described in vector form as follows:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_3 - 1x_5 \\ 1x_2 \\ 1x_3 \\ 1x_5 \\ 1x_5 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $x_2, x_3$ and $x_5$ are arbitrary. A basis for $\mathcal{N}(A)$ is:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$
(c): The set of vectors given in part (b) is a spanning set for \( \mathcal{N}(A) \). But that set is also linearly independent and hence a basis for \( \mathcal{N}(A) \). Here is a linearly dependent set of vectors which is a spanning set for \( \mathcal{N}(A) \):

\[
S = \begin{Bmatrix}
\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}
\end{Bmatrix}
\]

Notice that the fourth vector in the set \( S \) is just the sum of the second and third vectors in \( S \). Thus, the subspace of \( \mathbb{R}^5 \) spanned by \( S \) is exactly the same as the subspace spanned by just the first three vectors in \( S \). That subspace is \( \mathcal{N}(A) \). Therefore, \( S \) is a spanning set for \( \mathcal{N}(A) \). Since \( S \) is a linearly dependent set, it is not a basis for \( \mathcal{N}(A) \).

**QUESTION 2.**

(a): According to the answer for question 1, part (a), we have

\[
\mathcal{R}(A) = \text{Sp} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}
\]

\( \mathcal{R}(A) \) is a subspace of \( \mathbb{R}^3 \) and so \( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) is certainly in \( \mathcal{R}(A) \). The vectors

\[
5 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} \quad \text{and} \quad 5 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}
\]

are also in \( \mathcal{R}(A) \). To check whether the vector \( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \) is in \( \mathcal{R}(A) \), consider the vector equation

\[
x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\]
The corresponding augmented matrix is \[
\begin{bmatrix}
1 & 1 & 0 \\
2 & 1 & 1 \\
3 & 1 & 0
\end{bmatrix}
\]. This is row-equivalent to
\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & -1 & 1 \\
0 & -2 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & -1 \\
0 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & -1 \\
0 & 0 & -2
\end{bmatrix}
\]

Thus the vector equation has no solutions. This implies that \[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\] is not in \( \mathcal{R}(A) \).

There is no need to consider the other vectors in this question because they are vectors in \( \mathbb{R}^5 \) and so certainly cannot be in \( \mathcal{R}(A) \).

(b): Three of the given vectors are in \( \mathbb{R}^5 \). We could check whether or not they are in \( \mathcal{N}(A) \) by just checking whether or not they satisfy the matrix equation \( AX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \). However, since \( \mathcal{N}(A) = \mathcal{N}(E) \), we can also check the equivalent matrix equation \( EX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \). This is slightly simpler. We find that
\[
E \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix},
E \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
E \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

Thus the vectors \[
\begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}
\text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\] (of course!!) are in \( \mathcal{N}(A) \), but the vector \[
\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}
\] is not in \( \mathcal{N}(A) \).

**QUESTION 3.** The given information tells us that \( \text{dim(\( \mathcal{R}(A) \)) = 3} \) and \( \text{dim(\( \mathcal{N}(A) \)) = 4} \).
(a): In general, for any matrix $A$, we have $\dim(\mathcal{R}(A)) = \text{rank}(A)$. Thus, for the matrix $A$ considered in this question, we have $\text{rank}(A) = 3$.

(b): According to the dimension theorem, we have

\[ \dim(\mathcal{N}(A)) + \dim(\mathcal{R}(A)) = n. \]

Therefore, $n = 4 + 3 = 7$.

(c): We have $\text{rank}(A) \leq m$. Since $\text{rank}(A) = 3$, we have $m \geq 3$. Nothing more can be said about the value of $m$.

(d): We know that the null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. Hence, for the matrix $A$ considered in this question, $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^7$. The vector $w_3$ is in $\mathcal{N}(A)$ and so must also be in $\mathbb{R}^7$. Therefore, $w_3$ has 7 entries.

(e): The vector $u_2$ is in the range of $A$. Therefore, the matrix equation $AX = u_2$ must have at least one solution. This matrix equation is equivalent to a system of linear equations in 7 unknowns. The rank of the coefficient matrix $A$ is 3. Since $3 < 7$, that system of equations must have infinitely many solutions. Therefore, the number of solutions to the stated matrix equation is infinite.